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EXPONENTIAL FUNCTIONS AS BOUNDARY LAYER FUNCTIONS

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Abstract

We consider the singularly perturbed boundary value problems for which the boundary layers are described by exponential functions. The application of classical methods to obtain their approximatie solutions does not give satisfactory results. Hence, it is necessary to adapt the methods to the properties of exponential functions. We give a survey of the known procedures of adapting discrete and global methods. A new adaptation procedure is proposed for Shishkin meshes yielding uniform convergence of spline collocation procedures.

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1. Introduction

Models of many problems encountered in practice are described by differential equations (most often partial ones) with boundary conditions, containing a small parameter ε by the highest derivative. A change in the parameter yields abrupt changes in the solution. When the parameter becomes zero, the equation order is lowered and the boundary conditions in a general case cannot be satisfied. Normally, when $\varepsilon \neq 0$ but $\varepsilon << 1$, this deformation has consequence on the solution. Such a behaviour of the solution requires adaptation of numerical procedures, to ensure, if possible, that convergence is uniform with respect to the small parameter. The models in question are described by equations of different types [18],[19],[17], [4],[7].

The parabolic type ([4])

$$(\varepsilon^2 a(x,t)\frac{\partial^2}{\partial x^2} - p(x,t)\frac{\partial}{\partial t})u(x,t) = f((x,t),u(x,t)),$$

$$(x,t) \in G = D \times (0,T], D = (0,d),$$

$$u(x,t) = \phi(x,t), \quad (x,t) \in S, \quad S = \bar{G}\backslash G,$$

we encounter in considering, for example, the diffusion of a substance (e.g. pollution) in a homogeneous layer of solid material of thickness d. The boundary value problem for elliptic equations arise when diffusion processes in moving medium are modelled. The diffusion of substance in convective flow of an incompressible fluid in a two-dimension domain gives rise to the equation

$$-\varepsilon\Delta u(x) + \vec{v}(x)\cdot\vec{\nabla}u(x) = F(x), \quad x \in \Omega,$$

where $\vec{v}(x)$ and $F(x)$ are the velocity and source, respectively, $1/\varepsilon$ is the Peclet number. The boundary condition that describes the exchage of the substance with the surrounding environment has the form

$$-\alpha(u(x) - U(x)) - \frac{\partial}{\partial n}u(x) = 0,$$

where x belongs to the part of the boundary of the domain, $U(x)$ is a given function, and $\frac{\partial}{\partial n}$ is the outward normal derivative at the boundary. The condition on the remainder parts of the boundary are omitted.

Throughout the paper$M$ denotes any positive constant that may take different values in different formulas, but that are allways independent of ε and discretization mesh.

The simplest problem of this form is the process of stationary diffusion involving a reacting substance

$$\begin{cases} Ly = \varepsilon^2 y'' - b(x)y = f(x), \quad x \in I = [0,1], \\ u(0) = \alpha_0, \quad u(1) = \alpha_1. \end{cases}$$

(1)

ε - characterizes the diffusion coefficient; b(x) - characterizes the intensity of the substance decomposition. When $\varepsilon \to 0$ the diffusion boundary layer occurs at the boundary.

The problems that can arise may probably be best explained using the initial problem of the form [3]

$$(2) \qquad \begin{cases} Ly = \varepsilon y'(x) + y(x) = 0, & x \geq 0, \\ u(0) = 1. \end{cases}$$

If $\varepsilon \to 0$, than $y(x) \to 0$, so that the initial condition y(0)=1 cannot be satisfied. The solution to the problem is

$$y(x) = e^{-x/\varepsilon}.$$

Using the Euler scheme wtith step h we obtain the difference analogue

$$(3) \qquad \begin{cases} L_h y = \varepsilon D_+ u_i + u_i = 0, & x_i = ih, i \geq 0, \\ u_0 = 1. \end{cases}$$

The solution of the problem (3) is

$$u_i = (1 - \rho)^i, \quad \rho = h/\varepsilon.$$

For $i = 1$ and $\rho = 1$ we obtain

$$|y(h) - u_1| = e^{-1},$$

which means that the method does not converge when $h \to 0$. The operator L satisfies the maximum principle, whereas L_h does not satisfy discrete maximum principle for $\varepsilon > 1$. Then, the solution oscillates from point to point, which is not a property of an exact solution. If we take this discrete analogue

$$(4) \qquad \begin{cases} L_h y = \varepsilon D_+ u_i + u_{i+1} = 0, & x_i = ih, \quad i \geq 0, \\ u_0 = 1, \end{cases}$$

we obtain

$$u_1 = (1+\rho)^{-1} > 0,$$

but for $\rho = 1$ we have

$$|y(h) - u_1| = 1/2 - e^{-1},$$

and the convergence is not attained. The Padé approximation is taken as the solution, the discrete maximum principle is satisfied, but the convergence is not uniform. The approximation is good for $\varepsilon >> h$ and for $\varepsilon << h$. For $\rho > 3$ the error increases with decresing ρ. The difficulties are even more pronounced with boundary problems ([12],[3]), and especially with partial differential equations. To obtain approximate solutions to the above problems, the following three approaches are mainly used.

1. Asymptotic methods

They are based on the approximation of the solution with an asymptotic development to a certain degree of accuracy. The development is usually carried out with respect to the parameter ε, and the error is of the order ε^k, $k \in \boldsymbol{R}$. These methods belong to the domain of numerical mathematics, and they are not the subject of our concern.

2. Discrete methods

On the interval considered, a set of points (mesh) is determined, at which the solution approximation is sought.

3. Global approximation

The solution is sought in the form of a sufficiently smooth function $e(x)$, approximating $y(x)$ on the interval considered. However, many difference problems are generated by global methods, as they simultaneously give the solutions at the mesh points. Now we shall consider discrete methods.

2. Discrete Methods

Let us consider the problem (1). Let us suppose that $f, b \in C^2(I), b(x) \geq \beta > 0$. Then $y(x) \in C^4(I)$, which can be written in the form

$$y(x) = g(x) + w(x) + v(x),$$

where

$$|g^{(i)}(x)| \leq M, \quad i = 0(1)4, \quad v(x) = pe^{-\frac{b(0)x}{\varepsilon}}, \quad w(x) = qe^{-\frac{b(1)(1-x)}{\varepsilon}},$$

where p and q are the functions limited with respect to ε. When we apply the classical difference procedure based on polynomial approximations, the truncation error contains derivatives of the functions $v(x)$ and $w(x)$, and these derivatives become unbounded when $x \to 0$, which leads to an unbounded approximation error. In order to avoid this, we resort to fitting schemes or mesh fitting. Under the term "scheme fitting" we understand a change of the difference scheme coefficients to make the scheme more stable, and achieve that the truncation error becomes uniformly bounded with respect to ε, or both. Schemes of this type appear in the paper [1] and are analyzed in detail in the book [3]. Now, we will present the technique of fitting schemes.

2.1. Exponentialy fitted schemes

One of the ways of scheme fitting involves the introduction of "artificial viscosity" σ_i^2 by the highest derivative. For example, if for approximation of the second derivative we take the central difference quotient on a uniform mesh, we obtain the scheme

$$\text{(5)} \quad \begin{cases} L_h y = \varepsilon^2 D_+ D_- u_i + b(x_i) u_i = f(x_i), \qquad x_i = ih, \qquad 0 < i < n, \\ \\ u_0 = \alpha_0, \qquad u_n = \alpha_1 \end{cases}$$

We shall determine the fitting factor,σ_i, so that the truncation error is small for $v(x)$ and $w(x)$ when is supposed that $b(x) = \beta = const.$ Then $b(x)$ is replaced with $b(x_i)$, and we get

$$\sigma_i(\rho) = \frac{\rho^2 b(x_i)}{4\sin^2(\rho\sqrt{b(x_i)})/2}, \qquad \rho = h/\varepsilon,$$

$$|\sigma_i(\rho) - 1| \leq M\rho^2.$$

where M is a constant independent of h and ε. The scheme becomes stable and uniformly convergent, but the convergence decreases to one, because it involves a new error of the type

$$e^{-\frac{\sqrt{b(0)}x}{\varepsilon}} - e^{-\frac{\sqrt{b(x_i)}x}{\varepsilon}}.$$

To fit schemes it is necessary to know the asymptotic development of the solution and adjust the scheme in a way that the truncation error be equal to zero for the largest possible number of terms in the asymptotic development. This is difficult to achieve, and it leads to schemes with complicated coefficients. In [5] is given a fitting procedure enabling one to achieve an arbitrary order of accuracy. However, in the asymptotic development and cosequently in the scheme remain the integrals which can be solved by one of approximate methods. The method has to be chosen in such a way that a desired order of accuracy is preserved, which opens some new problems in the approximation. A similar situation arises with spline collocations, i.e. with the difference schemes derived from spline collocations. The requirement that the truncation error be equal to zero for the boundary layer functions yielded exponentially fitted spline difference schemes presented in [27], [29], and [30], converging uniformly with respect to the small parameter. The problem in this technique is in the calculation of the exponential functions in the coefficients.

Global uniform approximation is not obtained because of the difficulties with the polynomial approximation of exponential functions. Namely, on an equidistant mesh with a fixed-order polynomial spline it is not possible to achieve a global convergence.

2.2. Non-uniform Mesh

Another approach to this problem is mesh fitting ([6]). We take the schemes generated by polynomial approximations, and then adapt the scheme so to obtain uniform convergence with respect to the small parameter. This leads to the accumulation of points in the boundary layer, resulting in an increase of the system dimensions. On the other hand, the accumulation of points in the part where the function exhibits abrupt changes is of interest for a better solution approximation, so that the mesh fitting methods are interesting from this standpoint. These methods are more recent. There are two main types of meshes: meshes of Bakhalov and Shishkin type. The former are strictly directed, and the latter are piecewise equidistant.

Bakhvalov-type meshes

Bakhalov [2] constructed the function generating the mesh nodes. The nodes x_i are obtained as the values

$$\lambda(t_i), \quad t_i = ih, \quad h = 1/n,$$

where

$$\lambda(t) = \begin{cases} \psi(t) = -a\varepsilon \ln(1 - t/q), & t \in [0, \tau], \\ \psi(t) + \psi'(t)(t - \tau), & t \in [\tau, 1/2], \\ 1 - \lambda(1 - t) & t \in [1/2, 1], \end{cases}$$

$$q \in (0, 1), \quad a\varepsilon_0 \leq q, \quad \varepsilon \leq \varepsilon_0 << 1.$$

It has been shown that there is a unique point τ which is the abscissa of the contact point of the tangent from the point (1/2,1/2) to the curve $\psi(t), a$ is a constant. The point is obtained by an iterative procedure. Vulanovic [33] and Herceg [9] have introduced modification of the scheme aiming at an exact determination of the point τ. The modified function $\psi(t)$ is of the form

$$\psi(t) = a\varepsilon t/(q - t).$$

The function $\psi(t)$ in the Bakhalov mesh represents the inverse function of the boundary layer function and the function $\psi(t)$ in the modified mesh represents the inverse function for the Padé approximation of the boundary layer function. This means that for generating the meshes of this type it is necessary to know the boundary layer function and then its inverse function. The mesh in the layer is generated with respect to the inverse function, and outside of it the function of nodes distribution is continued by a tangent, or by a third-order polynomial when a higher accuracy is needed [34]. Similar meshes have been considered in the papers by Liseikin, Boglaev and many other authors.

Shishkin-type meshes

These meshes divide the interval into subintervals in which the solution has a characteristic behaviour (layers, smooth part, inflexion points). On each of these interval an equidistant mesh is formed. Thanks to the appropriately chosen division points, a uniform convergence is achieved. Let us present the Shishkin mesh for the example (1). Let be given a positive integer $n = 2^p$, $p \geq 2$. The interval $[0,1]$ is divided into the three subintervalas

$$[0,\delta], \quad [\delta, 1-\delta], \quad [1-\delta, 1].$$

The equidistant meshes on each of these subintervals, with $1 + n/4$ points in each of $[0,\delta]$ and $[1-\delta, 1]$, and $1 + n/2$ points in $[\delta, 1-\delta]$ is used. Set $b = min\{\beta, 1\}$. The transition point δ from the fine to the coarse mesh is defined by

$$\delta = min\{1/4, 4b^{-1}\varepsilon \ln n\}.$$

Set $i_0 = n/4$, then

$$x_{i_0} = \delta, \quad x_{n-i_0} = 1-\delta,$$

are the transition points and the mash spacing is given by

$$\tilde{h}_1 = h_i = 4\delta n^{-1} \quad i = 0, 1, .., i_0 - 1, n - i_0, ..., n-1,$$

and

$$\tilde{h}_2 = h_i = 2(1-2\delta)n^{-1} \quad i = i_0, ..., n - i_0 - 1.$$

We shall assume that $\delta = 4b^{-1}\varepsilon \ln n$ since in the oposite case the method can be analyzed using standard techniques. Thus we have that

$$\tilde{h}_1 = h_i = 16b^{-1}\varepsilon n^{-1} \ln n, \; i = 0, 1, .., i_0 - 1, n - i_0, ..., n-1,$$

and

$$\tilde{h}_2 = h_i = 2(1-2\delta)n^{-1}, \quad n^{-1} \leq h_i \leq 2n^{-1}, \quad i = i_0, .., n - i_0 - 1.$$

On the assumption that $\delta = 4b^{-1}\varepsilon \ln n$ for the scheme obtained when the second derivative is approximated with the second order difference approximation we obtain the error of uniform convergence of the order $O(n^{-2}\ln^2 n)$. The transition point has a key role in achieving a uniform convergence. For the majority of schemes the truncation error $\tau_j(v)$ corresponding to the function v(x) we have

$$|\tau_j(v)| \leq Me^{-b\frac{x_{i_0-1}}{2\varepsilon}} \leq Me^{-b\frac{x_{i_0}}{2\varepsilon}} e^{b\frac{h_{i_0}}{2\varepsilon}} \leq Mn^{-2}.$$

By choosing the constant at the transition point it is possible to achieve the order of convergence corresponding to the remaining part of the truncation error. Therefore, the inverse function of the boundary layer function plays again an essential role in forming the mesh yielding a uniform convergence, especially when we want to achieve a higher order of convergence. For the mentioned difference scheme on the Bakhalov mesh we obtain uniform approximation of the order $O(n^{-2})$. For a scheme of Hermitian type ([9],[34]) on the modified mesh of Bakhalov type the obtained uniform convergence is of the order $O(n^{-4})$, and on the mesh of Shishkin type with $\delta = min(1/4, 8b^{-1}\varepsilon \ln n)$ the uniform convergence order is $O(n^{-4}\ln^4 n)$([20], [21]). Therefore, the meshes of Bakhalov type give a higher degree of convergence, but they are more difficult for construction. Besides, by applying meshes of the Bakhalov type we cannot determine the solution at a point chosen in advance (additional interpolation is needed), which is often essential for practical applications. What makes special difficulties in the truncation error estimation from the one and the other mesh is the transition from the finer to coarser steps, especially when the aim is to obtain a higher degree of convergence. To achieve this, the polynomial degree is increased, by which the function of nodes outside the layer is continued with the Bakhalov meshes [34], or the assumptions are strengthened for the functions $b(x)$ and $f(x)$, and the like. In some cases use is made of more stringent estimation of the Jacobian matrix. Where is the problem? With the higher-order difference schemes and spline difference schemes, due to a high smoothness in the error estimation, we have expressions of the form

$$\frac{h_i^k}{\varepsilon^k} e^{-\frac{x_i b}{\varepsilon}},$$

which have to be estimated for $k \leq 2$. For $i = 1$, when the mesh is equidistant, it is possible to use the estimate

$$t^k e^{-t} \leq M(s) e^{-st}, \quad s \in (0,1), \quad t \in [0,\infty).$$

$M(s)$ is a constant dependent only on s. Thus we have

$$\frac{h_i^k}{\varepsilon^k} e^{-\frac{x_i b}{\varepsilon}} \leq M e^{\frac{h_i b}{2\varepsilon}}.$$

As a consequence of the Taylor expansion, in the truncation error appear expressions of the form

$$\frac{h_i^k}{\varepsilon^k} e^{-\frac{(x_i+\theta_i h_i)b}{\varepsilon}}, \quad 0 < \theta_i < 1.$$

For $i = 0$ the mentioned estimate leads to

$$\frac{h_i^k}{\varepsilon^k} e^{-\frac{\theta_i h_i b}{\varepsilon}} \leq \frac{h_i^k \theta_i^k}{\varepsilon^k \theta_i^k} e^{-\frac{\theta_i h_i) b}{\varepsilon}} \leq \frac{M}{\theta_i^k} e^{-\frac{\theta_i h_i) b}{2\varepsilon}}. \tag{6}$$

If θ_i is close to zero, the estimate is not good. With the Shishkin mesh at the transition to coarser steps we obtain expressions of the form

$$A = \frac{h_{i_0}^k}{k!\varepsilon^k} e^{-\frac{(x_{i_0}+\theta_{i_0}h_{i_0})b}{\varepsilon}}, \qquad x_{i_0} = b^{-1} m\varepsilon \ln n,$$

where $m \geq 2$. Further

$$|A| \leq \frac{h_{i_0}^k}{\varepsilon^k} e^{-\frac{\theta_{i_0}h_{i_0})b}{\varepsilon}} n^{-m}.$$

Therefore, it is necessary to estimate again the expression of the form (6). Hence, we shall analyze the quantity θ_{i_0}.

It is known [13] that in Taylor's expansion

$$f(x+h) = f(x) + hf'(x) + \frac{h^2}{2!} f''(x) + \cdots + \frac{x^p}{p!} f^{(p)}(x+\theta h)$$

holds

$$\lim_{h \to 0} \theta = \frac{1}{p+1}.$$

Hence, for example, for k=3 in (7) we have that $\theta_{i0} \to 1/4$, so that there is an n_0 such that for $n \geq n_0$ we have $\theta_i \geq 1/8$ and there holds the estimate

$$\frac{h_{i_0}^k}{\varepsilon^k} e^{-\frac{\theta_{i_0}h_{i_0}b}{\varepsilon}} \leq M e^{-\frac{\theta_{i_0}h_{i_0}b}{2\varepsilon}}.$$

Now, a question arises as to the magnitude of n_0 and if it depends only on ε. Unfortunately, the answer is affirmative. With decreasing ε increases n_0, so that the estimate is not uniform with respect to ε. We shall demonstrate this by expressing explicitly θ_{i_0}. Starting from the expression

$$e^{\frac{hb}{\varepsilon}} = \sum_{i=0}^{p-1} (-1)^i \frac{h^{p-1}b^{p-1}}{(p-1)!} + \frac{(-1)^p b^p}{p!\varepsilon^p} e^{\frac{-\theta hb}{\varepsilon}}, \qquad 0 < \theta < 1,$$

we obtain that

$$\theta = -\frac{\varepsilon}{hb}\left[(-1)^{p-1}(1-e^{-\frac{hb}{\varepsilon}})\frac{\varepsilon^p p!}{h^p b^p} + \sum_{i=1}^{p-1}(-1)^i \frac{\varepsilon^{p-i}p!}{h^{p-i}b^{p-i}}\right]. \tag{7}$$

By analyzing this expression we obtain that $\lim_{\varepsilon \to 0} \theta = 0$. Also

$$\lim_{\varepsilon \to 0} \frac{h}{\varepsilon} b e^{-\frac{\theta hb}{\varepsilon}} = p,$$

which can be used for error estimate with the schemes of lower order of accuracy. After all, this is obtained by taking the remainder in its integral form. Now we shall show how the expression for θ can be used to modify Shishkin's meshes to achieve uniform convergence with spline difference schemes. The same idea may also be used for other difference methods. With the spline difference scheme derived via a quadratic spline there appears an expression of the form

$$R_j = \frac{2h_{j-1}h_j^3}{3!(h_j + h_{j-1})}e^{-\frac{(x_j+\theta_j h_j)b}{\varepsilon^3}}, \quad 0 < \theta_j < 1.$$

Let the mesh be of Shishkin type and let $j = i_0$. In the error analysis for $0 \le i \le i_0$ we obtain that

$$|\tau_j(y)| \le Mn^{-4}\ln^4 n.$$

If we want the error at the point x_{i_0} satisfy the same estimate, we shall introduce the following constraint

$$|R_j| \le Mn^{-4}\ln^4 . \tag{8}$$

We will request the step h_{i_0} be such that the condition (8) be satisfied. Let $\tilde{x}_{i_0+1} = x_{i_0} + 2^{-k_0}$, where k_0 is the smallest k for which holds

$$\frac{n^{-1}\ln^{-3} n}{m^{-1} + 16\varepsilon n^{-1}\ln n}(\varepsilon(1 - e^{\frac{-b}{\varepsilon m}}) - m^{-1}b + \frac{m^{-2}b}{2\varepsilon}) \le 1,$$

where $m = 2^{-k_0}$. Then (8) is satisfied. This can be checked by substituting the corresponding quantities into R_j according to the Shishkin mesh, and substituting θ_j according to (7). If

$$\tilde{x}_{i_0+1} = x_{i_0} + h_{i_0}/2$$

in the already formed Shishkin mesh, the mesh remains unchanged. Then $n > n_0$, and the convergence order is attained. In the contrary case we shall add the point $\tilde{x}_{i_0+1}$ to the Shishkin mesh. The same we shall also do symmetrically at the other end of the interval, and carry out renumeration of the variables. Now the mesh will have $N = n + 2$ points. So, a question arises concerning the error estimate at the point x_{i_0} in the new mesh. It is easy to show that

$$|R_{i_0+1}| \le M|R_{i_0}|.$$

Namely, the above inequality is reduced to

$$e^{-\frac{2^{-k_0}b}{2eps}}(n^{-1}-2^{-k_0})^2\frac{n^{-3}}{\varepsilon}\leq$$

$$M\frac{n^{-5}\ln n}{2^{-k_0}+16n^{-1}\ln n}\left[(1-e^{-\frac{2^{-k_0}b}{\varepsilon}})-2^{-k_0}b+\frac{2^{-k_0}b^2}{2\varepsilon}\right],$$

where M is a constant independent of n, k_0 and ε. With the estimate of the other terms in the truncation error there are no any difficulty.

3. Global Methods

3.1. Collocation method

In order to obtain uniform convergence on the whole interval it is not sufficient to fit only the collocation conditions, as it has been done in the constructing of difference schemes. Polynomial collocation on an equidistant mesh cannot attain a uniform convergence. For this reason the methods are adapted to the functions of the boundary layer, by introducing them into the base of the function by which the collocation procedure is carried out. For example, the cubic spline is replaced by an exponential spline which can be written in the form

$$u(x)=span\{1,x,e^{-\frac{b(x_i)x}{\varepsilon}},e^{\frac{b(x_i)x}{\varepsilon}}\},\qquad x\in[x_j,x_{j+1}.]$$

Then, for the problem (1) the uniform convergence is of order two at the nodes ([11]) and first-order outside the mesh nodes on the equidistant mesh [23],[24],[16]. However, if non-equidistant meshes are applied, polynomial collocations can also give a uniform convergence. Cubic and quadratic splines attain a uniform convergence in the approximation of the solution to the problem (1) [22],[28]. On the Shishkin-type meshes the convergence is of order $O(n^{-2}\ln^2 n)$, and on the Bakhalov-type meshes it is of order $O(n^{-2})$ [25]. The above-mentioned modifications of the Shishkin mesh are more of "theoretical" than practical character. Practical applications indicate that for a small ε we usually start with a larger n in order to have a certain number of points in the boundary layer; this yields total number of points being such that $n \geq n_0$, and that the convergence is attained after a reasonable n_0. Spline functions give also convergence of derivatives, normalized flux, which is essential in practical applications on the previously mentioned models.

3.2. Method of finite elements

This is a global method which has recently been used most often (see [8], [15],[14]). The method consists of the following steps. We define the weak form of the problem (1): Find $y \in H^1(0,1)$ so that

$$B(y,v) = -\varepsilon(y',v') - (by,v) = (f,v),$$

for all $v \in H^1(0,1)$, where $H^1[0,1]$ is Sobolev's space with the norm $||u||_1 = ((u,u)+(u',u'))^{1/2}$, $(.,.)$ denotes the inner product in $L_2(0,1)$. If we choose subspaces S^h (trial space) and T^h (test space) from $H^1(0,1)$ we can define the problem: Find $u^h \in S^h$ such that

$$B(u^h,v^h) = (f,v^h), \tag{9}$$

for all $v^h \in T^h$ such that $v^h(0) = v^h(1) = 0$. Let $\{\phi_j\}_1^{n-1}$ and $\{\psi_j\}_1^{n-1}$ be a set of basis functions for S^h and T^h, respectively. Let $xj = h * j, j = 0(1)n, h = 1/n$. Then

$$u^h = \sum_{j=1}^{n-1} y_j\phi_j, \quad v^h = \sum_{j=1}^{n-1} v_j\psi_j.$$

Since

$$supp(\phi_j) = [x_{j-1}, x_{j+1}]; \phi(x_j) = 1; \sum_{j=1}^{n-1} \phi_j(x) = 1, x \in [x_1, x_{n-1}],$$

and the same relations are valid for the function ψ, we have $u^h(x_j) = u_j$. With this, (10) reduces to

$$\sum_{k=j-1}^{j+1} ((\phi'_k, -\varepsilon\psi'_j) + (\phi_k, -b(x)\psi_j))u_k = (f,\psi_j), j = 1(1)n-1, \tag{10}$$

$$u_0 = u_n = 0.$$

The test function we choose in the form

$$\psi_j(x) = \lambda\tilde{\psi}_j(x) + (1-\lambda)\dot{\psi}_j(x),$$

where

$$\tilde{\psi}_j(x) = \begin{cases} \tilde{e}_j(x - x_{j-1}) & \text{za} \quad x \in I_j \\ 1 - \tilde{e}_{j+1}(x_{j+1-x}) & \text{za} \quad x \in I_{j+1}, \end{cases}$$

$\tilde{e}_j(x) = \sinh(\tilde{\rho}_j x/h)/\sinh(\tilde{\rho}_j)$, $\tilde{\rho}_j = h\sqrt{\tilde{b}_j}/\varepsilon$, and

$$\dot{\psi}(x) = \begin{cases} \dot{e}_j(x - x_{j-1}) & \text{za} \quad x \in I_j \\ 1 - \dot{e}_{j+1}(-x + x_{j+1}) & \text{za} \quad x \in I_{j+1}, \end{cases}$$

and $\dot{e}_j(x) = \sinh(\dot{\rho}_j x/h)/\sinh(\dot{\rho}_j)$ $\dot{\rho}_j = h\sqrt{\dot{b}_j}/\varepsilon$, $\tilde{b}_j = (b_{j-1} + b_j)/2$, $\dot{b}_j = b(x_j - h/2)$, $I_j = [x_{j-1}, x_j]$ and λ is a real number. Note that the function $\tilde{e}_j(x)$ is the solution of the problem

$$-\varepsilon \tilde{e}_j'' + \tilde{b}_j \tilde{e}_j = 0$$

$$\tilde{e}_j(0) = 0, \quad \tilde{e}_j(h) = 1.$$

Similar folds for $\dot{e}_j$. Here $\dot{b}$ and $\tilde{b}$ are some aproximations for $b(x)$. For more details about test functions see [8] or [15].

In this way, irrespective of the trial space, the method converges uniformly at the nodes. In [32], for a suitable λ a uniform convergence of the second order is obtained. In order to obtain convergence outside the nodes for the equidistant meshes we have to use exponential functions as trial functions. For $\lambda = 1$ and the following trial functions

$$\phi_j(t) = \begin{cases} E_j(x - x_{j-1}), & x \in I_j = [x_{j-1}, x_j], \\ 1 - \phi_{j+1}(x) & x \in I_{j+1}, \\ j < n/2, \end{cases}$$

$$\phi_j(t) = \begin{cases} E_j(x - x_{j-1}), & x \in I_j = [x_{j-1}, x_j], \\ 1 - E^T_{j+1}(x) & x \in I_{j+1}, \\ j = n/2, \end{cases}$$

$$\phi_j(t) = \begin{cases} E^T_j(x - x_{j-1}), & x \in I_j = [x_{j-1}, x_j], \\ 1 - \phi_{j+1}(x) & x \in I_{j+1}, \\ j > n/2, \end{cases}$$

where $E_j(x)$ and $E_j^T(x)$ are respective solutions of

$$\begin{cases} \varepsilon(E_j)'' + \bar{b}_j(E_j)' = 0, \\ E_j(0) = 0, \quad E_j(h) = 1, \end{cases}$$

and

$$\begin{cases} \varepsilon(E_j^T)'' - \bar{b}_j(E_j^T)' = 0, \\ E_j^T(0) = 0, \quad E_j^T(h) = 1. \end{cases}$$

in [15] the global uniform convergence of the first order is achieved.

On the Shishkin mesh, the method of finite elements attains a uniform convergence with polynomial functions in the role of trial functions. This approach can be found in the majority of recent papers dealing with singular perturbations and finite elements, and the number of such papers is constantly increasing.

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