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# CONVERGENCE OF FINITE DIFFERENCE SCHEMES APPROXIMATING HYPERBOLIC PROBLEMS

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**Abstract.** We obtain the convergence rate estimates for the weak solution of hyperbolic initial-boundary value problem of order $2(s-k)/3$, but under conditions weaker than earlier. Also, a numerical example is presented.

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*Key words and Phrases:* boundary value problems, finite difference schemes, Sobolev spaces, interpolation of function spaces, convergence rate estimates

## 1. Introduction

For the elliptic boundary value problem, convergence rate estimates for finite difference schemes which are compatible with the smoothness of data

$$\|u-v\|_{W_{2,h}^k} \le Ch^{s-k}\|u\|_{W_2^s}, \quad s>k,$$

has been obtained in [1]. Here $u$ denotes the solution of the boundary value problem, $v$ denotes the corresponding discrete aproximation, $W_2^s$ is the standard Sobolev space and $W_{2,h}^k$ is discrete Sobolev space. The compatible estimates may also be derived in parabolic case [2]. But in the hyperbolic case, the usual estimates are not compatible with smoothness of data [3]:

$$\|u-v\|_{C_\tau(W_{2,h}^k)} \le Ch^{s-k-1}\|u\|_{W_2^s(Q)}, \quad s>k+1.$$

A few years ago, using interpolation theory (see [4]), B.S. Jovanović derived in [5] the convergence rate estimate of order $2(s-k)/3$ for a finite difference scheme in the case of homogenous wave equation with constant coefficients. The analogous estimate has been obtained in [6] when coefficients are functions from $C^k$–spaces.

We shall show in this paper that the same estimate holds although the coefficients are less smooth, i.e. if they are from $W_2^k$–spaces. This will also be illustrated with a numerical experiment.

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## 2. Some necessary function spaces

Let $L_q = L_q(0,1)\,(1 \le q \le \infty)$ be the Lebesgue spaces of integrable functions, $W_2^s = W_2^s(0,1)$ standard Sobolev spaces, $\mathcal{D}$ the spaces of infinitely differentiable functions with compact support in $(0,1)$ and $\overset{o}{W}{}_2^s$ be the closure of $\mathcal{D}$ in $W_2^s$. $(,)$ and $\|\ \|$ denote the inner product and the norm in $L_2$, respectively. Suppose $a \in L_\infty$ such that

$$a \ge a_0 \quad \text{in} \quad (0,1) \quad \text{a.e.}$$

For the operator $L : \overset{o}{W}{}_2^1 \to W_2^{-1}$ defined by $Lv = -(av')'$ there exist $0 < \lambda_1 < \lambda_2 < \ldots, \lim_k \lambda_k = \infty$ such that $L\varphi_k = \lambda_k \varphi_k\,(k \in N)$; The sequence of the eigenfunctions $(\varphi_k)_{k\in N} \subset \overset{o}{W}{}_2^1$ is an orthonormed basis of $L_2$ (see [7]). Introduce the spaces $V^\alpha\,(\alpha \ge 0)$ by $V^\alpha = \{v \in L_2 \;\big|\; \|v\|^2_{V^\alpha} = \sum_{k=1}^\infty \lambda_k^\alpha \tilde{v}_k^2 < \infty\}$, where $\tilde{v}_k = (v, \varphi_k)$ are the Fourier coefficients of $v$ in the basis $(\varphi_k)_{k\in N}$.

**Lemma 1.** *If $v \in L_2$ such that $(\forall \varphi \in \mathcal{D}) \left|\int_0^1 v\varphi'\,dx\right| \le K\|\varphi\|$, then $v \in W_2^1$ and $\|v'\| \le K$.*

*Proof.* See [8]. □

**Lemma 2.** *If $v \in W_2^2$ then $\|v'\|_{L_\infty} \le C\|v\|_{W_2^2}$.*

Proof. We cite here the Poincaré–Wirtinger inequality (see [9])

$$\|v - \overline{v}\|_{L_\infty} \le \|v'\|_{L_1} \quad (v \in W_1^1),$$

where $\overline{v} = \int_0^1 v\,dx$ is an average of $v$. It is easy now to verify our statement by putting $v'$ instead of $v$ in the last equation and applying the Cauchy–Schwarz inequality. □

**Lemma 3.** *If $v \in \overset{o}{W}{}_2^1$ is the solution of the variational problem*

(1) $$\int_0^1 av'\varphi'\,dx = \int_0^1 f\varphi\,dx \quad (\varphi \in \overset{o}{W}{}_2^1),$$

*then for $a \in W_2^3$, the relation*

(2) $$f \in W_2^i \Rightarrow v \in W_2^{i+2} \quad \text{and} \quad \|v\|_{W_2^{i+2}} \le C\|f\|_{W_2^i}$$

*holds for $i = 0, 1, 2$.*

*Proof.* The case $i = 0, 1$ is simple, but for $i = 2$ the situation is more complicated. So, let us assume that (2) holds for $i = 0, 1$. We shall prove it in the case $i = 2$. Suppose $\varphi \in \mathcal{D}$. Replacing $\varphi$ with $\left(\frac{\varphi}{a}\right)''$ in (1) we have

$$\int_0^1 av' \left(\frac{\varphi}{a}\right)''' dx = \int_0^1 f \left(\frac{\varphi}{a}\right)'' dx.$$

Applying partial integration one has

$$\int_0^1 (av')'' \left(\frac{\varphi}{a}\right)' dx = \int_0^1 f'' \frac{\varphi}{a} dx.$$

We know that $v \in W_2^3$ (because (2) holds for $i = 1$), so after a few obvious transformations we obtain

$$\begin{aligned}\int_0^1 v'''\varphi' \, dx &= \int_0^1 f''\varphi \, dx + \int_0^1 \frac{a'}{a^2}(a''v' + 2a'v'' + av''')\varphi \, dx \\ &- \int_0^1 \frac{a''v' + 2a'v''}{a} \varphi' \, dx.\end{aligned}$$

Some partial integrations applied to the last equation yield

$$\int_0^1 v'''\varphi \, dx = \int_0^1 f''\varphi \, dx + \int_0^1 \frac{a'}{a}(a''v' + 2a'v'' + av''')\varphi \, dx +$$

$$+ \int_0^1 \left( \frac{a''' + 3a''v'' + 2a'v'''}{a} - \frac{a'}{a^2} \frac{a''v' + 2a'v''}{a} \right) \varphi \, dx. \tag{3}$$

For example, let us estimate the term $\int_0^1 \frac{a'a''}{a^2} v'\varphi \, dx$. Because $\frac{a'a''}{a^2} \in L_\infty$, we have

$$\begin{aligned}\left| \int_0^1 \frac{a'a''}{a^2} v'\varphi \, dx \right| \leq \left\| \frac{a'a''}{a^2} \right\|_{L_\infty} \int_0^1 |v'\varphi| \, dx &\leq C\|v'\| \, \|\varphi\| \\ &\leq C\|v\|_{W_2^2} \|\varphi\| \leq C\|f\| \, \|\varphi\|\end{aligned}$$

(the last inequality follows from (2) for $i = 0$).

All the other terms can be estimated similarly except the term $\int_0^1 \frac{a'''v'}{a}\varphi \, dx$. But $v' \in L_\infty$ and taking into account Lemma 2, we obtain

$$\begin{aligned}\left| \int_0^1 \frac{a'''v'}{a}\varphi \right| dx \leq \|v'\|_{L_\infty} \int_0^1 \left| \frac{a'''}{a}\varphi \right| dx &\leq \|v\|_{W_2^2} \left\| \frac{a'''}{a} \right\| \|\varphi\| \\ &\leq C\|f\| \, \|\varphi\|.\end{aligned}$$

Estimating all the other terms in the right-hand side of (3) one obtains

$$\left| \int_0^1 v'''\varphi' \, dx \right| \leq C\|f\|_{W_2^2} \|\varphi\|.$$

Then Lemma 1 implies (2) for $i = 2$. □

The two following statements can be established in the perfectly same way as it was done in [10].

**Theorem 1.** *Suppose $a \in W_2^3$ and $a \geq a_0$ in $(0,1)$. Then*

$$V^1 = \overset{o}{W}{}_2^1, \quad V^2 = W_2^2 \cap \overset{o}{W}{}_2^1, \quad \overset{o}{W}{}_2^3 \subset V^3 \subset W_2^3, \quad \overset{o}{W}{}_2^4 \subset V^4 \subset W_2^4.$$

**Proposition 1** *Suppose $a \in W_2^3$ and $a \geq a_0$ in $(0,1)$. Then*

$$c\|v\|_{W_2^i} \leq \|v\|_{V^i} \leq C\|v\|_{W_2^i} \quad (v \in V^i)$$

*for $i = 1, 2, 3, 4$.*

## 3. Hyperbolic problem and discretisation

Consider the initial-boundary value problem for the homogenous second-order hyperbolic equation in the domain $Q = (0,1) \times (0,T]$:

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial}{\partial x}\left(a(x)\frac{\partial u}{\partial x}\right), \quad (x,t) \in Q$$

$$(4) \qquad u(0,t) = u(1,t) = 0, \quad t \in [0,T]$$

$$u(x,0) = u_0(x), \quad \frac{\partial u}{\partial t}(x,0) = u_1(x), \quad x \in (0,1).$$

There is the unique weak solution of this problem for $u_0 \in V^1$, $u_1 \in V^0$ (see [11]). In [6] was also shown that if $u_0 \in V^\alpha$, $u_1 \in V^{\alpha-1}$ and $a \in L_\infty$, $a \geq a_0 > 0$ in $(0,1)$ then

$$\max_{t\in[-T,T]} \|\partial^l u/\partial t^l\|_{V^{\alpha-l}} \leq C(\|u_0\|_{V^\alpha} + \|u_1\|_{V^{\alpha-1}}),$$

for $1 \leq \alpha \leq 4$, $l \in Z$, $0 \leq l \leq \alpha$, wherefrom, using Theorem 1 and Proposition 1 follows that if $a \in W_2^3$ and $a \geq a_0 > 0$ in $(0,1)$ then

$$(5) \qquad \max_{t\in[-T,T]} \|\partial^l u/\partial t^l\|_{W_2^{\alpha-l}} \leq C(\|u_0\|_{V^\alpha} + \|u_1\|_{V^{\alpha-1}}),$$

for $1 \leq \alpha \leq 4$, $l \in Z$, $0 \leq l \leq \alpha$. The extension of $\partial^l u/\partial t^l$ on $[-T,T]$ can be done in a natural way using the Fourier series (see also [6]).

Let $\overline{\omega}_h$ be a uniform mesh on $[0,1]$ with the stepsize $h = 1/n$, $\omega_h = \overline{\omega}_h \cap (0,1)$ and $\omega^- = \omega_h \cup \{0\}$. We set $\overset{o}{W}{}_{2,h}^1$ to be the space of all functions defined on $\overline{\omega}_h$ vanishing at 0 and 1. Introduce finite differences in $x$:

$$v_x = (v(x+h) - v(x))/h, \quad v_{\overline{x}} = (v(x) - v(x-h))/h.$$

We define the following discrete norms

$$\|v\|_h = \Big(h \sum_{x\in\omega_h} v^2(x)\Big)^{1/2}, \quad \|[v\|_h = \Big(h \sum_{x\in\omega_h^-} v^2(x)\Big)^{1/2},$$

$$\|v\|_{W^1_{2,h}} = (\|v\|^2 + \|[v_x\|^2)^{1/2}.$$

Also introduce the operator $L_h : \overset{o}{W}{}^1_{2,h} \to \overset{o}{W}{}^1_{2,h}$ by

$$L_h v = \begin{cases} -\frac{1}{2}[(av_x)_{\overline{x}} + (av_{\overline{x}})_x], & x \in \omega_h \\ 0, & x \in \{0,1\} \end{cases}$$

Let $\overline{\omega}_h$ be a uniform mesh on $[-\tau/2, T]$ with the stepsize $\tau = T/(m-1/2)$, $\omega_\tau = \overline{\omega}_\tau \cap (0,T)$, and $\omega_\tau^- = \omega \cup \{-\tau/2\}$. We shall also use the following notations:

$$v = v(t), \quad \hat{v} = v(t+\tau), \quad \check{v} = v(t-\tau), \quad v^j = v((j-1/2)\tau),$$

$$\overline{v} = (v+\hat{v})/2, \quad v_t = (\hat{v}-v)/\tau, \quad v_{\overline{t}} = (v-\check{v})/\tau.$$

For the functions defined on $\overline{\omega}_h \times \overline{\omega}_\tau$ we define the norms

$$\|v\|_{C_\tau(W^1_{2,h})} = \max_{t\in\omega_\tau^-} \|v(\cdot,t)\|_{W^1_{2,h}}, \quad \|v\|_{L_{q,\tau}(L_{2,h})} = \Big(\tau \sum_{t\in\omega_\tau} \|v(\cdot,t)\|_h^q\Big)^{1/q}.$$

One can easily deduce

**Lemma 4.** *For $v \in \overset{o}{W}{}^1_{2,h}$ the inequality $\|v\|_{(I+0.25\tau^2(\sigma-1/4)L_h)} \le C\|v\|_h$ holds if one of the following two conditions is satisfied:*

(i) *If $\sigma > 1/4$, then $\tau/h < C$, where $C$ is an arbitrary constant;*

(ii) *If $\sigma < 1/4$, then $\tau/h \le 4\sqrt{\frac{1-s_0}{(1-4\sigma)c_1^2}}$ for some $s_0 \in (0,1)$, where $c_1$ is a constant depending only on the function $a$.*

Let $S_x$ and $S_t$ denote the Steklov smoothing operators in $x$ and $t$:

$$S_x f(x,t) = \frac{1}{h}\int_{x-h/2}^{x+h/2} f(s,t)\,ds, \quad S_t f(x,t) = \frac{1}{\tau}\int_{t-\tau/2}^{t+\tau/2} f(x,\eta)\,d\eta.$$

For the approximation of (4) we shall use a weighted finite difference scheme:

$$v_{t\overline{t}} = -L_h(\sigma\hat{v} + (1-2\sigma)v + \sigma\check{v}),$$

$$v(0,t) = v(1,t) = 0, \quad t \in \overline{\omega}_\tau,$$

$$v^0 = u_0 - (\tau/2)S_x^2 u_1, \quad v^1 = u_0 + (\tau/2)S_x^2 u_1.$$

Finally, we have

**Theorem 2.** *Suppose $a \in W_2^3$ and $a \geq a_0$ in $(0,1)$, $u$ is the weak solution of (4), $v$ is its discrete approximation and let one of the conditions in Lemma 4 is satisfied. Then for the error $z = u - v$ the following estimates hold:*

(*i*) $\|\overline{z}\|_{C_\tau(W^1_{2,h})} \leq C(h+\tau)^{2(s-1)/3}(\|u_0\|_{V^s} + \|u_1\|_{V^{s-1}})$,

*if* $u_0 \in V^s$, $u_1 \in V^{s-1}$, $1 \leq s \leq 4$;

(*ii*) $\|\overline{z}\|_{C_\tau(W^1_{2,h})} \leq C(h+\tau)^{2(s-1)/3}(\|u_0\|_{W_2^s} + \|u_1\|_{W_2^{s-1}})$,

*if* $u_0 \in \overset{o}{W}{}_2^s$, $u_1 \in \overset{o}{W}{}_2^{s-1}$, $1 \leq s \leq 4$, $s \neq$ *integer* $+ 1/2$.

Proof. Starting with (5) and applying identical techniques that we used for deriving the analogous esimates in [6], we prove this theorem. □

## 4. Numerical examples

A series of initial–boundary value problems of the form

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial}{\partial x}\Big(a(x)\,\frac{\partial u}{\partial x}\Big), \qquad (x,\,t) \in Q = (0,\,1) \times (0,\,T)\,.$$

$$u(0,\,t) = u(1,\,t) = 0\,, \qquad u(x,\,0) = u_0(x)\,, \qquad \frac{\partial u(x,\,0)}{\partial x} = 0$$

were solved numerically. Input data were taken in the form

$$u_0(x) = \psi_{s_0}(x)\,, \qquad a(x) = 1 + \psi_{s_1}(x)\,,$$

where

$$\psi_s(x) = \begin{cases} 0\,, & x \in (0,\,1/2) \\ 4^{s+2.5}\,(x-0.5)^{s-0.5}\,(1-x)^3\,, & x \in (1/2,\,1)\,. \end{cases}$$

We have $\psi_s(x) \in W_2^{s-\varepsilon}(0,\,1)$ for all $\varepsilon > 0$.

The problem was solved for various values of parameters $s_0$ and $s_1$, using symmetric difference scheme with the weight $\sigma = 1/4$, on the sequence of refined meshes. The rate of convergence was determined numerically, using Runge's rule.

Obtained convergence rates for $T = 1/2$, $s_1 = 3$ and $1 \leq s_0 \leq 4$ are presented in the following table.

| Mesh | $s_0$ | | | | | | |
|---|---|---|---|---|---|---|---|
| | 4.0 | 3.5 | 3.0 | 2.5 | 2.0 | 1.5 | 1.0 |
| $16 \times 16$ | -0.163 | -0.221 | -0.226 | -0.211 | -0.335 | -0.660 | -1.375 |
| $32 \times 32$ | 0.510 | 0.714 | 0.798 | 0.877 | 0.934 | 0.341 | -0.165 |
| $64 \times 64$ | 1.481 | 1.553 | 1.674 | 1.542 | 0.860 | 0.492 | 0.028 |
| $128 \times 128$ | 1.861 | 1.831 | 1.640 | 1.178 | 0.755 | 0.244 | -0.129 |
| $256 \times 256$ | 1.951 | 1.890 | 1.589 | 0.958 | 0.423 | 0.148 | -0.188 |
| $512 \times 512$ | 1.994 | 1.889 | 1.434 | 1.004 | 0.662 | 0.307 | -0.097 |
| $\frac{2}{3}(s_0 - 1)$ | 2.000 | 1.667 | 1.333 | 1.000 | 0.667 | 0.333 | 0.000 |

We note that the numerical convergence rates weakly depend on the smoothness of the coefficient $a(x)$. The obtained convergence rates for $T = 1/2$, $s_0 = 4$ and $0.5 \leq s_1 \leq 3$ are presented in the following table.

| Mesh | $s_1$ | | | | | |
|---|---|---|---|---|---|---|
| | 3.0 | 2.5 | 2.0 | 1.5 | 1.0 | 0.5 |
| $16 \times 16$ | -0.163 | -0.254 | -0.286 | -0.142 | -0.146 | -0.094 |
| $32 \times 32$ | 0.510 | 0.619 | 0.653 | 0.503 | 0.351 | 0.134 |
| $64 \times 64$ | 1.481 | 1.466 | 1.375 | 1.141 | 0.917 | 1.103 |
| $128 \times 128$ | 1.861 | 1.869 | 1.831 | 1.706 | 1.410 | 1.061 |
| $256 \times 256$ | 1.951 | 1.960 | 1.965 | 1.915 | 1.679 | 1.138 |
| $512 \times 512$ | 1.994 | 2.006 | 2.009 | 2.002 | 1.837 | 1.398 |

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