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SOME PROPERTIES OF SUBSETS, ALMOST CLOSED MAPPINGS AND PARACOMPACTNESS

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Abstract. The purpose of the present paper is to study some properties of α-Hausdorff subset, almost closed mappings and closed graphs.

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1. Preliminaries

Our notation is standard. No separation properties are assumed for spaces unless explicitly stated.

A subset A of a space X is regular open iff $IntClA = A$. A subset A of a space X is regular closed iff $ClIntA = A$, [9].

A subset A of a space X is α-paracompact (α-nearly paracompact) with respect to a subset B iff for every open (regular open) cover $\mathcal{U} = \{U_i : i \in I\}$ of A there is an open family $\mathcal{V} = \{V_j : j \in J\}$ such that:

1. $\mathcal{V}$ refines $\mathcal{U}$,
2. $A \subset \cup\{V_j : j \in J\}$,
3. $\mathcal{V}$ is locally finite at each point $x \in B$.

Subsets A and B of a space X are mutually α-paracompact (α-nearly paracompact) iff the subset A is α-paracompact (α-nearly paracompact) with respect to the subset B and B is α-paracompact (α-nearly paracompact) with respect to the subset A [5].

A subset A of a space X is α-Hausdorff iff for any two points a, b of a space X, where $a \in A$ and $b \in X\backslash A$, there are disjoint open sets U and V containing a and b respectively.

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A subset A of a space X is α-regular (α-almost regular) iff for any point $a \in A$ and any open (regular open) set U containing a, there is an open set V such that $a \in V \subset ClV \subset U$, [7].

A nonempty proper subset A of a space X is M_N iff:

1. A is an α-Hausdorff α-nearly paracompact with respect to $X \backslash A$,
2. Any two distinct points of A can not be strongly separated by open neighbourhoods [3].

A mapping $f : X \to Y$ is almost closed iff for every regular closed set F in X, the set $f(F)$ is closed, [9].

A mapping $f : X \to Y$ has a closed graph $G(f)$ iff $G(f) = \{(x, f(x)) : x \in X\}$ is closed in $X \times Y$, [1].

2. Results

Theorem 2.1. *If E is an α-Hausdorff retract of a space X, then E is a closed set in X.*

Proof. Suppose that E is an α-Hausdorff retract with the retraction $r : X \to E$. We are going to prove that $X \backslash E$ is an open set.

If $X \backslash E = \emptyset$, the theorem is trivial. Thus we suppose that $X \backslash E \neq \emptyset$. Let a be any point of $X \backslash E$. Then the point $b = r(a)$ is in E and we have $b \neq r(a)$. Since $b \in E$, $a \in X \backslash E$, E is an α-Hausdorff then there are the open sets U and V such that $a \in U$, $b \in V$, $U \cap V = \emptyset$. Since r is continuous and $E \cap V$ is an open set of the subspace E it follows that $r^{-1}(E \cap V)$ is an open set containing a. Let

$$W = U \cap r^{-1}(E \cap V).$$

Then W is an open set of X such that $a \in W$, $V \cap W = \emptyset$, and $r(W) \subset V$. This implies that $r(x) \neq y$ for each $x \in W$, hence $W \subset X \backslash E$. This proves that $X \backslash E$ is an open set, therefore E is closed. □

Lemma 2.1. *If $f : X \to Y$ is an almost closed mapping of a space X into a space Y, then for any subset B with $B \cap f(X) \neq \emptyset$ and any regular open set U containing $f^{-1}(B)$, there is an open set V of Y such that*

$$B \subset V \quad \textit{and} \quad f^{-1}(B) \subset f^{-1}(V) \subset U.$$

Proof. Since U is regular open, then the set $X \backslash U$ is regular closed. Suppose that $X \backslash U \neq \emptyset$ (if $U = X$ the theorem is trivial). Since f is an almost closed mapping it follows that $f(X \backslash U)$ is closed in Y.

From $f^{-1}(B) \subset U$ it follows that $B \subset Y \backslash f(X \backslash U) = V$. V is an open set containing B such that $f^{-1}(B) \subset f^{-1}(V) \subset U$. Hence the result. □

Remark 2.1. *The converse in Lemma 2 is not always true (the converse in Lemma 2 is true if $f : X \to Y$ is surjection - Lemma 3 in [8]). It can be seen from the following example.*

Example 2.1. *Let $X = \{a, b, c, d\}$ and*

$$\tau_X = \{\emptyset, \{a\}, \{a, b\}, \{c, d\}, \{a, c, d\}, X\}.$$

Let $Y = \{m, n\}$ be endowed by the topology

$$\tau_Y = \{\emptyset, \{m\}, Y\}.$$

Let $f : X \to Y$ be a mapping defined by $f(a) = f(b) = f(c) = f(d) = m$.

For $B = \{m\}$ or $B = Y$, $f^{-1}(B) = X$.

$U = X$ is a regular open set containing $f^{-1}(B)$. There is an open set $V = Y$ such that $U = X = f^{-1}(B) = f^{-1}(V)$.

The mapping f is not almost closed since for a regular closed set $\{a, b\}$ of X, $f(\{a, b\}) = m$ is not closed in Y.

Theorem 2.2. *Let $f : X \to Y$ be an almost closed mapping of a space X into a space Y such that for each $y \in f(X)$ $f^{-1}(y)$ is an α-Hausdorff α-nearly paracompact subset with respect to $X \backslash f^{-1}(y)$. Then f has a closed graph.*

Proof. Let $(x, y) \notin G(f)$ be any point. Suppose that $Y \backslash f(X) \neq \emptyset$. Let $y \in f(X)$. Since $y \neq f(x)$, then $x \notin f^{-1}(y)$. Since $f^{-1}(y)$ is an α-Hausdorff α-nearly paracompact subset with respect to $X \backslash f^{-1}(y)$, then, by theorem 2.2 in [5] there are regular open disjoint sets U and V such that

$$x \in U, \qquad f^{-1}(y) \subset V.$$

Since f is almost closed, there is an open neighbourhood N of y such that

$$f^{-1}(y) \subset f^{-1}(N) \subset V.$$

Thus $U_1 = U \times N$ is an open neighbourhood of a point (x, y) such that $U_1 \cap G(f) = \emptyset$.

Now, let $y \in Y \backslash f(X)$. Since f is an almost closed mapping, then $f(X)$ is closed. Let $V = Y \backslash f(X)$. Since for each point $x \in X$, $f(x) \notin V$, it follows that the set $U_2 = X \times V$ is an open neighbourhood of (x, y) such that $U_2 \cap G(f) = \emptyset$.

From this two facts it follows that there is an open neighbourhood W of a point (x, y) such that $(x, y) \subset W \subset (X \times Y) \backslash G(f)$, hence $(X \times Y) \backslash G(f)$ is open. Thus $G(f)$ is closed. Hence the result. □

The following theorem was proved in [5], theorem A. Let $f : X \to Y$ be an almost closed mapping of a space X onto a space Y such that the family

$$\{f^{-1}(y) : y \in Y\}$$

consists of α-Hausdorff subsets which are mutually α-nearly paracompact, then Y is Hausdorff.

If $f : X \to Y$ is an almost closed mapping of a space X into a space Y such that the family

$$\{f^{-1}(y) : y \in f(X)\}$$

consists of α-Hausdorff subsets which are mutually α-nearly paracompact, then $f(X)$ is not always α-Hausdorff what we can see from the following example.

Example 2.2. *Let* $X = \{a, b, c, d\}$ *and*

$$\tau_X = \{\emptyset, \{a\}, \{a, b\}, \{c, d\}, \{a, c, d\}, X\}.$$

Let $Y = \{m, n\}$ and

$$\tau_Y = \{\emptyset, \{n\}, Y\}.$$

Let the mapping $f : X \to Y$ be defined by

$$f(a) = f(b) = f(c) = f(d) = m.$$

The mapping $f : X \to Y$ is an almost closed mapping such $f^{-1}(m) = X$ is an α-Hausdorff α-nearly paracompact set of X. The set $\{m\}$ is not α-Hausdorff because the points m and n can not be strongly separated by open neighbourhoods.

Theorem 2.3. *Let $f : X \to Y$ be an almost closed mapping of a space X into a space Y such that for each $y \in f(X)$ $f^{-1}(y)$ is an α-Hausdorff α-nearly paracompact subset with respect to $X \backslash f^{-1}(y)$. If $Y \backslash f(X)$ is an M_N subset then:*

1. *$f(X)$ is α–Hausdorff.*

2. *Every two points of $f(X)$ can be strongly separated by open neighbourhoods.*

3. *If $f(X)$ is an M_N subset, then $f(X)$ contains only one point.*

Proof.

1. Since f is an almost closed mapping the set $f(X)$ is closed. Since $Y \backslash f(X)$ is a M_N set, then, by Theorem 2.2 in [5], $Y \backslash f(X)$ is closed.

   Let $a \in f(X)$, $b \in Y \backslash f(X)$ be any two points. The sets $U = f(X)$, $V = Y \backslash f(X)$ are disjoint open neighbourhoods of a and b respectively. Hence the result.

2. Let y_1 and y_2 be any distinct points of $f(X)$. By Theorem 2.4 in [5] there exist disjoint regular open neighbourhoods U_1 and U_2 of $f^{-1}(y_1)$ and $f^{-1}(y_2)$ respectively. Since f is almost closed, there are open sets V_1 and V_2 containing y_1 and y_2 respectively such that

   $$f^{-1}(y_1) \subset f^{-1}(V_1) \subset U_1; \quad f^{-1}(y_2) \subset f^{-1}(V_2) \subset U_2.$$

   Let $V_1' = V_1 \cap f(X)$ and $V_2' = V_2 \cap f(X)$. V_1' and V_2' are disjoint open neighbourhoods of the points y_1 and y_2 respectively. Hence the result.

3. Since $f(X)$ is an M_N set, then by 2) it follows that $f(X)$ contains only one point. □

If $f : X \to Y$ is an almost closed mapping of a space X into a space Y, such that for each point $y \in f(X)$ $f^{-1}(y)$ is an M_N subset, then Y is not obviously Hausdorff. This fact we can see from the following example.

Example 2.3. *Let $X = \{a, b, c, d\}$ be endowed by the topology*

$$\tau_X = \{\emptyset, \{a, b\}, \{c, d\}, X\},$$

and $Y = \{m, n, p\}$ be endowed by the topology

$$\tau_Y = \{\emptyset, \{m, p\}, \{n, p\}, \{p\}, Y\}.$$

Let $f : X \to Y$ be a mapping defined by

$$f(a) = f(b) = m; \quad f(c) = f(d) = n.$$

The space Y is not Hausdorff (the points m, n can not be strongly separated by open neighbourhoods). The mapping f is almost closed, such that:

1. *$f^{-1}(m) = \{a, b\}$ is an M_N subset of the space X.*
2. *$f^{-1}(n) = \{c, d\}$ is an M_N subset of the space Y.*

Theorem 2.4. *If $f : X \to Y$ is an almost closed mapping of a space X into a space Y such that $f(X)$ is open and for each point $y \in f(X)$ $f^{-1}(y)$ is an M_N subset of a space X, then any two distinct points of $f(X)$ can be strongly separated by open neighbourhoods.*

Proof. The proof is omitted. It is similar to the proof of theorem 2.3. □

Example 2.4. *Let $X = \{a, b, c, d\}$ be endowed by the topology $\tau_X = \{\emptyset, \{a, b\}, \{c, d\}, X\}$ and $Y = \{m, n, p, q\}$ be endowed by the topology $\tau_Y = \{\emptyset, \{m\}, \{n\}, \{m, n\}, \{p, q\}, \{m, p, q\}, \{n, p, q\}, Y\}$. Let $f : X \to Y$ be a mapping defined by*

$$f(a) = f(b) = m; \quad f(c) = f(d) = n.$$

The mapping $f : X \to Y$ is almost closed so that $f^{-1}(m) = \{a, b\}$ $f^{-1}(n) = \{c, d\}$ are two M_N subsets and $f(X) = \{m, n\}$ is a clo-open subset of a space Y. The points $m, n \in f(X)$ can be strongly separated by open neighbourhoods. ($m \in \{m\}, n \in \{n\}, \{m\} \cap \{n\} = \emptyset$).

Theorem 2.5. *If $f : X \to Y$ is an almost closed mapping of a space X into a compact space Y such that the family*

$$\{f^{-1}(y) : y \in f(X)\}$$

consists of α-Hausdorff subsets which are mutually α-nearly paracompact, then f is continuous.

Proof. Suppose that f is not continuous at some point $x \in X$. Let $\mathcal{U}(x)$ denote the family of all open neighbourhoods of x. Let $y = f(x)$. Since f is not continuous at x, then there is an open neighbourhood V of y such that for every $U \in \mathcal{U}(x)$ $f(U) \cap (Y \backslash V) \neq \emptyset$.

Thus

$$\mathcal{A} = \{f(ClU) \cap (Y \backslash V) : U \in \mathcal{U}(x)\}$$

is a family of closed subsets of Y. This family must have the finite intersection property (if there is a finite number of open sets $U_1, U_2, \ldots, U_n$ such that

$$\bigcap_{i=1}^{n} (f(ClU_i) \cap (Y \backslash V)) = \emptyset, \quad \text{then}$$

$\bigcap\limits_{i=1}^{n} U_i$ is an open set containing x and

$$(Y \backslash V) \cap f(\bigcap_{i=1}^{n} U_i) \subset \bigcap_{i=1}^{n} (Y \backslash V) \cap f(ClU_i) = \emptyset$$

which is a contradiction).

Since Y is compact, there is a point $y_0 \in \cap\{A : A \in \mathcal{A}\}$. Thus we have $y_0 \in Y \backslash V$ and hence $x \notin f^{-1}(y_0)$. Since $f^{-1}(y_0)$ is an α-Hausdorff subset of X which is α-nearly paracompact with respect to each point $y \in X \backslash f^{-1}(y_0)$, then, by Theorem 2.2 in [5], it follows that there are disjoint regular open sets U_x and U_0 such that

$$x \in U_x \quad \text{and} \quad f^{-1}(y_0) \subset U_0.$$

From

$$ClU_x \cap f^{-1}(y_0) \subset ClU_x \cap U_0 = \emptyset$$

we have

$$y_0 \notin f(ClU_x).$$

On the other hand, since U_x belongs to $\mathcal{U}(x)$, we have

$$y_0 \in f(ClU_x) \cap (Y \backslash V) \subset f(ClU_x).$$

This is a contradiction. Hence f must be continuous at x. Thus, f is continuous. □

Theorem 2.6. *Let $f : X \to Y$ be an almost closed mapping of a space X into a space Y such that for each point $y \in f(X) f^{-1}(y)$ is closed. If A is an α-regular α-paracompact subset with respect to $X \backslash A$, then $f(A)$ is closed.*

Proof. The proof is ommited. It is similar to the proof of Theorem 2.2. in [6]. □

Theorem 2.7. *Let $f : X \to Y$ be an almost closed mapping of a space X into a space Y such that $G(f)$ is closed. If A is an α-regular α-paracompact subset with respect to $X \backslash A$, then $f(A)$ is closed.*

Proof. Since the mapping f has a closed graph, then $f^{-1}(y)$ is closed for each $y \in f(X)$. □

Corollary 2.1. *Let $f : X \to Y$ be an almost closed mapping of a space X into a space Y such that $G(f)$ is closed. If every closed subset A of X is α-regular α-paracompact with respect to $X \backslash A$, then f is closed.*

Theorem 2.8. *Let $f, g : X \to Y$ be continuous mappings of a space X into a space Y such that $f(X) = g(X)$ and for each point $y \in f(X)$ $f^{-1}(y)$ is an M_N subset of X. Then:*

1. *The set $A = \{x \in X : f(x) = g(x)\}$ is closed.*
2. *If there is a dense subset $B \subset X$ such that for each point $x \in B$, $f(x) = g(x)$, then $f = g$.*

Proof.

1. Let $A \neq \emptyset$, and $A \neq X$ (the theorem is trivial if $A = \emptyset$ or $A = X$).

   Let $x \in X \backslash A$ be any point. Then $f(x) \neq g(x)$.

   There is a point $x_1 \in X$ such that $f(x_1) = g(x)$.

   By Lemma 3.1 in [5], there are disjoint regular open neighbourhoods U_1 and U_2 of $f^{-1}(f(x))$ and $f^{-1}(g(x))$ respectively. Since f is almost closed, there are open sets V_1 and V_2 containing $f(x)$ and $g(x)$ respectively such that

   $$f^{-1}(f(x)) \subset f^{-1}(V_1) \subset U_1$$
   $$f^{-1}(g(x)) \subset f^{-1}(V_2) \subset U_2.$$

   It follows that the sets

   $$W_1 = V_1 \cap f(X) \quad \text{and} \quad W_2 = V_2 \cap f(X)$$

   are disjoint sets containing $f(x)$ and $g(x)$ respectively.

   Since f is continuous at x it follows that there is an open set H_1 containing x such that $f(H_1) \subset V_1$ so, $f(H_1) \subset W_1$.

   Similarly, since g is continuous at x it follows that there is an open set H_2 containing x such that

   $$g(H_2) \subset V_2 \text{ so, } g(H_2) \subset W_2.$$

   Hence, for each point $z \in H = H_1 \cap H_2$, $f(z) \neq g(z)$. Since H is an open neighbourhood of x such that $H \cap A = \emptyset$, hence $X \backslash A$ is open, i.e. A is closed.

2. It follows that from 1., B is closed, hence $B = X$. □

Example 2.5. *Let* $X = \{a, b, c. d, e\}$ *be endowed by the topology*

$$\tau_X = \{\emptyset, \{a, b\}, \{c, d\}, \{e\}, \{a, b, e\}, \{c, d, e\}, X\}$$

and $Y = \{m, n, p, q\}$ *be endowed by the topology*

$$\tau_y = \{\emptyset, \{m, n, p\}, \{q\}, Y\}.$$

Let the mappings $f : X \to Y$, $g : X \to Y$ be defined by

$$f(a) = f(b) = m; \quad f(c) = f(d) = n; \quad f(e) = p$$

$$g(a) = g(b) = m, \quad g(c) = g(d) = p, \quad g(e) = n.$$

The mappings f and g are almost closed so that

$$f^{-1}(m) = \{a, b\}, f^{-1}(n) = \{c, d\}, f^{-1}(p) = e$$

are the M_N subsets of X.

The set $A = \{a, b\}$ is closed.

$$f(a) = g(a) = f(e) = g(e) = m.$$

$$(f(A) = g(A) = \{m\}).$$

The space Y is not Hausdorff (the points m, n can not be strongly separated by open neighbourhoods).

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