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THE INJECTIVE HULL AND THE *bc*-HULL OF A TOPOLOGICAL SPACE

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Abstract

A close connection between the notion of the *bc*-hull and the notion of the injective hull (cf. the definitions below) of a topological T_0-space is established.

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Bounded complete domains (shortly, *bc*-domains) [1] or, which is the same, complete A_0-spaces [2] form a subcartesian closed full subcategory of the category TOP_0 of topological T_0-spaces. This subcategory is important for denotational semantics.

The author [3] introduced the notion of the *bc*-hull of a topological space X as follows. A homeomorphic embedding $\lambda : X \to B$ of a space X in a *bc*-domain B is called the *bc-hull* of X if the following conditions are satisfied:

(1) *Universality.* For any continuous mapping $f : X \to B'$ from X to a *bc*-domain B' there exists a continuous mapping $f^* : B \to B'$ such that $f^*\lambda = f$.

(2) *Minimality.* If $f : B \to B$ is a continuous mapping and $f\lambda = \lambda$, then $f = id_B$.

In [3], the existence of the *bc*-hull of an α-space (cf. the definition below) is established.

With each topological (T_0-)space X we associate binary relations $\preceq_X$ and $\prec_X$ on X defined as follows. For $\xi_0, \xi_1 \in X$ we set

$$\xi_0 \preceq_X \xi_1 \;\Leftrightarrow\; \text{for any open subset } V \subseteq X \text{ from } \xi_0 \in V \text{ it follows that } \xi_1 \in V (\Leftrightarrow \xi_0 \in cl\{\xi_1\}, \text{ where } cl \text{ denotes the closure});$$
$$\xi_0 \prec_X \xi_1 \;\Leftrightarrow\; \xi_1 \in int\{\xi \mid \xi_0 \preceq_X \xi_1\}, \text{ where } int \text{ denotes the interior.}$$

The relation $\preceq_X$ is a partial order on X and the relation $\prec_X$ is transitive. The relations $\preceq_X$ and $\prec_X$ are connected as follows:

$$\xi_0 \prec_X \xi_1 \;\Rightarrow\; \xi_0 \preceq_X \xi_1,$$
$$\xi_0 \preceq_X \xi_0' \prec_X \xi_1' \preceq_X \xi_1 \;\Rightarrow\; \xi_0 \prec_X \xi_1.$$

A space X is called an *α-space* if for any open subset $V \subseteq X$ and point $\xi \in V$ there exists a point $\xi' \in V$ such that $\xi' \prec_X \xi$.

The following characterization of *bc*-domains (cf. [2, Proposition 2, §4]) is essential for the further considerations.

Proposition 1. *For a topological T_0-space X the following conditions are equivalent:*

(1) *X is a bc-domain,*

(2) *for any topological space Y, everywhere dense subspace Y_0, and continuous mapping $f_0 : Y_0 \to X$ there exists a continuous mapping $f : Y \to X$ such that $f_0 = f|_{Y_0}$.*

Remark 2. The set of extensions f of f_0 in condition (2) has the largest element, i.e., there exists a continuous mapping $f^* : Y \to X$ such that $f^*|_{Y_0} = f_0$ and for any continuous mapping $f : Y \to X$ such that $f|_{Y_0} = f_0$ and any $\eta \in Y$ we have $f(\eta) \preceq_X f^*(\eta)$.

Let X be a subspace of Y. The space Y is called an *essential extension* of X if for any continuous mapping $f : Y \to Z$ from the fact that $f|_X$ is a homeomorphic embedding of X in Z it follows that f is a homeomorphic embedding of Y in Z.

Proposition 3. *If $Y \supseteq X$ is the bc-hull of X, where λ is the identity mapping id_X, then Y is an essential extension of X.*

Lemma 4. *The space X is everywhere dense in Y, i.e., $clX = Y$.*

Proof. By [2, Proposition 5, §3], $Y_0 = clX \subseteq Y$ is a complete A_0-space, i.e., Y_0 is a bc-domain. The space Y is the bc-hull of X. By the universality condition, there exists a continuous mapping $f : Y \to Y_0$ such that $f|_X = id_X$. By the minimality condition, $f = id_Y$, i.e., $Y = Y_0 = clX$. □

Proof of Proposition 3. Let $f : Y \to Z$ be a continuous mapping such that $f|_X$ is a homeomorphic embedding of X in Z. Since any (T_0-)space is homeomorphically embedded in a bc-domain, without loss of generality, we assume that Z is a bc-domain. Let Z_0 be the closure of $f(X)$ in Z. By Lemma 4, we have $f(Y) \subseteq Z_0$. Indeed, if $\eta \in Y$ is an element such that $f(\eta) \in Z \setminus Z_0$, then $\eta \in f^{-1}(Z \setminus Z_0)$. Since $V = f^{-1}(Z \setminus Z_0)$ is a nonempty open subset, $V \cap X = \emptyset$, which is impossible. The space Z_0 is a bc-domain and $X_0 = f(X)$ is everywhere dense in Z_0. Let $g_0 : X_0 \to X \subseteq Y$ be a homeomorphism such that $fg_0 = id_{X_0}$ and $g_0(f|_X) = id_X$. By Proposition 1, there exists a continuous mapping $g : Z_0 \to Y$ extending g_0. The continuous mapping $gf : Y \to Y$ is such that $(gf)|_X = g_0(f|_X) = id_X$. Hence $gf = id_Y$ and f is a homeomorphic embedding of Y in $Z_0 \subseteq Z$. □

Remark 5. Proposition 3 gives the positive answer to Question 1 in [3]. An answer to Question 2 in [3] is also positive. Indeed, by the construction of the bc-hull $H_{bc}(X)$ of an α-space X, there exists a bc-domain B such that $X \subseteq H_{bc}(X) \subseteq B$ and X is a smooth subspace of B. Therefore, X is a smooth subspace of any intermediate space.

As is shown in [4], for any T_0-space X there exists "the largest" essential extension λX. If λX is an injective space, then λX is called the *injective hull* of X. It is convenient to use the following obvious characterization of the injective hull:

A T_0-space Y including X as a subspace is the injective hull of X if and only if Y is injective and is an essential extension of X.

The following theorem establishes a close connection between the notion of the injective hull and the notion of the bc-hull.

Theorem 6. *A topological space X possesses the bc-hull if and only if X possesses the injective hull.*

Proof. Let $Y \supseteq X$ be the injective hull of X, and let Y_0 be the closure of X in Y. By [2, Proposition 5, §3], Y_0 is a *bc*-domain. Since X is everywhere dense in Y_0, the space Y_0 satisfies the universality condition in view of Proposition 1. The minimality condition for Y_0 holds in view of Proposition 3 and the following lemma.

Lemma 7. *Let Z be an essential extension of X, $X \subseteq Z$. If f is a continuous mapping from Z to Z such that $f|_X = id_X$, then $f = id_Z$.*

Proof. Let $Z_0 \supseteq Z$ be the largest essential extension of Z. Then Z_0 is the largest essential extension of X. The space Z_0 is also the largest essential extension of $f(Z)$, $X \subseteq f(Z) \subseteq Z \subseteq Z_0$. Since Z is an essential extension of X and $f|_X = id_X$, we conclude that f is a homeomorphism from X to $f(X)$. Since the largest essential extension is unique, there exists a homeomorphism g from Z_0 onto Z_0 such that $g|_Z = f$ (consequently, $g|_X = id_X$). If $f \neq id_Z$, then $g \neq id_{Z_0}$. Thus, it suffices to prove the lemma under the assumption that Z is a maximal essential extension of X. As is noticed in [5], for any point $\zeta \in Z$ we have $\zeta = \sup_Z X_\zeta$, where $X_\zeta = \{\xi \mid \xi \in X, \xi \preceq_Z \zeta\}$. Since f is monotone, $f(\zeta) \succeq \sup f(X_\zeta) = \sup X_\zeta = \zeta$. The mapping f^{-1} is also a homeomorphism from Z such that $f^{-1}|_X = id_X$. Hence $f^{-1}(\zeta) \succeq \zeta$ for any point $\zeta \in Z$. Thus, if $f \neq id_Z$, then there exists a point ζ such that $f(\zeta) > \zeta$. We have $\zeta > f^{-1}(\zeta) \succeq \zeta$, which is a contradiction. Thus, $f = id_Z$. □

Corollary 8. *Let $X \subseteq Y_0$ and $X \subseteq Y_1$ be essential extensions of X. Then there exists at most one continuous mapping $f : Y_0 \to Y_1$ such that $f|_X = id_X$.*

Let $Y \supseteq X$ be the *bc*-hull of X. By Proposition 3, Y is an essential extension of X. If Y is an injective space, then Y is the injective hull of X. Assume that the space Y is not injective. Consider the extension $Y^\top$ obtained from Y by adding the new isolated largest element $\top$.

Lemma 9. *The space $Y^\top$ is injective. This space is an essential extension of Y.*

Proof. Let X_0 be a subspace of X_1, $g_0 : X_0 \to Y^\top$ a continuous mapping, and X_2 the closure of $g_0^{-1}(Y)$ in X_1. Then $X_1 \setminus X_2$ is an open subset, $g_0|_{g_0^{-1}(Y)}$

is a continuous mapping from $g_0^{-1}(Y)$ to Y, and $g_0^{-1}(Y)$ is everywhere dense in X_1. By Proposition 1, there exists a continuous mapping $g_2 : X_2 \to Y$ extending $g_0|_{g_0^{-1}(Y)}$. We define a mapping g_1 from X_1 to $Y^\top$ by setting $g_1(\xi) = g_2(\xi)$ for $\xi \in X_2$ and $g_1(\xi) = \top$ for $\xi \in X_1 \setminus X_2$. It is easy to verify that g_1 is continuous and $g_1|_{X_0} = g_0$. Thus, $Y^\top$ is an injective space.

Since the space Y is not injective, there is no largest element in Y. Therefore, there exist inconsistent elements η_0 and η_1, i.e., there exists no element $\eta \in Y$ such that $\eta_0 \preceq_Y \eta$ and $\eta_1 \preceq_Y \eta$. We show that this implies the existence of open subsets U_0 and U_1 such that $\eta_0 \in U_0$, $\eta_1 \in U_1$, and $U_0 \cap U_1 = \emptyset$. Indeed, since Y is an α-space, we have $\eta_0 = \sup\{\eta_0' \mid \eta_0' \prec_Y \eta_0\}$ and $\eta_1 = \sup\{\eta_1' \mid \eta_1' \prec_Y \eta_1\}$. If every pair $\eta_0' \prec_Y \eta_0$, $\eta_1' \prec_Y \eta_1$ is consistent, then the family $\{\eta_0' \vee \eta_1' \mid \eta_0' \prec_Y \eta_0, \eta_1' \prec_Y \eta_1\}$ is directed. But the existence of $\eta = \sup\{\eta_0' \vee \eta_1' \mid \eta_0' \prec_Y \eta_0, \eta_1' \prec_Y \eta_1\}$ contradicts the fact that η_0 and η_1 are inconsistent. Let $\eta_0'(\prec_Y \eta_0)$ and $\eta_1'(\prec_Y \eta_1)$ be inconsistent. Then $U_0 = int\{\eta \mid \eta_0' \prec_Y \eta\}$ and $U_1 = int\{\eta \mid \eta_1' \prec_Y \eta\}$ satisfy the required conditions.

Now, we will prove that $Y^\top$ is an essential extension of X. Let $f : Y^\top \to Z$ be a continuous mapping such that $f|_Y$ is a homeomorphic embedding of Y in Z. Since $f(Y)$ is homeomorphic to Y, there is no largest element in $f(Y)$. Hence $f(\top) \notin f(Y)$. Thus, f is a one-to-one mapping. It suffices to prove that $f(\top)$ is an isolated point of $f(Y^\top)$. Since $f(U_0)$ and $f(U_1)$ are open subsets of $f(Y)$, there exist open subsets V_0 and V_1 of Z such that $V_0 \cap f(Y) = f(U_0)$ and $V_1 \cap f(Y) = f(U_1)$. We have $(V_0 \cap V_1) \cap f(Y) = f(U_0) \cap f(U_1) \cap f(Y) = f(U_0 \cap U_1) = \emptyset$. Since $f(\top) \in V_0 \cap V_1$, we conclude that $f(\top)$ is an isolated point of $f(Y^\top)$. The lemma is proved. □

Lemma 9 completes the proof of the theorem. □

Thus, a T_0-space X possesses the bc-hull if and only if X possesses the injective hull; moreover, the injective hull coincides with the bc-hull or is obtained from the bc-hull by adding the new isolated largest element.

In [5], the following characterization of spaces that possess the injective hull is obtained:

A T_0-space X possesses the injective hull if and only if for any open subset $U \subseteq X$ and point $\xi \in U$ there exists a finite set $\xi_0, \ldots, \xi_n$ of points in X and a family of open sets $U_0, \ldots, U_n$ such that $\xi_i \prec_X \xi$, $\xi_i \in U_i$ for all $i \preceq n$, and $\bigcap_{i \preceq n} U_i \subseteq U$.

In conclusion, we present a simple example of a space that satisfies the conditions of the above characterization and is not an α-space.

Let S be an infinite set, and let $P(S)$ be the family of all subsets of S endowed with the Scott topology. Consider the subspace

$$X = \{S_0 \mid S_0 \subseteq S,\ S_0 \text{ is infinite or it contains at most one element}\}$$

of $P(S)$. The injective hull of X is $P(S)$, whereas X is not an α-space.

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