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A GENERALIZATION OF A FIXED POINT THEOREM OF CIRIC

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Abstract. We give a generalization of Ćirić's theorem for a set-valued mapping on a complete metric space.

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1. Preliminaries

First, we give some basic definitions and notations which will be used in the sequel. Let (X, d) be a metric space and $B(X)$ be the set of all nonempty, bounded subsets of X. A set-valued mapping F on X is a point to set the correspondence $x \to Fx$ such that Fx is a nonempty, bounded subset of X for each $x \in X$. We denote such a mapping by $F : X \to B(X)$. For any $x \in X$ and $A, B \in B(X)$, we have the following notations:

$$\begin{aligned} d(x, A) &= \inf\{d(x, a) : a \in A\}, \\ \delta(A, B) &= \sup\{d(a, b) : a \in A,\ b \in B\}. \end{aligned}$$

The function δ satifies the following conditions:

(i) $\delta(A, B) = \delta(B, A) \geq 0$ and $\delta(A, B) = 0$ implies that $A = B$ and this set consists of only one point.

(ii) $\delta(A, B) \leq \delta(A, C) + \delta(C, B)$ for $A, B, C \in B(X)$. Also, if $A = \{a\}$ we write $\delta(A, B) = \delta(a, B)$ and furthermore, if $B = \{b\}$, we write $\delta(A, B) = \delta(a, b) = d(a, b)$.

A sequence $\{A_n\}$ of sets in $B(X)$ is said to converge to the subset A of X if the following two conditions are satisfied:

(1) for each point a in A, there is a sequence $\{a_n\}$ such that $a_n \in A_n$ for all n and $a_n \to a$.

(2) for every $\epsilon > 0$, there exists an integer N such that $A_n \subseteq A_\epsilon$ for all $n \geq N$, where A_ϵ is the union of all open spheres with centres in A and radius ϵ. In such a case A is said to be the limit of the sequence $\{A_n\}$ and we write $\lim A_n = A$ or $A_n \to A$.

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The mapping $F : X \to B(X)$ is said to be continuous at $x_0 \in X$ if whenever $\{x_n\}$ is a sequence of points in X converging to x, the sequence $\{Fx_n\}$ in $B(X)$ converges to Fx in $B(X)$. We say that F is a continuous mapping of X into $B(X)$ if F is continuous at each point x in X.

The usual definition of a fixed point x of a set-valued mapping F is that $x \in Fx$.

2. Main Results

Proposition 2.1. Let F be a mapping of a metric space (X, d) into $B(X)$ satisfying the inequality

$$\begin{aligned} \delta(Fx, Fy) \quad &\leq \quad a\max\{d(x,y), \delta(x,Fx), \delta(y,Fy), \tfrac{1}{2}[d(x,Fy) + d(y,Fx)]\} + \\ (1) \qquad & \qquad + b\max\{\delta x, Fx), \delta(y,Fy)\} + c[d(x,Fy) + d(y,Fx)] \end{aligned}$$

for all x, y in X, where $a \geq 0,\ b > 0,\ c > 0$ with $a + b + 2c = 1$, then

$$(2) \qquad \inf\{\delta(x, Fx) \,:\, x \in X\} = 0.$$

Proof. Let x_0 be an arbitrary point in X and consider the sequence $\{x_n\}$ defined by $x_{n-1} \in Fx_n$ for $n = 1, 2, \ldots$. Then by (1)

$$\begin{aligned} \delta(x_{n-1}, Fx_{n-1}) \quad &\leq \quad \delta(F(x_n, Fx_{n-1}) \\ &\leq \quad a\max\{d(x_n, x_{n-1}), \delta(x_n, Fx_n), \delta(x_{n-1}, Fx_{n-1}), \\ &\qquad\qquad \tfrac{1}{2}[d(x_n, Fx_{n-1}) + d(x_{n-1}, Fx_n)]\} \\ &+ b\max\{\delta(x_n, Fx_n), \delta(x_{n-1}, Fx_{n-1})\} + c[d(x_n, Fx_{n-1}) + d(x_{n-1}, Fx_n)]. \end{aligned}$$

Since

$$\begin{aligned} d(x_n, Fx_{n-1}) \quad &\leq \quad d(x_n, x_{n-1}) + d(x_{n-1}, Fx_{n-1}) \\ &\leq \quad \delta(x_n, Fx_n) + \delta(x_{n-1}, Fx_{n-1}) \end{aligned}$$

we have

$$\begin{aligned} \delta(x_{n-1}, Fx_{n-1}) \quad &\leq \quad \delta(F(x_n, Fx_{n-1}) \\ &\leq a\max\{\delta(x_n, Fx_n), \delta(x_{n-1}, Fx_{n-1}), \tfrac{1}{2}[\delta(x_n, Fx_n) + \delta(x_{n-1}, Fx_{n-1})]\} \\ &+ b\max\{\delta(x_n, Fx_n), \delta(x_{n-1}, Fx_{n-1})\} + c[\delta(x_n, Fx_n) + \delta(x_{n-1}, Fx_{n-1})]. \end{aligned}$$

If $\delta(x_{n-1}, Fx_{n-1}) > \delta(x_n, Fx_n)$ for some n, then we have

$$\begin{aligned} \delta(x_{n-1}, Fx_{n-1}) \quad &\leq \quad (a+b)\delta(x_{n-1}, Fx_{n-1}) + 2c\delta(x_{n-1}, Fx_{n-1}) \\ &= \quad \delta(x_{n-1}, Fx_{n-1}), \end{aligned}$$

a contradiction. Thus $\delta(x_{n-1}, Fx_{n-1}) \leq \delta(x_n, Fx_n)$ and so

$$(3) \qquad \delta(x_n, Fx_n) \leq \delta(x_0, Fx_0)$$

for all positive integers n.

Using (1) we have

$$\begin{aligned}\delta(x_{n+1}, Fx_{n+2}) &\leq \delta(Fx_n, Fx_{n+2})\\ &\leq a\max\{d(x_n, x_{n+2}), \delta(x_n, Fx_n), \tfrac{1}{2}[d(x_n, Fx_{n+2}) + d(x_{n+2}Fx_n)]\} +\\ &\qquad b\delta(x_n, Fx_n) + c[d(x_n, Fx_{n+2}) + d(x_{n+2}, Fx_n)]\\ &\leq a\max\{\delta(x_n, Fx_{n+1}), \delta(x_n, Fx_n), \tfrac{1}{2}[d(x_n, Fx_{n+2}) + \delta(x_n, Fx_n)]\} +\\ &\qquad b\delta(x_n, Fx_n) + c[d(x_n, Fx_n) + d(x_n, Fx_n)]\end{aligned}$$

since by (3)

$$d(x_{n+2}, Fx_n) \leq d(x_{n+1}, x_{n+2}) \leq \delta(x_{n+1}, Fx_{n+1}) \leq \delta(x_n Fx_n)$$

and

$$\begin{aligned}d(x_n, Fx_{n+2}) + \delta(x_n, Fx_n) &\leq d(x_n, x_{n+1}) + \delta(x_{n+1}, Fx_{n+2}) + \delta(x_n, Fx_n)\\ &\leq 2\delta(x_n, Fx_n) + \delta(x_{n+1}, Fx_{n+2}).\end{aligned}$$

Further,

$$\begin{aligned}\delta(x_{n+1}, Fx_{n+2}) &\leq 2a\delta(x_n, Fx_n) + b\delta(x_n, Fx_n) + 2c\delta(x_n, Fx_n) + c\delta(x_{n+1}, Fx_{n+2})\\ &= (1+a)\delta(x_n, Fx_n) + c\delta(x_{n+1}, Fx_{n+2})\end{aligned}$$

since

$$\begin{aligned}d(x_n, Fx_{n+2}) + \delta(x_n, Fx_n) &\leq d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) +\\ &\qquad +\delta(x_{n+2}, Fx_{n+2}) + \delta(x_n, Fx_n)\\ &\leq 4\delta(x_n, Fx_n).\end{aligned}$$

Hence

(4) $$\delta(x_{n+1}, Fx_{n+2}) \leq (1+a)/(1-c)\delta(x_n, Fx_n) \leq (2-b)\delta(x_n, Fx_n).$$

Using (1), (3) and (4) we have

$$\begin{aligned}\delta(x_{n+2}, Fx_{n+2}) &\leq \delta F(x_{n+1}, x_{n+2})\\ &\leq a\max\{\delta(x_n, Fx_n), \tfrac{1}{2}\,\delta(x_{n+1}, Fx_{n+2})\} +\\ &\qquad +b\delta(x_n, Fx_n) + c\delta(x_{n+1}, Fx_{n+2})\\ &\leq a\delta(x_n, Fx_n) + b\delta(x_n, Fx_n) + c(2-b)\delta(x_n, Fx_n)\end{aligned}$$

and hence

(5) $$\delta(x_{n+2}, Fx_{n+2}) \leq (1-bc)\delta(x_n, Fx_n).$$

It is easily shown by induction that (5) implies that

(6) $$\delta(x_n, Fx_n) \leq (1-bc)^{n/2}\delta(x_0, Fx_0).$$

Since by hypothesis $0 < bc < 1$, we conclude from (6) that (2) holds. This completes the proof.

Definition 2.1. Let (X, d) be a metric space. If $F : X \to B(X)$ is a mapping, then F is asymptotically regular if for each $x \in X$ there exists a sequence $\{x_n\}$ with $x_{n-1} \in Fx_n$ $(n = 0, 1, 2, \ldots)$ such that $d(x_n, x_{n-1}) \to 0$ as $n \to \infty$.

Because of the definition of x_n in the proof of Proposition 2.1, we observe that by (6)

$$d(x_n, x_{n-1}) \leq \delta(x_n, Fx_n) \to 0 \quad \text{as} \quad n \to \infty$$

so that we have

Proposition 2.2. Let $F : X \to B(X)$ be a mapping satisfying (1), where $a \geq 0,\ b > 0,\ c > 0$ and $a + b + 2c = 1$. Then F is asymptotocally regular at any point in X.

Note that if F satisfies (1), where a, b, c are non-negative real numbers such that $a + b + 2c = 1$, then F may be discontinuous at some points but we have the following result.

Proposition 2.3. If F satisfies (1), where $a \geq 0,\ b \geq 0,\ c > 0$ and $a + b + 2c = 1$ and if F has a fixed point such that $Fp = \{p\}$, then F is continuous at p.

Proof. Suppose $x_n \to p$. Then from (1)

$$\begin{aligned}
\delta(Fx_n, Fp) &\leq a \max\{d(x_n, p), \delta(x_n, Fx_n), \tfrac{1}{2}[d(x_n, p) + d(p, Fx_n)]\} + \\
&\quad + b\delta(x_n, Fx_n) + c[d(x_n, p) + d(p, Fx_n)] \\
&\leq a[d(x_n, p) + \delta(Fp, Fx_n)] + (b + c)[d(x_n, p) + \delta(Fp, Fx_n)] \\
&\leq (a + b + c)d(x_n, p) + (a + b + c)\delta(Fp, Fx_n)
\end{aligned}$$

and so

$$\delta(Fx_n, Fp) \leq (a + b + c)/(1 - a - b - c)d(x_n, p).$$

Since $a + b + c = 1 - c < 1$, we have

$$\delta(Fx_n, Fp) \leq (1 - c)/cd(x_n, p).$$

It follows that $Fx_n \to Fp$, proving that F is continuous at p.

We now prove the following fixed point theorem.

Theorem 2.1. Let (X, d) be a complete metric space and $F : X \to B(X)$ a mapping satisfying (1), where $a \geq 0,\ b > 0,\ c > 0$ and $a + b + 2c = 1$. Then F has a fixed point p. Further $Fp = \{p\}$ and p is the unique fixed point of F whenever $a + 2c < 1$ and at this point F is continuous.

Proof. Let $x = x_0$ be an arbitrary point in X. Define $\{x_n\}$ as in Proposition 2.1. Then, by (6), since $1 - bc < 1$, we have for arbitrary $\epsilon > 0$

$$d(x_m, x_n) \leq \delta(Fx_{m-1}, Fx_{n-1}) < \epsilon$$

for m, n greater than some N. It follows that $\{x_n\}$ is a Cauchy sequence in the complete metric space X and so has a limit p in X. Using (1), we have

$$\begin{aligned}
\delta(p, Fp) &\leq \delta(p, x_{n-1}) + \delta(x_{n-1}, Fp) \\
&\leq d(p, x_{n-1}) + \delta(Fx_n, Fp) \\
&\leq d(p, x_{n-1}) + a\max\{d(x_n, p), \delta(x_n, Fx_n), \delta(p, Fp), \\
&\quad \tfrac{1}{2}[d(x_n, Fp) + d(p, Fx_n)]\} \\
&\qquad +b\max\{\delta(x_n, Fx_n), \delta(p, Fp)\} + c[d(x_n.Fp) + d(p, Fx_n)].
\end{aligned}$$

Letting $n \to \infty$, we get

$$\delta(p, Fp) \leq a\delta(p, Fp) + b\delta(p, Fp) + c\delta(p, Fp) = (1 - c)\delta(p, Fp)$$

which implies that $\delta(p, Fp) = 0$, since $c > 0$. Thus $Fp = \{p\}$.

Now suppose that F has a second fixed point w so that w is in Fw. Then by (1)

$$\delta(Fw, Fw) \leq a\delta(Fw, Fw) + b\delta(Fw, Fw) < \delta(Fw, Fw),$$

since $a + b < 1$. It follows that $\delta(Fw, Fw) = 0$ and so Fw consists of the single point w. Then using (1) we have

$$d(p, w) = \delta(Fp, Fw) \leq (a + 2c)d(p, w).$$

It follows that $p = w$ and so is the unique fixed point of F. This completes the proof of the theorem.

In Theorem 2.1, if F is a single-valued mapping T, we obtain the following result of Ćirić [1].

Corollary 2.1. Let T be a mapping of a complete metric space (X, d) into itself satisfying the inequality

$$\begin{aligned}
d(Tx, Ty) &\leq a\max\{d(x, y), d(x, Tx), d(y, Ty), \tfrac{1}{2}[d(x, Ty) + d(y, Tx)]\} \\
&\qquad +b\max\{d(x, Ty), d(y, Ty)\} + c[d(x, Ty) + d(y, Tx)]
\end{aligned}$$

for all x, y in X, where $a \geq 0,\ b > 0,\ c > 0$ and $a + b + 2c = 1$. Then T has a fixed point in X, which is unique whenever $a + 2c < 1$, and at this point T is continuous.

References

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