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PSEUDO-OPERATIONS ON FINITE INTERVALS

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Abstract. A characterization of associative, commutative, monotone operation with neutral element on finite interval of reals is given. Using these results a complete characterization of one class of semirings on finite intervals is obtained.

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1. Introduction

Many different kinds of operation defined on subsets of real numbers play fundamental roles in many important fields as for example in fuzzy set theory, fuzzy logic, neural nets, operation research, optimization problems, differential equations, etc. Special intention is paid to operations defined on interval of reals. The examples are t-norms and t-conorms which act on the interval $[0, 1]$, (see Schweizer and Sklar [14]), pseudo-additions and pseudo-multiplications in the sense of Sugeno and Murofushi [15] which act on the interval $[0, \infty]$ or in the sense of E. Pap [10] which act on the interval $[a, b](-\infty \leq a < b < +\infty)$, compensatory operators (Klement, Mesiar, Pap [5]), and uni-norms (Fodor, Yager, Rybalov [3]).

In this paper we investigate pseudo-operations on a finite interval. The first result in this direction was given by Mostert and Schields [9] and then the problem was investigated in more detail by Ling [6] (see also Aczél [1]). We use all these characterizations and, combining them with the representation theorems by ordinal sums (Schweizer and Sklar [14], Fuchs [4]) and the newest results of Klement, Mesiar, Pap [5] and Fodor, Yager, Rybalov [3], we obtain the complete description of pseudo-operation on finite intervals. These results enable us to completely characterize one class of semirings on finite intervals. The notion of semiring today plays an important role in many branches of mathematics, for example: optimization theory, and automate theory, nonlinear differential equations (see Maslov, Samborskij [7], E. Pap [12]). There are still two open problems. The first, to characterize semirings on infinite intervals.

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The second problem is to consider generalization of semirings in the sense that the distributive law of pseudo-multiplication with respect to pseudo-addition holds only in a restricted domain of the interval (conditional distributivity). Such situations often occur in the integration theory.

2. Associative binary operations on intervals

First, we give some general definitions. Let A be an arbitrary non-empty set.

Definition 1. *Let R be a binary operation on A and $a \in A$.*

a) The function $v_a : A \to A$ defined by $v_a(x) = R(a,x)$ is the vertical section of R.

b) The function $h_a : A \to A$ defined by $h_a(x) = R(x,a)$ is the horizontal section of R.

c) The function $\delta : A \to A$ defined by $\delta(x) = R(x,x)$ is the diagonal section of R.

d) The R-powers of a are the elements of A given recursively by

$$a^{(1)} = a \quad , \quad a^{(n+1)} = R(a^{(n)}, a)$$

for all $n \in \mathbb{N}$.

We have

$$\delta^n(x) = x^{(2^n)},$$

for $n \in \mathbb{N}$, $x \in A$.

Definition 2. *The element $u \in A$ is a null element or the annihilator of the operation $R : A^2 \to A$ if*

$$(\forall x \in A)(R(x,u) = R(u,x) = u).$$

The neutral element and annihilator (if they exist) of a binary operation are uniquely determined and they are always different (for $card(A) > 1$).

Definition 3. *The element $x \in A$ is idempotent in relation to the operation $R : A^2 \to A$ if*

$$R(x,x) = \delta(x) = x.$$

Definition 4. *The operation $R_2 : A^2 \to A$ is distributive in relation to the operation $R_1 : A^2 \to A$ if*

$$(\forall x,y,z \in A)\ R_2(R_1(x,y),z) = R_1(R_2(x,z), R_2(y,z)),$$

holds.

Now we shall restrict ourselves on operations defined on the intervals of reals.

Definition 5. *Let $A = [a, b]$ be an interval of the set of real numbers $\mathbb{R}$. An operation $R : A^2 \to A$ is said to be nondecreasing in each coordinate if*

$$(\forall x_1, x_2, y_1, y_2 \in A)(x_1 \leq x_2 \wedge y_1 \leq y_2 \Rightarrow R(x_1, y_1) \leq R(x_2, y_2)).$$

We have by [14] the following theorem.

Theorem 1. *Let $[a, b]$ be a closed interval and $R : [a, b]^2 \to [a, b]$ an associative operation nondecreasing in each coordinate and having a neutral element $e = b$. Then*

i) *All vertical sections v_x, all horizontal sections h_x and the diagonal section δ of R are nondecreasing functions on $[a, b]$.*

ii) *The endpoint a is an annihilator of R;*

iii) $(\forall x, y \in [a, b])\ R(x, y) \leq \min(x, y)$.

iv) $(\forall x \in [a, b])\ \delta(x) \leq x$.

v) *If $\lim_{t \to x-0} \delta(t) = x$, for some x in $(a, b]$, then $\delta(x) = x$.*

Analogously, we have in the following theorem.

Theorem 2. *Let $[a, b]$ be a closed interval and $R : [a, b]^2 \to [a, b]$ an associative operation which is nondecreasing in each coordinate and which has a neutral element $e = a$. Then*

i) *All vertical sections v_x, all horizontal sections h_x and the diagonal section of δ are nondecreasing functions on $[a, b]$.*

ii) *The endpoint b is an annihilator of R.*

iii) $(\forall x, y \in [a, b])\ R(x, y) \geq \max(x, y)$.

iv) $(\forall x \in [a, b])\ \delta(x) \geq x$.

v) *If $\lim_{t \to x+0} \delta(t) = x$, for some x in $[a, b)$, then $\delta(x) = x$.*

Remark. The endpoints a and b are idempotent elements, because the neutral element and annihilator are idempotent.

Definition 6. *Let $\{(A_\alpha, R_\alpha)\}_{\alpha \in K}$ be a family of semigroups indexed by a set $K \subset \mathbb{R}$. For all α, β, γ in K let these semigroups satisfy the following compatibility conditions*

(i) *If $\alpha < \beta < \gamma$ and $A_\alpha \cap A_\gamma \neq \emptyset$, then $A_\beta = A_\alpha \cap A_\gamma$.*

(ii) If $\alpha < \beta$ and $x \in A_\alpha \cap A_\beta$, then x is the unique element from $A_\alpha \cap A_\beta$ which is the neutral of R_α and the annihilator of R_β.

If $A = \cup_\alpha A_\alpha$ and $R : A \times A \to A$ is a binary operation defined by: For $x \in A_\alpha$ and $y \in A_\beta$,

$$R(x,y) = \begin{cases} x, & \alpha < \beta \\ R_\alpha(x,y), & \alpha = \beta \\ y, & \alpha > \beta \end{cases},$$

then the semigroup (A, R) is called $t-$ordinal sum of the family of semigroups $\{(A_\alpha, R_\alpha)\}$.

Note that the compatibility conditions guarantee that the binary operation R is well defined on A.

Analogously, we give the following definition.

Definition 7. *Let $\{(A_\alpha, R_\alpha)\}$ be a family of semigroups indexed by the set $K \subset \mathbb{R}$. For each α, β, γ from K these semigroups satisfy the following compatibility conditions.*

(i) If $\alpha < \beta < \gamma$ and $A_\alpha \cap A_\gamma \neq \emptyset$, then $A_\beta = A_\alpha \cap A_\gamma$.

(ii) If $\alpha < \beta$ and $x \in A_\alpha \cap A_\beta$, then x is the unique element from $A_\alpha \cap A_\beta$ which is the annihilator of R_α and the neutral element of R_β.

If $A = \cup_\alpha A_\alpha$ and $R : A \times A \to A$ is a binary operation defined by: For $x \in A_\alpha$ and $y \in A_\beta$,

$$R(x,y) = \begin{cases} y, & \alpha < \beta \\ R_\alpha(x,y), & \alpha = \beta \\ x, & \alpha > \beta \end{cases},$$

then the semigroup (A, R) is called $s-$ordinal sum of the family of semigroups $\{(A_\alpha, R_\alpha)\}$.

Definition 8. *We say that the operation $R : [a,b]^2 \to [a,b]$ satisfies the condition of the diagonal if from $\delta(x) = x$ follows that v_x and h_x are continuous functions.*

Let $[a,b]$ be a closed finite interval of $\mathbb{R}$. We denote by $\mathcal{T}[a,b]$ the class of all associative binary operations R on $[a,b]$, nondecreasing in each coordinate, whose neutral element is the endpoint b and which satisfies the condition of the diagonal and the condition $v)$ from Theorem 2. We denote by $\mathcal{S}[a,b]$ the class of all associative binary operations R on $[a,b]$, nondecreasing in each coordinate, whose neutral element is the endpoint a, and which satisfies the condition of the diagonal and the condition $v)$ from Theorem 1.

We introduce an important property of a binary operation

Definition 9. *Let R be an associative binary operation on the closed interval $[a,b]$, nondecreasing in each coordinate and whose neutral element is b. Then R is an Archimedean if*

$$(\forall x, y \in (a,b))(\exists m \in \mathbb{N})\; x^{(m)} < y.$$

Definition 10. *Let R be an associative binary operation on the closed interval $[a,b]$, nondecreasing in each coordinate and whose neutral element is a. Then R is an Archimedean if*

$$(\forall x, y \in (a,b))(\exists m \in \mathbb{N})\; x^{(m)} > y.$$

We have by [14].

Theorem 3. *Let R be a binary operation on the interval $[a,b]$.*

i) For $R \in \mathcal{T}[a,b]$, the necessary and sufficient condition for R to be Archimedean is that the following holds

$$(\forall x \in (a,b))\; \delta(x) < x.$$

ii) For $R \in \mathcal{S}[a,b]$, the necessary and sufficient condition for R to be Archimedean is that the following holds

$$(\forall x \in (a,b))\; \delta(x) > x.$$

We have by [14] the following characterization.

Theorem 4. *Let $R \in \mathcal{T}[a,b]$. Then we have*

(i) If $\delta(x) < x$ for all $x \in (a,b)$, then R is an Archimedean on $[a,b]$ and conversely.

(ii) If $\delta(x) = x$ for all $x \in [a,b]$, then R is the restriction of min *to $[a,b]$ and conversely.*

(iii) Otherwise, the semigroup $([a,b],R)$ is a t-ordinal sum of semigroups $((A_\alpha, T_\alpha))$ where each A_α is a closed subinterval of $[a,b]$, each T_α belongs to $\mathcal{T}[A_\alpha]$ and T_α is either Archimedean on A_α or the restriction of min *to A_α.*

Analogously, we have

Theorem 5. *Let $R \in \mathcal{S}[a,b]$. Then we have*

(i) If $\delta(x) > x$ for all $x \in (a,b)$, then S is an Archimedean on $[a,b]$ and conversely.

(ii) If $\delta(x) = x$ for all $x \in [a,b]$, then S is the restriction of max *to $[a,b]$ and conversely.*

(iii) Otherwise, the semigroup $([a,b],R)$ is an s-ordinal sum of semigroups $((A_\alpha, S_\alpha))$ where each A_α is a closed subinterval of $[a,b]$, each S_α belongs to $\mathcal{S}[A_\alpha]$, and S_α is either Archimedean on A_α or the restriction of max *to A_α.*

3. Pseudo-operations on finite intervals

Definition 11. *We call the binary operation* $* : [a,b] \times [a,b] \to [a,b]$ *a pseudo-operation, if it is an associative and commutative operation which has a neutral element* e *and it is nondecreasing in each coordinate.*

We introduce special kinds of pseudo-operations.

Definition 12. *The pseudo-operation on the interval* $[a,b]$ *is a generalized t-norm on the interval* $[a,b]$ *if* b *is its neutral element.*

The pseudo-operation on the interval $[a,b]$ *is a generalized t-conorm on the interval* $[a,b]$ *if* a *is its neutral element.*

Remark. The preceding notions in the case $[a,b] = [0,1]$ coincide with the usual notions of t-norms and t-conorms, respectively.

It is interesting to consider the case when $e \in (a,b)$.

Theorem 6. *Let* $*$ *be an arbitrary pseudo-operation on a finite interval* $[a,b]$ *with a neutral element* $e \in (a,b)$. *Then by* $*_T = *|_{[a,e]}$ *a generalized t-norm on the interval* $[a,e]$ *is defined, and by* $*_S = *|_{[e,b]}$, *a generalized t-conorm on the interval* $[e,b]$ *is defined.*

Proof. Let $x, y \in [a,e]$. The operation $*_T$ is a restriction of $*$, therefore, $a \le x *_T y$ and the monotonicity of $*$ implies $x, y \le e \Rightarrow x *_T y = x * y \le e * e = e$, as well, i.e. $*_T$ is the closed operation. The closeness of the operation $*_S$ is proved similarly.

The associativity, commutativity, and monotonicity are invariant with respect to the operation; they are also valid as universal properties for restriction of this operation. Clearly, e is a neutral element for $*_T$ and $*_S$. □

Analogously as in the paper [3], we can prove the following lemmas.

Lemma 1. *If* $*$ *is the pseudo-operation on the interval* $[a,b]$ *and* $\min(x,y) \le e \le \max(x,y)$, *then*

$$\min(x,y) \le x * y \le \max(x,y), \quad x, y \in [a,b].$$

Proof. We suppose $x \le e \le y$. Then $x * y \le e * y = y = \max(x,y)$ and $x * y \ge x * e = x = \min(x,y)$. □

Lemma 2. *For each pseudo-operation* $*$ *on the interval* $[a,b]$, *the element* $a * b$ *is its annihilator, i.e.,*

$$(\forall x \in [a,b])\; a * b = (a * b) * x.$$

Proof. Let $x \in [e,b]$. Then Theorem 2 implies that b is an annihilator for $*_S$, i.e., $b = b *_S x = b * x$. Hence

$$a * b = a * (b * x) = (a * b) * x.$$

If $x \in [a, e]$, then we have by Theorem 1 that a is an annihilator for $*_T$, i.e. $a = x *_T a = x * a$. Therefore,

$$a * b = (x * a) * b = x * (a * b) = (a * b) * x.$$

□

Corollary 1. *If $*$ is a pseudo-operation on the interval $[a, b]$, then*

$$a * b \in \{a, b\}.$$

Proof. Since the neutral element is never an annihilator, we have $a * b \neq e$. If $a * b \in (e, b]$, then by Lemma 2 we have $a * b = (a * b) * b = b$, while b is the annihilator for $*$ in interval $[e, b]$.

Similarly, for $a * b \in [a, e)$ it follows $a * b = (a * b) * a = a$. □

Let $*$ be a partially continuous pseudo-operation, i.e., $x * b$ is a continuous function on $[a, e)$, and if $a * b = a$ holds, then pseudo-operation is called pseudo-operation of the first kind. If $x * a$ is a continuous function on $(e, b]$, and if $a * b = b$ holds, then $*$ is called the pseudo-operation of the second kind.

Theorem 7.

i) If $*$ is the pseudo-operation of the first kind, then

$$(\forall x \in [a, e)) \; x * b = x.$$

ii) If $*$ is the pseudo-operation of the second kind, then

$$(\forall x \in (e, b]) \; x * a = x.$$

Proof. i) By the continuity of the function $s(x) = x * b$ and using that $s(a) = a * b = a$ and $s(e) = e * b = b$, it follows that for each $x \in (a, e) \subset [a, b]$ there exists $z \in (a, e)$ so that $x = s(z) = z * b$. Now we have

$$x * b = (z * b) * b = z * (b * b) = z * b = x.$$

ii) Similarly, for all $x \in (e, b) \subset [a, b]$ there exists $z \in (e, b)$ such that $x = z * a$ and

$$x * a = (z * a) * a = z * (a * a) = z * a = x.$$

□

Let $*$ be a pseudo-operation of the first kind. If $x \leq e \leq y$ then by Theorem 7, $x * b = x$. Now by Lemma 1 it follows $x \leq x * y \leq y$. Hence $x \leq x * y \leq x * b = x$, i.e., $x * y = x$. Similarly for $y \leq e \leq x$ it follows $x * y = y$, so we have $\min(x, y) \leq e \leq \max(x, y)$ and so

$$x * y = \min(x, y).$$

Let $*$ be a pseudo-operation of the second kind. Analogously, if $\min(x,y) \leq e \leq \max(x,y)$ we have

$$x * y = \max(x,y).$$

Hence using Theorem 6, the following theorem on the representation of pseudo-operations on a finite interval follows.

Theorem 8. *Let $* : [a,b]^2 \to [a,b]$ be the pseudo-operation with a neutral element $e \in (a,b)$.*

i) If $*$ is the pseudo-operation of the first kind, then

$$x * y = \begin{cases} x *_T y, & x,y \in [a,e] \\ x *_S y, & x,y \in [e,b] \\ \min(x,y), & \min(x,y) \leq e \leq \max(x,y) \end{cases}.$$

ii) If $*$ is the pseudo-operation of the second kind, then

$$x * y = \begin{cases} x *_T y, & x,y \in [a,e] \\ x *_S y, & x,y \in [e,b] \\ \max(x,y), & \min(x,y) \leq e \leq \max(x,y) \end{cases}.$$

4. Semirings on finite intervals

Let $\oplus$ and $\otimes$ be two pseudo-operations defined on the finite interval $[a,b]$ $(a < b)$. Let us denote their neutral elements with $\mathbf{0}$ and $\mathbf{1}$, respectively.

Definition 13. *Let $\otimes$ be a pseudo-operation which is distributive with respect to the pseudo-operation $\oplus$ and $\mathbf{0}$ is annihilator for the operation $\otimes$, then we say that the operations $\oplus$ and $\otimes$ are pseudo-addition and pseudo-multiplication, respectively, and the structure $([a,b],\oplus,\otimes)$ is called a semiring.*

The equality $\mathbf{0} = \mathbf{1}$ is impossible, because $x = x \otimes \mathbf{1} = x \otimes \mathbf{0} = \mathbf{0}$, for all $x \in [a,b]$, therefore $\mathbf{0} \neq \mathbf{1}$.

Theorem 9. *Let $([a,b],\oplus,\otimes)$ be a semiring. Then either $\oplus$ is a generalized t-conorm and $\otimes$ a generalized t-norm or $\oplus$ is a generalized t-norm and $\otimes$ a generalized t-conorm.*

Proof. Suppose that $\mathbf{1} \in (a,b)$. If $\mathbf{0} \in [a,\mathbf{1})$, by Theorem 1 we have that a is an annihilator for the restriction of the operation $\otimes$ on the interval $[a,\mathbf{1}]$, so $a = \mathbf{0}$, since the annihilator is uniquely determined. Similarly, from Theorem 2 we have that b is an annihilator, for the restriction of the operation $\otimes$ on the interval $[\mathbf{1},b]$, then it follows that $b = \mathbf{0}$. Contradiction, because $a \neq b$.

Therefore, $\mathbf{1} \in \{a,b\}$. If $\mathbf{1} = b$, by Theorem 1, it follows $\mathbf{0} = a$, i.e., $\oplus$ is a generalized t-conorm and $\otimes$ a generalized t-norm. Similarly, if $\mathbf{1} = a$, then $\mathbf{0} = b$, i.e., $\oplus$ is a generalized t-norm and $\otimes$ a generalized t-conorm. $\square$

Theorem 10. *If* $\oplus \in \{\max, \min\}$, *and* $\otimes$ *is an arbitrary pseudo-operation on the interval* $[a,b]$, *then the distributive law* ($\otimes$ *with respect to* $\oplus$) *is valid.*

Proof. Suppose $x \leq y$. Hence $x \otimes z \leq y \otimes z$. If $\oplus = \max$, then we have

$$(x \oplus y) \otimes z = \max(x,y) \otimes z = y \otimes z = \max(x \otimes z, y \otimes z) = (x \otimes z) \oplus (y \otimes z).$$

We have similarly for $\oplus = \min$ that

$$(x \oplus y) \otimes z = \min(x,y) \otimes z = x \otimes z = \min(x \otimes z, y \otimes z) = (x \otimes z) \oplus (y \otimes z).$$

□

Let $[a,b]$ be a finite interval. For the semiring $([a,b], \oplus, \otimes)$ we say that it is partially continuous, if $\oplus$ satisfies the condition of the diagonal and either the condition $v)$ from Theorem 2 or the condition $v)$ from Theorem 1. Then $\oplus \in \mathcal{T}[a,b] \cup \mathcal{S}[a,b]$.

We have now the main result.

Theorem 11. *If* $[a,b]$ *is a finite interval, then* $([a,b], \oplus, \otimes)$ *is a partially continuous semiring if and only if either* $\oplus = \max$ *and* $\otimes$ *is a generalized* $t-$*norm or* $\oplus = \min$ *and* $\otimes$ *is a generalized* $t-$*conorm.*

Proof. The endpoints a and b are idempotent elements with respect to $\oplus$, and Theorem 9 implies that $\mathbf{1} \oplus \mathbf{1} = \mathbf{1}$. Hence, if we choose that $x = y = \mathbf{1}$, for $z \in [a,b]$ we have

$$(x \oplus y) \otimes z = (x \otimes z) \oplus (y \otimes z) \Leftrightarrow z = z \oplus z,$$

i.e., all elements of the interval $[a,b]$ are idempotent with respect to $\oplus$. Therefore, it follows from Theorems 5(ii) and 4(ii) that either $\oplus = \max$ or $\oplus = \min$. If $\oplus = \max$ then $\mathbf{0} = a$, $\mathbf{1} = b$, i.e., $\otimes$ is a generalized $t-$norm. If $\oplus = \min$ then $\mathbf{0} = b$, $\mathbf{1} = a$, i.e. $\otimes$ is a generalized $t-$conorm.

Conversely, by Theorem 10 the distributivity of the operation $\otimes$ with respect to the operation $\oplus \in \{\max, \min\}$ follows. It is obvious that $\max \in \mathcal{S}[a,b]$, $\min \in \mathcal{T}[a,b]$. Let us prove that $\mathbf{0} \otimes x = \mathbf{0}$, $x \in [a,b]$. Let $\oplus = \max$ and $\otimes$ be a generalized $t-$norm. Then, $\mathbf{0} = a$ and $\mathbf{1} = b$, and by Theorem 1 $ii)$ we obtain that a is an annihilator for $\otimes$, i.e. $\mathbf{0} \otimes x = \mathbf{0}$. The proof is similar when $\oplus = \min$ and $\otimes$ is a generalized $t-$conorm. Therefore, $([a,b], \oplus, \otimes)$ is a partially continuous semiring. □

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