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MATRIX SPLITTING PRINCIPLES

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Abstract

The paper gives a systematic analysis of the convergence conditions used in comparison theorems, proven for a few types of matrix splittings representing a large class of applications. The central idea of this analysis is the scheme of condition implications derived from the properties of regular splittings of a monotone matrix $A = M_1 - N_1 = M_2 - N_2$. Equivalence of some conditions are an autonomous character of the conditions $M_1^{-1} \geq M_2^{-1} \geq 0$ and $A^{-1}N_2 \geq A^{-1}N_1 \geq 0$ are pointed out.

Key words and phrases: linear equation systems, convergence conditions, comparison theorems, regular splittings, non-negative splittings, weak splittings.

1. Introduction

Let us consider the iterative solution of the linear equation system

$$Ax = b, \tag{1}$$

where $A \in \mathbb{R}^{n\times n}$ is a nonsingular matrix and $x, b \in \mathbb{R}^n$.

Traditionally, a large class of iterative methods for solving equation (1) can be formulated by means of a suitable splitting of the matrix A

$$A = M - N \quad \text{with} \quad M \text{ -- nonsingular}, \tag{2}$$

and the approximate solution $x^{(t)}$ is generated as follows

$$Mx^{(t+1)} = Nx^{(t)} + b, \quad t \geq 0$$

or equivalently,

$$x^{(t+1)} = M^{-1}Nx^{(t)} + M^{-1}b, \quad t \geq 0, \tag{3}$$

where the starting vector $x^{(0)}$ is given.

The convergence analysis of the above method is based on the spectral radius of the iteration matrix $\varrho(M^{-1}N)$. The iterative method is convergent to the unique solution

$$x = A^{-1}b \tag{4}$$

for each $x^{(0)}$, if and only if $\varrho(M^{-1}N) < 1$. For large values of t, the solution error decreases in magnitude approximately by a factor of $\varrho(M^{-1}N)$ at each iteration step; the smaller is $\varrho(M^{-1}N)$, the quicker is the convergence. Thus, the evaluation of an iterative method focuses on two issues: M should be chosen as an easily invertible matrix and $\varrho(M^{-1}N)$ should be as small as possible.

Definition 1.1. *The decomposition $A = M - N$ is called a* convergent splitting *of A, if A and M are nonsingular matrices, and $\varrho(M^{-1}N) < 1$.*

General properties of a splitting of A (not necessarily convergent), useful for proving comparison theorems, are given in the following lemma [15].

Lemma 1.1. *Let $A = M - N$ be a splitting of A. If A and M are nonsingular matrices, then*

$$M^{-1}NA^{-1} = A^{-1}NM^{-1}, \tag{5}$$

the matrices $M^{-1}N$ and $A^{-1}N$ commute, and the matrices NM^{-1} and NA^{-1} also commute.

Proof. From the definition of the splitting of A, it follows that

$$M^{-1} = (A+N)^{-1} = A^{-1}(I+NA^{-1})^{-1} = (I+A^{-1}N)^{-1}A^{-1} \tag{6}$$

or

$$A^{-1} = M^{-1} + M^{-1}NA^{-1} = M^{-1} + A^{-1}NM^{-1} \tag{7}$$

which implies that

$$M^{-1}NA^{-1} = A^{-1}NM^{-1}.$$

Hence

$$M^{-1}NA^{-1}N = A^{-1}NM^{-1}N \quad \text{or} \quad NM^{-1}NA^{-1} = NA^{-1}NM^{-1}. \qquad \square$$

From the above lemma, the following corollary can be deduced.

Corollary 1.1 *Let $A = M - N$ be a splitting of A. If A and M are nonsingular matrices, then both matrices $M^{-1}N$ and $A^{-1}N$ (or NM^{-1} and NA^{-1}) have the same eigenvectors.*

Historically, the idea of splittings of matrices has its scientific origin in the regular splitting theory introduced in 1960 by Varga [9] and extended in 1973 by results of the author's thesis [10] (recalled in [15]). These first results, given as comparison theorems for regular splittings of monotone matrices and proven with natural hypotheses by means of the Perron-Frobenius theory of nonnegative matrices [9], have been useful tools in the convergence analysis of some iterative methods for solving systems of linear equations [10-13,15,16].

Further extensions for regular splittings have been obtained by Csordas and Varga [2] in 1984, and from this time a renewed interest in comparison theorems, proven under progressively weaker hypotheses for different splittings, has been permanently observed in the literature. These new results lead to successive generalizations and were accompanied with an increased complexity in the verification of hypotheses. Therefore, some comparison theorems may have more theoretical than practical significance. Theorems proven under different hypotheses, for a few types of splittings of monotone matrices representing a large class of applications, have been reviewed in [19].

The main objective of this paper is a systematic analysis of the convergence conditions derived from their implications for the regular splitting case and discussed in the next section. Further generalizations are presented in Section 3.

2. Regular Splitting Theory

At the beginning we recall the basic results of Varga.

Definition 2.1. *([9])*
The decomposition $A = M - N$ is called a regular splitting *of A, if M is a nonsingular matrix with $M^{-1} \geq 0$ and $N \geq 0$.*

Theorem 2.1. *([9]) Let $A = M - N$ be a regular splitting of A. If $A^{-1} \geq 0$, then*

$$\varrho(M^{-1}N) = \frac{\varrho(A^{-1}N)}{1 + \varrho(A^{-1}N)} < 1. \tag{8}$$

Conversely, if $\varrho(M^{-1}N) < 1$, then $A^{-1} \geq 0$.

Theorem 2.2. *([9]) Let $A = M_1 - N_1 = M_2 - N_2$ be two regular splittings of A, where $A^{-1} \geq 0$. If $N_2 \geq N_1$, then*

$$\varrho(M_1^{-1}N_1) \leq \varrho(M_2^{-1}N_2).$$

In particular, if $N_2 \geq N_1$ with $N_2 \neq N_1$, and if $A^{-1} > 0$, then

$$\varrho(M_1^{-1}N_1) < \varrho(M_2^{-1}N_2).$$

Theorem 2.2 allows us to compare spectral radii of iteration matrices only in the Jacobi and Gauss-Seidel methods [9]. The excellent convergence properties of iterative AGA algorithms [10,11,13,15] encouraged further studies one of the results being the following theorem.

Theorem 2.3. *([10,15]) Let $A = M_1 - N_1 = M_2 - N_2$ be two regular splittings of A, where $A^{-1} \geq 0$. If $M_1^{-1} \geq M_2^{-1}$, then*

$$\varrho(M_1^{-1}N_1) \leq \varrho(M_2^{-1}N_2).$$

In particular, if $A^{-1} > 0$ and $M_1^{-1} > M_2^{-1}$, then

$$\varrho(M_1^{-1}N_1) < \varrho(M_2^{-1}N_2).$$

The main result of the application of this theorem is the generalization of the Stein-Rosenberg theorem for the iterative prefactorization methods in an irreducible case [10,15].

It is easy to verify that for regular splittings of a monotone matrix A (i.e., $A^{-1} \geq 0$),

$$A = M_1 - N_1 = M_2 - N_2, \tag{9}$$

the assumption of Theorem 2.2

$$N_2 \geq N_1 \geq 0 \tag{10}$$

implies the equivalent condition

$$M_2 \geq M_1 \not\geq 0 \tag{11}$$

but the last inequality implies the condition of Theorem 2.3, i.e.,

$$M_1^{-1} \geq M_2^{-1} \geq 0. \tag{12}$$

From the inequality (10) one obtains the inequality $A^{-1}N_2 \geq A^{-1}N_1 \geq 0$. Since by the relation (8) $\varrho(M^{-1}N)$ is a monotone function with respect to $\varrho(A^{-1}N)$, the result of Theorem 2.2 follows immediately.

In the case of the proof of Theorem 2.3, the condition (12) can be expressed as follows

$$(I + A^{-1}N_1)^{-1}A^{-1} \geq A^{-1}(I + N_2A^{-1})^{-1} \tag{13}$$

which, after relevant multiplications, is equivalent to

$$A^{-1}N_2A^{-1} \geq A^{-1}N_1A^{-1} \geq 0. \tag{14}$$

¿From the above inequality, one obtains

$$A^{-1}N_2A^{-1}N_1 \geq (A^{-1}N_1)^2 \geq 0, \tag{15}$$

and

$$(A^{-1}N_2)^2 \geq A^{-1}N_1A^{-1}N_2 \geq 0. \tag{16}$$

Hence,

$$\varrho^2(A^{-1}N_2) \geq \varrho(A^{-1}N_1A^{-1}N_2) = \varrho(A^{-1}N_2A^{-1}N_1) \geq \varrho^2(A^{-1}N_1) \tag{17}$$

which gives us

$$\varrho(A^{-1}N_2) \geq \varrho(A^{-1}N_1) \tag{18}$$

and by the result (8), the inequality

$$\varrho(M_1^{-1}N_1) \leq \varrho(M_2^{-1}N_2) \tag{19}$$

can be deduced.

In the case of the strict inequality in (12), similar considerations lead to the strict inequality in (19) [10,15].

On the other hand, from the inequality (10), one obtains

$$A^{-1}N_2 \geq A^{-1}N_1 \geq 0, \tag{20}$$

which implies the inequalities (14), (15) and (16), and additionally

$$A^{-1}N_1A^{-1}N_2 \geq (A^{-1}N_1)^2 \geq 0, \tag{21}$$

and

$$(A^{-1}N_2)^2 \geq A^{-1}N_2A^{-1}N_1 \geq 0. \tag{22}$$

The inequality (11) gives us that

$$A^{-1}M_2 \geq A^{-1}M_1 \geq 0, \tag{23}$$

since for each regular splitting of A

$$A^{-1}M = I + A^{-1}N, \tag{24}$$

hence, it is evident that both conditions (20) and (23) are equivalent.

Each of the above conditions, except (16) and (22), leads to proving the inequality (19), however, as can be shown on simple examples of regular splittings the reverse implications may fail. Thus, the above inequalities being progressively weaker conditions to those used as the hypotheses in comparison theorems, provide successive generalizations of results. The conditions (14); (15), (16), (21) and (22); (20); and (23) were considered by Csordas and Varga [2]; Beauwens [1]; Marek [4]; and Song [8] respectively.

The scheme of implications of the above conditions is demonstrated in Figure 1. Both conditions D) and E) are equivalent by the relation (24).

The conditions C) and D) imply the condition G) equivalent, again by (24), to the condition F). The condition C) implies indirectly only the conditions H1) and H2), whereas the condition E) implies directly all conditions H1), H2), H3) and H4).

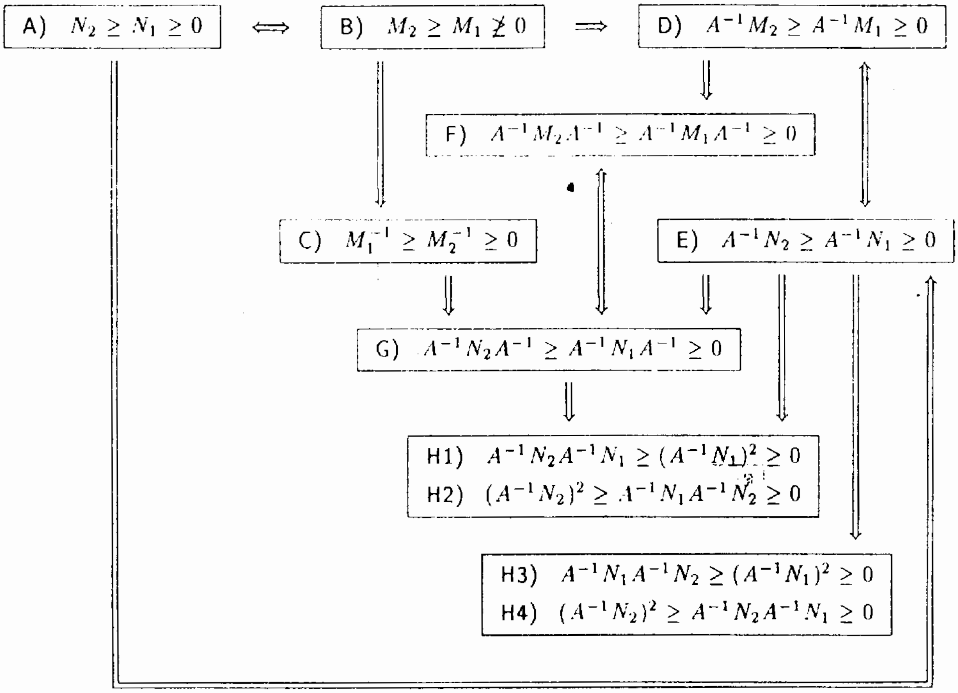

Figure 1. *Scheme of condition implications for regular splittings of* $A = M_1 - N_1 = M_2 - N_2$, *where* $A^{-1} \geq 0$.

It seems to be interesting to ask, does a dependence exist between the condition C) playing the essential role in the conjugate type iterative solvers [14], and the condition E)? To give the answer to the above question, let us consider for the following matrix [15]

$$(25) \qquad A = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ -1 & 0 & 2 \end{bmatrix} \quad \text{where } A^{-1} = \begin{bmatrix} 2 & 2 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

some regular splittings of $A = M_i - N_i$.

$$M_1 = \begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & -1 \\ -1 & 0 & 2 \end{bmatrix}, \quad N_1 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{where} \tag{26}$$

$$M_1^{-1} = \begin{bmatrix} 1 & 1 & 0 \\ \frac{1}{2} & \frac{3}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{bmatrix}, \quad M_1^{-1}N_1 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & \frac{1}{2} \\ 0 & 0 & \frac{1}{2} \end{bmatrix}, \quad A^{-1}N_1 = \begin{bmatrix} 0 & 0 & 2 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}. \tag{27}$$

$$M_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 2 \end{bmatrix}, \quad N_2 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \quad \text{where} \tag{28}$$

$$M_2^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & \frac{1}{2} \\ 0 & 0 & \frac{1}{2} \end{bmatrix}, \quad M_2^{-1}N_2 = \begin{bmatrix} 0 & 1 & 0 \\ \frac{1}{2} & 0 & 0 \\ \frac{1}{2} & 0 & 0 \end{bmatrix}, \quad A^{-1}N_2 = \begin{bmatrix} 1 & 2 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix}. \tag{29}$$

$$M_3 = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 2 \end{bmatrix}, \quad N_3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \quad \text{where} \tag{30}$$

$$M_3^{-1} = \begin{bmatrix} 1 & 1 & \frac{1}{2} \\ 0 & 1 & \frac{1}{2} \\ 0 & 0 & \frac{1}{2} \end{bmatrix}, \quad M_3^{-1}N_3 = \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ \frac{1}{2} & 0 & 0 \\ \frac{1}{2} & 0 & 0 \end{bmatrix}, \quad A^{-1}N_3 = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}. \tag{31}$$

$$M_4 = \begin{bmatrix} 2 & -1 & 0 \\ 0 & 1 & -1 \\ -1 & 0 & 2 \end{bmatrix}, \quad N_4 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{where} \tag{32}$$

$$M_4^{-1} = \begin{bmatrix} \frac{2}{3} & \frac{2}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{4}{3} & \frac{2}{3} \\ \frac{1}{3} & \frac{1}{3} & \frac{2}{3} \end{bmatrix}, \quad M_4^{-1}N_4 = \begin{bmatrix} \frac{2}{3} & 0 & 0 \\ \frac{1}{3} & 0 & 0 \\ \frac{1}{3} & 0 & 0 \end{bmatrix}, \quad A^{-1}N_4 = \begin{bmatrix} 2 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}. \tag{33}$$

As can be easily noticed for

$M_1^{-1} \geq M_2^{-1}$ ($M_2 \not\geq M_1$), $\varrho(M_1^{-1}N_1) = \frac{1}{2} < \varrho(M_2^{-1}N_2) = \frac{1}{\sqrt{2}}$ whereas $A^{-1}N_2 \not\geq A^{-1}N_1$

and for

$A^{-1}N_4 \geq A^{-1}N_3$, $\varrho(M_3^{-1}N_3) = \frac{1}{2} < \varrho(M_4^{-1}N_4) = \frac{2}{3}$ whereas $M_3^{-1} \not\geq M_4^{-1}$.

Thus, the above regular splitting examples show us that both conditions C) and E) have an autonomous character, and there is no even a precursor relation between them.

Some results for the condition C) and regular splittings of monotone matrices, derived with a different fineness of block partitions, have been recently obtained in [17].

3. Non-negative Splitting Theory

In fact, the conditions of the regular splitting of a monotone matrix $A = M - N$

$$(I) \qquad M^{-1} \geq 0,$$
$$(II) \qquad N \geq 0$$

imply

$$(IIIa) \qquad M^{-1}N \geq 0,$$
$$(IVa) \qquad A^{-1}N \geq 0$$

and extra conditions

$$(IIIb) \qquad NM^{-1} \geq 0,$$
$$(IVb) \qquad NA^{-1} \geq 0$$

important in convergence analysis as well. Thus, the principle of regular splitting is based on six conditions.

Both matrices $M^{-1}N$ and NM^{-1} (as well as $A^{-1}N$ and NA^{-1}) have the same eigenvalues because they are similar matrices. It may occur that for the splittings (9) none of conditions given in Figure 1 is not satisfied, but the following lemma holds

Lemma 3.1. *([15]) Let $A = M_1 - N_1 = M_2 - N_2$ be two regular splittings of A, where $A^{-1} \geq 0$. If $M_1^{-1}N_1 \geq N_2M_2^{-1} \geq 0$ or $A_1^{-1}N_1 \geq N_2A_2^{-1} \geq 0$, then*

$$\varrho(M_1^{-1}N_1) \leq \varrho(M_2^{-1}N_2).$$

The first extension of the regular splitting case is due to Ortega and Rheinboldt [7] who introduced the class of weak regular splittings, based on the conditions (I), $(IIIa)$ and $(IIIb)$, for which Theorems 2.2 and 2.3 hold. However, it is necessary to mention that some authors [3,5,8], using the same name "weak regular splitting", restrict this definition to its weaker version based on the conditions (I) and $(IIIa)$ only. In this case of weak regular splitting, Elsner [3] showed that the assumption $M_1^{-1} \geq M_2^{-1} \geq 0$ may be not a sufficient hypothesis for ensuring the inequality (19) and he stated result of Theorem 2.3 for the case when one of splittings is regular one. This means that Elsner restored the need of the condition $(IIIb)$ sticking originally of the Ortega and Rheinboldt's definition. This topic is discussed in detail in [18].

In two last decades a renewed interest of comparison theorems, proven for different types of splittings and assumptions, is observed in the literature [1-6,8,15]. The Varga's definition of regular splitting became the standard terminology in the literature, whereas other splittings are usually defined as a matter of author's taste. The definitions of splittings, with progressively weaking conditions and consistent from the viewpoint of names, are collected in the following definition [15].

Definition 3.1. *Let $M, N \in \mathbb{R}^{n\times n}$. Then the decomposition $A = M - N$ is called:*

- **(a)** *a regular splitting of A if $M^{-1} \geq 0$ and $N \geq 0$.*
- **(b)** *a non-negative splitting of A if $M^{-1} \geq 0$, $M^{-1}N \geq 0$ and $NM^{-1} \geq 0$.*
- **(c)** *a weak non-negative splitting of A if $M^{-1} \geq 0$ and either $M^{-1}N \geq 0$ (the first type) or $NM^{-1} \geq 0$ (the second type).*
- **(d)** *a weak splitting of A if M is nonsingular and either $M^{-1}N \geq 0$ (the first type) or $NM^{-1} \geq 0$ (the second type). In particular a given weak splitting can be both types.*
- **(e)** *a convergent splitting of A if $\varrho(M^{-1}N) = \varrho(NM^{-1}) < 1$.*

The definition assumed in item *(b)* is equivalent to the definition of weak regular splitting of A introduced originally by Ortega and Rheinboldt [7]. However, as was mentioned, the case with $M^{-1} \geq 0$ and $M^{-1}N \geq 0$ without the condition $NM^{-1} \geq 0$ (corresponding to weak non-negative splitting of the first type) is also defined as a weak regular splitting of $A = M - N$ by other authors, but in this case it is necessary to use additional assumptions in comparison theorems. It should be noticed that the use of the Ortega and Rheinboldt's terminology "weak regular" in item *(b)* causes a confusion with using the splitting name in item *(c)*. Therefore, it seems that assuming the term "non-negative" allows us to avoid this confusion. The definition of the first type weak splitting of A has been introduced by Marek and Szyld [5], but it is again called "splitting of positive type by Marek [4] and non-negative splitting" by Song [8].

The splittings defined in the successive items of Definition 3.1 extend successively a class of splittings of $A = M - N$ for which the matrices N and M^{-1} may lose the properties of non-negativity. Distinguishing both types of weak non-negative and weak splittings leads to further extensions allowing us to analize cases when $M^{-1}N$ may have negative entries, if NM^{-1} is a non-negative matrix, for which Lemma 3.1 may be used as well.

Conditions ensuring that a splitting of a nonsingular matrix $A = M - N$ will be convergent are unknown in a general case. As was pointed out in [15], the splittings defined in the first three items of Definition 3.1 are convergent if and only if $A^{-1} \geq 0$.

The properties of weak non-negative splittings are extensively analyzed in [15] for the conditions of implication scheme demonstrated in Figure.

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