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ON ACCELERATION OF SOLVING SINGULARLY PERTURBED BOUNDARY VALUE PROBLEM

Dragoslav Herceg[1], **Helena Maličić**[1]

Abstract. We consider numerical methods for solving discrete analogue of nonlinear singularly perturbed boundary value problem using a combination of the Hermite scheme and standard central difference scheme on mesh of Bakhvalov type. The obtained nonlinear system is solved by Newton's method and also transformed and split in two systems. New algorithms involve solving Newtonian linear systems with symmetric and positive definite matrix using conjugate gradient and Orthomin methods.

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1. Introduction

This paper is concerned with the following singularly perturbed boundary value problem

$$-\varepsilon^2 u'' + c(x,u) = 0, \quad x \in [0,1], \qquad u(0) = u(1) = 0, \tag{1}$$

where $\varepsilon \in (0, \varepsilon_0)$, $\varepsilon_0 \ll 1$, is a small perturbation parameter. This kind of problem often arises in practice (see [7], [9]). Under smoothness assumptions on c and standard condition

$$0 < \gamma^2 \le c_u(x,u), \quad x \in I, \quad u \in R, \quad I = [0,1], \tag{2}$$

the solution u_ε to (1) has in general two boundary layers at $x = 0$ and $x = 1$. Because of this property, for a start we have to choosing an appropriate mesh, meaning that it has to provide a considerable amount of mesh points in these boundary layers. One of the possibilities is to use some of the well known Shishkin-type meshes, [10], [13], whose advantage is their simplicity of discretization as they are piecewise equidistant. But since a Bakhvalov-type mesh produces much better numerical results (for the transition from a layer to the outside region is smooth), and it is not that much complicated as the

[1]Institute of Mathematics, University of Novi Sad, Trg Dositeja Obradovića 4, 21000 Novi Sad, Yugoslavia, e-mail: {hercegd, helena}@unsim.im.ns.ac.yu

former one, we chose a nonequidistant discretization of the mesh of this type, given by a suitable mesh generating function (see [11]). The second step in solving (1) is choose the type of approximation to $u''(x)$ at the mesh points. We shall use a nonequidistant generalization of the fourth order three-point finite difference scheme, known as the Hermite scheme, [4]. Because of the stability reasons, the Hermite scheme is abandoned at some points and the standard second order central difference scheme is applied. Nevertheless, the combination of this kind has the fourth order uniform accuracy (throughout the paper, the term "uniform" means "uniform in ε"). Various, but similar types of meshes and approximations are discussed in [4], [13], [11], [12].

Choosing a mesh generating function and scheme for approximation to $u''(x)$ at the mesh points, we obtain a discrete analogue

$$F(y) = 0, \tag{3}$$

where $F : R^{n+1} \to R^{n+1}$ is a nonlinear mapping of a special form. The solution $w = [w_0, w_1, \ldots, w_n]^\top$ of (3) is a numerical approximation to u_ε.

A most popular method for solving (3) is Newton's method, which generates the sequence $\{w^k\}$ of approximations to the exact solution w, where the well known iterative step of Newton's method is defined as

$$\begin{aligned} J(w^k)\, s^k &= -F(w^k), \\ w^{k+1} &= w^k + s^k, \quad k = 0, 1, \ldots . \end{aligned} \tag{4}$$

Although Newton's method has local quadratic convergence, there are two difficulties that appear in solving linear system in every Newton iteration. First, the coefficients of the mapping F and its Jacobian matrix J are cumbersome to calculate as they depend on a small perturbation parameter ε, and second, convergence of (4) can be very local. We try to overcome these difficulties by constructing a very good starting vector w^0, for which the method (4) will in just a few iterations satisfy the termination criterion

$$\max\left\{\left\|w^k - w^{k-1}\right\|_\infty, \left\|F(w^k)\right\|_\infty\right\} \le \delta.$$

We shall transform (3) and split the obtained system in two systems with smaller dimensions on which we also apply Newton's method. This idea is first used in [5], to obtain exact solution of the corresponding Newtonian linear systems. Here we shall present some acceleration using conjugate gradient (CG) and Orthomin methods (see [1]) obtaining thus a kind of inexact Newton's method.

In general, CG and Orthomin methods solve linear system $Ax = b$, $A \in R^{n,n}$, $x, b \in R^n$, where the matrix A in CG must be symmetric and positive definite (we shall call such matrices *spd*). For $x^0 \in R^n$, the sequence of vectors $\{x^k\}, k \in N$, is formed and it is known that in exact arithmetic there exists $m \in N, m \le n$, such that $x^m = x = A^{-1}b$. Considering the rounding errors, x^m

will not be equal to the exact solution $A^{-1}b$ and our procedure of forming the vectors x^k becomes iterative.

The algorithm for CG and Orthomin procedures is the following :

1. choose $x^0 \in R^n$, compute $r^0 = b - Ax^0$ and set $p^0 = r^0$,
2. if $r^0 = 0$, then set $m = 0$ and $x = x^0$,
3. if $r^0 \neq 0$, then for $k = 1, 2, \dots$
   - 3.1 compute Ap^{k-1} ,
   - 3.2 compute a_{k-1} ,
   - 3.3 $x^k = x^{k-1} + a_{k-1}p^{k-1}$,
   - 3.4 $r^k = r^{k-1} - a_{k-1}Ap^{k-1}$,
   - 3.5 compute b_{k-1} ,
   - 3.6 $p^k = r^k + b_{k-1}p^{k-1}$,

where for CG

$$a_{k-1} = \frac{\langle r^{k-1}, r^{k-1}\rangle}{\langle p^{k-1}, Ap^{k-1}\rangle} , \quad b_{k-1} = \frac{\langle r^k, r^k\rangle}{\langle r^{k-1}, r^{k-1}\rangle} ,$$

and for Orthomin procedure

$$a_{k-1} = \frac{\langle r^{k-1}, Ap^{k-1}\rangle}{\langle Ap^{k-1}, Ap^{k-1}\rangle} , \quad b_{k-1} = \frac{\langle Ar^k, Ap^{k-1}\rangle}{\langle Ap^{k-1}, Ap^{k-1}\rangle} .$$

During the implementation of these algorithms, we limit the number of iterations to $k_{\max}$ and terminate the iteration when

$$\frac{\|r^k\|_2}{\|b\|_2} < tol. \tag{5}$$

The paper is organized as follows. The next section introduces mesh generating function and mesh points and describes the way of obtaining a discrete analogue of the main problem using a combination of mentioned schemes. Sections 3 and 4 give theoretical and practical justification of forming the starting approximation for (4). Here, transformations and constructing w^0 are presented. Finally, in section 5 we give some numerical examples which demonstrate the effectiveness of this procedure, especially when the exact solution of (1) is symmetric on $[0, 1]$.

2. Mesh generating function and discrete analogue

Throughout the paper we shall assume the following two hypotheses on the problem (1), cf. [9], [11] :

PRESUMPTION 1. Let the reduced problem $c(x,u)=0$, $x\in I$, have a $C^2(I)$ solution u_0. Then there exist $C^2(I)$ functions d_1 and d_2, independent of ε, such that

$$d_i(x)\geq d_*>0,\quad i=1,2,\quad x\in I,$$

$$d_1(t)\geq -u_0(t),\quad d_2(t)\geq u_0(t),\quad t\in\{0,1\}.$$

PRESUMPTION 2. Let for the functions

$$y(x):=u_0(x)-d_2(x)\quad\text{and}\quad z(x):=u_0(x)+d_1(x)$$

and set

$$W=\{(x,u):x\in I,\quad y(x)\leq u\leq z(x)\}$$

holds

$$c\in C^4(W),$$

$$c^*\geq c_u(x,u)\geq c_*\geq\gamma^2>0,\quad (x,u)\in W.$$

Theorem 2.1. *[11] There exists a sufficiently small ε_0 such that for $\varepsilon\in(0,\varepsilon_0]$ the problem (1) has a unique solution u_ε which satisfies $(x,u_\varepsilon(x))\in W$ and $u_\varepsilon\in C^6(I)$.*

We shall consider the mesh generating function given in [11] by

$$\lambda(t)=\begin{cases}\mu(t):=\dfrac{a\varepsilon t}{q-t}, & t\in[0,\alpha],\\[2mm] \pi(t), & t\in[\alpha,0.5],\\[2mm] 1-\lambda(1-t), & t\in[0.5,1],\end{cases}$$

where $\pi(t)$ is a third order polynomial,

$$\pi(t):=\omega(t-\alpha)^3+\frac{1}{2}\mu''(\alpha)(t-\alpha)^2+\mu'(\alpha)(t-\alpha)+\mu(\alpha).$$

Here q is a parameter from $\left(\sqrt[3]{\varepsilon_0},0.5\right)$ and $\alpha=q-\sqrt[3]{\varepsilon}>0$, where we assume that $\varepsilon_0<\frac{1}{8}$. The coefficient ω is determined from $\pi\left(\frac{1}{2}\right)=\frac{1}{2}$,

$$\omega=\frac{0.5-a\left[q(0.5-\alpha)^2+q(0.5-\alpha)\sqrt[3]{\varepsilon}+\alpha\sqrt[3]{\varepsilon^2}\right]}{(0.5-\alpha)^3}.$$

The parameter a is chosen so that $\omega \geq 0$ (such a parameter independent of ε, obviously exists).

For $h = \frac{1}{n}$, n is an even number, $n \geq 2$, the set of mesh points will be

$$I_h = \{x_i = \lambda(ih), \quad i = 0, 1, \ldots, n\},$$

and let

$$h_i = x_i - x_{i-1}, \quad i = 1, 2, \ldots, n.$$

For $x_i \in I_h'$, where

$$I_h' = \left\{x_i \in I_h : \rho_i \leq 1, \frac{\sqrt{5}-1}{2} \leq \frac{h_i}{h_{i+1}} \leq \frac{\sqrt{5}+1}{2}\right\},$$

$$\rho_i = c^* \frac{(h_{i+1} + h_i)\,|h_{i+1} - h_i| + h_{i+1}h_i}{12\varepsilon^2}, \quad i = 1, 2, \ldots, n-1,$$

we approximate $u''(x_i)$ using the difference scheme from [4]:

$$\begin{aligned} F_i := \quad & \varepsilon^2 (a_1(i)\, w_{i-1} + a_0(i)\, w_i + a_2(i)\, w_{i+1}) + \\ & + b_1(i)\, c_{i-1} + b_0(i)\, c_i + b_2(i)\, c_{i+1} = 0, \end{aligned} \tag{6}$$

where

$$c_i = c(x_i, w_i)$$

and the coefficients are

$$a_1(i) = \frac{-2}{h_i(h_i + h_{i+1})}, \quad a_0(i) = \frac{2}{h_i h_{i+1}}, \quad a_2(i) = \frac{-2}{h_{i+1}(h_i + h_{i+1})},$$

$$b_1(i) = -a_1(i)\, \frac{h_i^2 - h_{i+1}^2 + h_{i+1}h_i}{12},$$

$$b_0(i) = a_0(i)\, \frac{h_i^2 + h_{i+1}^2 + 3h_{i+1}h_i}{12},$$

$$b_2(i) = -a_2(i)\, \frac{h_{i+1}^2 - h_i^2 + h_{i+1}h_i}{12}.$$

If $I_h \backslash I_h'$ is non-empty, we mentioned above that we shall then use central difference scheme at those points, so the differential equation from (1) at $x_i \in I_h \backslash I_h'$ is

$$F_i := \varepsilon^2 (a_1(i)\, w_{i-1} + a_0(i)\, w_i + a_2(i)\, w_{i+1}) + c_i = 0.$$

Now the discrete analogue of problem (1) has the form

$$
(7)\qquad
\begin{aligned}
F_0 :=\quad & w_0 = 0, \\
F_i :=\quad & \varepsilon^2 \left(a_1(i)\, w_{i-1} + a_0(i)\, w_i + a_2(i)\, w_{i+1}\right) + \\
& +b_1(i)\, c_{i-1} + b_0(i)\, c_i + b_2(i)\, c_{i+1} = 0, \\
& i = 1, 2, \ldots, n-1, \\
F_n :=\quad & w_n = 0.
\end{aligned}
$$

Of course, for $x_i \in I_h \backslash I_h'$, holds $b_1(i) = b_2(i) = 0$ and $b_0(i) = 1$.

If the nonlinear function F from (3) is given by

$$F = (F_0, F_1, \ldots, F_n),$$

then the solution $w^* = [w_0^*, w_1^*, \ldots, w_n^*]^\top$ to (7), i.e. to $F(w) = 0$, is the approximation to the exact solution u_ε of (1) on the mesh I_h. The following theorem (without proof) gives the error estimate obtained using the Hermite scheme.

Theorem 2.2. *[11] Let*

$$u_{\varepsilon,h} = \left[u_\varepsilon(x_0), u_\varepsilon(x_1), \ldots, u_\varepsilon(x_n)\right]^\top.$$

If ε_0 be sufficiently small, then for $\varepsilon \in (0, \varepsilon_0]$ it holds

$$\left\|u_{\varepsilon,h} - w^*\right\|_\infty \leq Mh^4,$$

where $M > 0$ is a constant independent of ε and h.

Uniqueness of w^* and local convergence of Newton's method (4) follow from [8].

3. Numerical solving of the discrete analogue

Our aim is to find a good starting vector for the Newton method (4), using the discrete analogue (7) with

$$(8)\qquad b_1(i) = b_2(i) = 0, \quad b_0(i) = 1, \quad i = 1, 2, \ldots, n-1.$$

Let us denote in this case the corresponding system of nonlinear equations with $\Phi(w) = 0$. Now, the Jacobian matrix is a tridiagonal matrix

$$J(w) = \left[a_1(i), a_0(i) + c_u(x_i, w_i), a_2(i)\right].$$

Theorem 3.1. *[5] Suppose that the condition (8) is satisfied. Then the system $\Phi(w) = 0$ has a unique solution w^{**} and there exists $\delta > 0$ such that Newton's iterative sequence converges to w^{**} for every w^0 which satisfies $\|w^0 - w^{**}\| \le \delta$.*

Theorem 3.2. *[12] Let the condition (8) be satisfied and let u_ε be the solution to the problem (1),*

$$u_{\varepsilon,h} = \left[u_\varepsilon(x_0), u_\varepsilon(x_1), \ldots, u_\varepsilon(x_n)\right]^\top,$$

and let w^{**} be the solution to $\Phi(w) = 0$. Then

$$\|u_{\varepsilon,h} - w^{**}\| \le Mh^2,$$

where $M > 0$ is a constant independent of ε and h.

The vector w^{**} or its good approximation, say w^0, can be used as a starting approximation for Newton's method for numerical solving of (7). Naturally, we want to obtain the vector w^0 at the lowest computation cost possible.

4. Transformation of the system

The system $\Phi(w) = 0$ can be written as

$$\Phi(w) = Aw + d(x, w), \tag{9}$$

where $w \in R^{n+1}$, $d(x,w) = [0, c_1, c_2, \ldots, c_{n-1}, 0]^\top$ and

$$A = \begin{bmatrix} 1 & & & & & & \\ a_1(1) & a_0(1) & a_2(1) & & & & \\ & a_1(2) & a_0(2) & a_2(2) & & & \\ & & \ddots & \ddots & \ddots & & \\ & & & & a_1(n-1) & a_0(n-1) & a_2(n-1) \\ & & & & & & 1 \end{bmatrix}.$$

Since coefficients of the matrix A depend on ε and solving the corresponding Newtonian linear system can be very complicated, we are going to transform the linear part of the system in such a way that instead of A we shall get an almost constant matrix H. New matrix H will be very close to the tridiagonal matrix $[-1, 2, -1]$ arising from the discretization of classical BVP with $\varepsilon = 1$. In fact, H will differ from $[-1, 2, -1]$ in a few rows.

For the given ε, a and q, choose m such that $m = \lfloor n\alpha \rfloor - 1$ and let D and L be diagonal matrices given with

$$\begin{aligned} D &= diag(d_0, d_1, \ldots, d_n), \\ L &= diag(l_0, l_1, \ldots, l_n), \end{aligned}$$

where

$$d_i = d_{n-i} = \begin{cases} 1, & i = 0, \\ s_i, & i = 1, 2, \ldots, \frac{n}{2}, \end{cases}$$

$$l_i = l_{n-i} = (aqh)^2 \begin{cases} 1, & i = 0, \\ s_{i-1}s_is_{i+1}, & i = 1, 2, \ldots, \frac{n}{2}, \end{cases}$$

and

$$s_i = \begin{cases} \dfrac{1}{q - ih}, & i = 0, 1, \ldots, m+1, \\ 1, & i = m+2, m+3, \ldots, \frac{n}{2} + 1. \end{cases}$$

Then (9) is equivalent to the system

$$Ht + Ld(x, Dt) = 0 \tag{10}$$

where $H = LAD$ and $t = D^{-1}w$. The matrix H is a tridiagonal matrix with the form of the matrix A with $\alpha_1(i), \alpha_0(i)$ and $\alpha_2(i)$ instead of $a_1(i), a_0(i)$ and $a_2(i)$, where

$$\alpha_0(i) = \alpha_0(n-i) = \begin{cases} 2, & i = 1, 2, \ldots, m, \\ l_i \cdot a_0(i) \cdot d_i, & i = m+1, m+2, \ldots, \frac{n}{2}, \end{cases}$$

$$\alpha_2(i) = \begin{cases} -1, & i = 1, 2, \ldots, m \\ l_i \cdot a_2(i) \cdot d_{i+1}, & i = m+1, m+2, \ldots, n-m-1, \\ -1, & i = n-m, n-m+1, \ldots, n, \end{cases}$$

and

$$\alpha_1(i) = \alpha_2(n-i).$$

Jacobian matrix for the system (10) is

$$J(t) = H + L \cdot d_u(x, Dt) \cdot D,$$

with the diagonal matrix

$$d_u(x, Dt) = diag(1, c_u(x_1, w_1), c_u(x_2, w_2), \ldots, c_u(x_{n-1}, w_{n-1}), 1),$$

so that Newton's method becomes

(11)
$$J\left(t^k\right)s^k = -L\Phi\left(w^k\right),$$
$$w^{k+1} = Ds^k + w^k, \quad k = 0,1,\ldots.$$

Since a unique solution $u_0 \in C^2(I)$ of the reduced problem

$$c(x,u) = 0$$

approximates u_ε quite well on the interval $[s, 1-s]$ for $s < 0.5$, in fact, for some constant $M > 0$

$$|u_\varepsilon(x) - u_0(x)| \le M\left(\exp(-\frac{s\gamma}{\varepsilon}) + \varepsilon^2\right),$$

the starting vector w^0 for Newton's iterative method will be constructed in the following way:

(12)
$$w^0 = \left[0, w_1^0, w_2^0, \ldots, w_{n-1}^0, 0\right]^\top,$$

with

$$z_1 = \left[0, w_1^0, \ldots, w_m^0\right]^\top,$$

$$w_i^0 = u_0(x_i), \quad i = m+1, m+2, \ldots, n-m-1,$$

$$z_2 = \left[w_{n-m}^0, w_{n-m+1}^0, \ldots, 0\right]^\top,$$

where z_1 and z_2 are the solutions of the discrete analogues with (8) of the problems

(13)
$$-\varepsilon^2 u'' + c(x,u) = 0, \quad x \in [0, x_m],$$
$$u(0) = 0, \quad u(x_m) = u_0(x_m),$$

and

(14)
$$-\varepsilon^2 u'' + c(x,u) = 0, \quad x \in [1-x_m, 1],$$
$$u(1-x_m) = u_0(1-x_m), \quad u(1) = 0.$$

The discrete analogues for (13) and (14) can be transformed to the form (10) and then solved by Newton's method. After the transformation these systems yield discrete analogues with the $(m+1)\times(m+1)$ matrix

$$H = \begin{bmatrix} 1 & & & & & \\ -1 & 2 & -1 & & & \\ & -1 & 2 & -1 & & \\ & & \ddots & \ddots & \ddots & \\ & & & -1 & 2 & -1 \\ & & & & & 1 \end{bmatrix}$$

as a linear part. Of course, we can not prove second order convergence uniform in ε for an approximation consisting of these three parts, but this is the way we can at least obtain a very good initial approximation for application of Newton's method to (7).

The corresponding Newtonian linear systems have tridiagonal matrices with the diagonal elements $2 + l_i c_u(x_i, w_i) d_i$ and -1 of the diagonal. Since $s_i > 0$ and hypotheses 2 holds, these matrices are *spd*, so we can implement CG and Orthomin methods.

Theorem 4.1. *[3] Let $m = \lfloor n\alpha \rfloor - 1$. If the vector w^0 is given by (12), then*

$$\left\|u_{\varepsilon,h} - w^0\right\|_\infty \leq M\left(h^2 + \varepsilon^2\right),$$

where M is a constant independent of ε and h.

5. Numerical examples

Here we shall present results of numerical experiments which include procedures described in the previous sections applied on (1).

Let l^k and d^k be Newton iterates obtained by solving discrete analogue for (13) and (14) respectively, with the initial vectors

$$z_l^0 = [0, u_0(x_1), u_0(x_2), \ldots, u_0(x_m)]^\top,$$

$$z_d^0 = [u_0(x_{n-m}), u_0(x_{n-m+1}), \ldots, u_0(x_{n-1}), 0]^\top,$$

such that it holds

$$\max\left\{\left\|l^k - l^{k-1}\right\|_\infty, \left\|F\left(l^k\right)\right\|_\infty\right\} \leq 10^{-10},$$

$$\max\left\{\left\|d^k - d^{k-1}\right\|_\infty, \left\|F\left(d^k\right)\right\|_\infty\right\} \leq 10^{-10}.$$

In all experiments we took $q = 0.48$, $a = 1$ and $\varepsilon = 10^{-12}$ using the following methods :

- Newton's method applied to system (7) with

$$w^0 = [0, u_0(x_1), u_0(x_2), \ldots, u_0(x_{n-1}), 0]^\top$$

and the termination criterion

$$\max\left\{\left\|w^k - w^{k-1}\right\|_\infty, \left\|F\left(w^k\right)\right\|_\infty\right\} \leq 10^{-10}. \tag{15}$$

- A2 - accelerated Newton's method with w^0 given by (12), with l^2 instead of z_1 and d^2 instead of z_2, and the termination criterion (15).

- A4 - accelerated Newton's method with w^0 given by (12), with l^4 instead of z_1 and d^4 instead of z_2, and the termination criterion (15).

- A - accelerated Newton's method with w^0 given by (12), with l^k instead of z_1 and d^k instead of z_2, and the termination criterion (15).

- Combining CG and Orthomin methods with A2 and A4, we obtain CA2(tol), CA4(tol) and OA2(tol), OA4(tol), where tol is defined in (5) and takes values 0.5, 0.05 and 0.005 (except CA4(0.005) and OA4(0.005)). In all cases the initial vector for CG and Orthomin methods is zero vector. The second termination criterion is $k_{\max}$. We took 100 iterations for the examples with symmetric solution and 50 iterations for the others.

For the problems with symmetric solution, $u(t) = u(1-t)$, we took

$$d^k = \left[l_m^k, l_{m-1}^k, \ldots, l_1^k, 0\right]^\top$$

and these results are marked SA2, SA4, SA, CSA2(tol), CSA4(tol), OSA2(tol) and OSA4(tol).

The presented tables show the total CPU time needed to satisfy the described criterion expressed in percentages relative to CPU time for Newton's method (which is evidently 100%). All experiments were performed using package *Mathematica 3.0.*

Symmetric problems. Here are presented the results for the main problem (1) with the following functions c and their corresponding solutions u_0 of the reduced problem.

$$\begin{aligned}
c_1(x,u) &= \left(u^2+u-0.75\right)\left(u^2+u-3.75\right), \\
u_0(x) &= -1.5; \\
c_2(x,u) &= \frac{u-1}{5-4u}, \\
u_0(x) &= 1; \\
c_3(x,u) &= 0.5\sin\left(1-\frac{\exp(-\frac{x}{\varepsilon})+\exp(-\frac{1-x}{\varepsilon})}{1+\exp(-\frac{1}{\varepsilon})}\right)-0.5\sin u+u-1, \\
u_0(x) &= 1.
\end{aligned}$$

Function c_1 is from [6] and [4], c_2 is from [7] and c_3 is artificially constructed.

Example 1

n	2^5	2^6	2^7	2^8	2^9	2^{10}	2^{11}	2^{12}	2^{13}	2^{14}
SA2	61	62	55	53	48	43	50	41	38	36
SA4	78	74	70	66	60	54	54	49	38	33
SA	72	74	70	68	61	56	55	49	37	34
CSA2(0.5)	72	68	64	59	53	46	50	44	43	40
CSA4(0.5)	89	81	70	70	61	52	51	43	41	37
CSA2(0.005)	67	65	56	52	46	43	42	36	33	31
CSA4(0.05)	89	78	74	66	57	48	48	40	37	32
CSA2(0.005)	72	68	57	55	49	43	46	43	44	37
OSA2(0.5)	72	78	62	59	53	47	50	45	43	40
OSA4(0.5)	84	74	69	63	58	49	50	42	37	33
OSA2(0.05)	67	65	57	53	47	41	42	37	34	31
OSA4(0.05)	89	84	80	68	60	49	50	42	37	33
OSA2(0.005)	72	65	59	54	50	44	47	45	47	39

Example 2

n	2^5	2^6	2^7	2^8	2^9	2^{10}	2^{11}	2^{12}		
SA2	56	61	62	59	55	50	50	[illegible]		
SA4	67	67	61	55	50	45	41	[illegible]		
SA	72	73	65	61	56	49	49	40		
CSA2(0.5)	78	82	81	83	77	71	75	68	[illegible]	
CSA4(0.5)	78	68	64	57	51	44	43	35	37	
CSA2(0.05)	67	74	60	58	52	47	48	42	47	43
CSA4(0.05)	78	74	70	88	95	100	86	52	64	51
CSA2(0.005)	67	67	65	64	63	80	73	56	57	75
OSA2(0.5)	78	82	81	83	77	71	77	69	80	76
OSA4(0.5)	78	73	65	59	53	45	43	36	39	34
OSA2(0.05)	61	65	64	59	53	48	49	43	48	44
OSA4(0.05)	89	82	86	95	135	136	122	87	75	57
OSA2(0.005)	72	67	70	65	68	76	85	63	98	87

Example 3

n	2^5	2^6	2^7	2^8	2^9	2^{10}	2^{11}	2^{12}	2^{13}	2^{14}
SA2	68	67	59	55	51	47	43	39	36	36
SA4	81	76	71	66	62	57	53	48	35	33
SA	87	73	71	67	64	58	53	48	36	34
CSA2(0.5)	81	76	73	68	64	60	55	52	50	48
CSA4(0.5)	93	87	82	75	70	63	55	48	43	41
CSA2(0.05)	75	73	62	55	51	47	42	38	37	34
CSA4(0.05)	106	83	91	95	114	110	97	61	49	46
CSA2(0.005)	93	73	79	77	92	97	85	67	56	59
OSA2(0.5)	81	84	75	71	67	63	57	53	50	49
OSA4(0.5)	93	87	79	74	70	63	54	48	44	41
OSA2(0.05)	87	76	61	57	52	57	43	48	47	44
OSA4(0.05)	106	93	104	111	171	142	126	106	81	62
OSA2(0.005)	100	73	78	73	86	110	103	78	74	64

Nonsymmetric problems. Here are also given the functions c and the solutions u_0 of the reduced problem.

$$\begin{aligned} c_4\,(x,u) &= u - u^2 - x - x^2 + 2ux, \\ u_0(x) &= x; \end{aligned}$$

$$\begin{aligned} c_5\,(x,u) &= -u^2 + 1 + x, \\ u_0(x) &= -\sqrt{1+x}. \end{aligned}$$

Function c_4 is given in [10] and c_5 is from [2]. These examples have a layer of exponential type at $x = 1$.

Example 4

n	2^5	2^6	2^7	2^8	2^9	2^{10}	2^{11}	2^{12}	2^{13}	2^{14}
A2	69	77	57	59	49	47	47	36	39	36
A4	87	87	69	72	62	59	52	44	38	36
A	82	87	69	71	63	60	51	44	39	36
CA2(0.5)	81	77	69	70	60	57	56	50	55	53
CA4(0.5)	87	87	74	77	67	69	68	62	65	56
CA2(0.05)	82	77	65	68	59	71	62	65	64	55
CA4(0.05)	111	114	115	166	132	139	100	85	89	62
CA2(0.005)	87	80	69	87	85	83	83	66	65	57
OA2(0.5)	86	87	75	79	66	62	59	62	66	86
OA4(0.5)	94	90	77	91	94	108	64	69	80	96
OA2(0.05)	82	80	65	69	62	74	78	65	66	90
OA4(0.05)	107	104	108	154	133	136	122	91	84	102
OA2(0.005)	81	84	74	97	98	106	96	73	69	90

Example 5

n	2^5	2^6	2^7	2^8	2^9	2^{10}	2^{11}	2^{12}	2^{13}	2^{14}
A2	84	74	71	64	61	54	45	40	40	37
A4	106	99	97	91	83	75	61	55	48	45
A	106	99	97	89	83	75	64	56	47	45
CA2(0.5)	86	77	82	76	68	62	57	55	58	52
CA4(0.5)	114	103	106	96	90	75	73	71	69	55
CA2(0.05)	86	77	84	87	97	92	77	72	76	56
CA4(0.05)	135	140	185	211	190	151	137	106	89	65
CA2(0.005)	92	85	106	138	123	123	100	83	75	58
OA2(0.5)	86	77	83	76	68	71	66	56	71	76
OA4(0.5)	114	107	111	102	97	100	117	91	94	92
OA2(0.05)	94	85	91	83	92	102	100	83	79	82
OA4(0.05)	149	166	221	254	267	233	187	133	100	99
OA2(0.005)	100	92	108	161	152	153	132	97	82	82

We conclude that the combining the transformation and splitting described in this paper with CG and Orthomin methods, the number of Newton iterations is significantly reduced and CPU time for these procedures in most cases decreased by the dimension of the system. Such a behaviour is more expressed in symmetric cases. We expect that in solving partial differential equations (for example, the diffusion equation), our procedure will overcome usual difficulties.

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