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A SURVEY OF ALGEBRA OF TOURNAMENTS

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Abstract

The present paper surveys the history of algebraic represetations of complete directed graphs, known in graph theory as tournaments, or equivalently, relational structures with a trichotomous binary relation. Essentially, two kinds of algebraizations of tournaments were studied in the literature: algebras with one binary operation (called groupoids of tournaments) and algebras with two binary operations (weakly associative lattices). Different properties of these algebras have been explored by many authors. Also, varieties generated by algebraic versions of tournaments attracted a certain interest in universal algebra, primarily such questions as equations satisfied by tournaments and finite base problems, congruences, simple and subdirectly irreducible algebras, homomorphisms and, especially, automorphism groups, decidability problems, etc. A selection of results concerning these problems are also presented in the current survey.

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1. Introduction

As an object of the graph theory, a *tournament* is a complete directed graph, i.e. a digraph in which every pair of different vertices is connected by exactly one directed edge. Certainly, it will make no major difference (moreover, it will make the algebraic treatment easier) if we consider tournaments to have loops, that is, edges leading from a vertex to itself. To be more precise, we define tournaments as relational structures $\mathbf{T} = \langle T, \to \rangle$, where T is a nonempty set and $\to$ is a relexive trichotomous binary relation, which means that for each $x, y \in T$ such that $x \neq y$, exactly one of the assertions $x \to y$, $y \to x$ is true.

The fact that $x \to y$ usually reads as "x beats y" in accordance with the terminology of sports, which, among others, gave motivation for the concept of tournament. Sometimes, the universe T of a tournament is required to be finite (in order to meet intuitive expectations), but some authors do not follow this restriction, considering some algebraic and model-theoretic properties of infinite graphs, such as in [6].

In 1965, Z. Hedrlín observed that any tournament $\mathbf{T}$ can be transformed into a groupoid by defining the multiplication operation for all $x, y \in T$ by $xy = yx = x$ if and only if $x \to y$. In particular, this yields $x^2 = x$ for all $x \in T$. It takes only a short reflection to see that this is, in fact, a bijective correspondence between the class of all tournaments and all idempotent and commutative groupoids satisfying $xy \in \{x, y\}$ for all x and y, the latter being special cases of what Ježek and Kepka in [15] called *quasitrivial algebras*. The groupoids obtained in the way just described are called *groupoids of tournaments*, or simply *tournaments* for short, whenever it makes no confusion. In addition, it should be mentioned that credits for the above ideas belong to Chvátal as well, whose paper [3] was also dealing with the subject. Since then, a systematic study of algebraic properties of directed graphs was carried out, especially in the early seventies by the graph theory group at the Charles Univesity, Prague, led at the time by Hedrlín, Nešetřil, Sabidussi, Čulik and Pultr. Their work yielded successful applications in various branches of mathematics (see, for example, [13]).

An alternative approach to algebraization of tournaments was almost simultaneously proposed by Fried [8] and Skala [27]. Namely, while the result of the operation in groupoids of tournaments applied to x and y is the "winner" of the "match" of x vs. y, in this case we introduce two binary operations which will denote both the "winner" and the "looser". More formally, we represent a tournament $\mathbf{T} = \langle T, \rightarrow \rangle$ by an algebra $\mathbf{A_T} = \langle T, \vee, \wedge \rangle$ such that if $x \rightarrow y$, then $x \vee y = y \vee x = x$ and $x \wedge y = y \wedge x = y$. Both of these operations are idempotent and commutative, but not associative except for transitive tournaments. However, a weak form of the associative law is satisfied, as well as the absorption law, thus the described algebras belong to a class named by Fried and Grätzer *weakly associative lattices*. We will not be dealing with this kind of algebras associated to tournaments until Section 7, where the results concerning tournaments as weakly associative lattices will be presented.

The rest of the paper is organized as follows. Section 2 starts with definitions of basic notions used throughout the survey and gives a brief account on some results of a general character, while its main part is devoted to tournaments having no nontrivial congruences, i.e. tournaments which are simple algebras. Their properties are also brought to connection with homomorphisms of tournaments and Hamiltonian cycles. Section 3 deals with charaterizations of congruence lattices of tournaments. In Section 4, the focus is on groups of automorphisms of tournaments. Questions that are motivated by equations satisfied by (finite) tournaments (more generally, their logical aspect) are the subject of Sections 5 and 6. Most of the results of these sections are, in fact, the contents of two recent papers by Crvenković, Dolinka and Marković [4] and Ježek, Marković, Maróti and McKenzie [16]. Finally, as already mentioned, in Section 7 we survey contributions that are related to represetations of tournaments with two binary operations. The latter is, in fact, a story of the variety generated by *the triangle*: the cycle with three elements. Also, we give, as an appendix, the summary list of open problems raised through the exposition in these seven sections.

2. Simple tournaments

Let $\mathbf{T} = \langle T, \rightarrow^{\mathbf{T}} \rangle$ and $\mathbf{S} = \langle S, \rightarrow^{\mathbf{S}} \rangle$ be two arbitrary tournaments. A mapping $\varphi : T \mapsto S$ is called *tournament homomorphism* if for all $x, y \in T$, $x \rightarrow^{\mathbf{T}} y$ implies $\varphi(x) \rightarrow^{\mathbf{S}} \varphi(y)$. Some special kinds of homomorphisms, such as *endomorphisms*, *automorphisms* and *isomorphisms* of tournaments are defined in a usual way. The set of all homomorphisms from $\mathbf{T}$ to $\mathbf{S}$ is denoted by $Hom(\mathbf{T}, \mathbf{S})$. Of course, this set is never empty, since every constant mapping $T \mapsto S$ is a homomorphism of tournaments. If $\mathbf{T} = \mathbf{S}$, we obtain the monoid of endomorphisms of **T** under composition of mappings, denoted by **End(T)**. The group part of this monoid consists precisely of all bijective endomorphisms, that is, automorphisms of **T** and this group, often called *the symmetry group of* **T**, we denote by **Aut(T)**.

If a tournament **S** is a subtournament of **T**, we write $\mathbf{S} \leq \mathbf{T}$. Clearly, any nonempty subset M of T induces a subtournament with M as its universe. This subtournament is usually denoted by $\mathbf{T}|_M$. Moreover, if we have a surjective homomorphism $\varphi : \mathbf{T} \mapsto \mathbf{S}$, then **S** can be regarded as a subtournament of **T** by considering a set of representatives of the family $\{\varphi^{-1}(s) : s \in S\}$. Hence, every epimorphism of tournaments is a retraction, see [14].

The ambigous use of the notion of the homomorphism of tournaments in the sense of both graph theory and algebra is justified by the following observation.

Proposition 2.1. [24] *The mapping* $\varphi : T \mapsto S$ *is a homomorphism of tournaments* **T** *and* **S** *considered as graphs if and only if it is a homomorphism of the corresonding groupoids of tournaments.* □

Naturally, this will yield the equivalence of the concept of congruence of tournaments respectively as graphs and groupoids. Namely, an equivalence relation θ on the set T is a *congruence* of the tournament **T** if for all $x, x', y \in T$, $\langle x, x' \rangle \in \theta$ and $x \to y$ implies $x' \to y$. That the definition just given exactly matches the notion of the congruence of a groupoid of tournament is shown by the proposition below.

Proposition 2.2. [24] *Let* **T** *be a tournament, M and N congruence classes of* **T** *for some congruence θ and $x, x' \in M$, $y, y' \in N$. Then $x \to y$ if and only if $x' \to y'$.* □

From the universal-algebraic standpoint, the construction of direct product of tournaments would be also interesting. However, it is easy to show that a direct product of groupoids representing tournaments (even of a two finite ones) need not be a groupoid of a tournament (a quite precise argument is presented, for example, in Subotić [28]). Therefore, the class of (finite) tournaments is *not* a variety. Throughout the paper, the variety of groupoids generated by all tournaments will be denoted by $\mathcal{T}$. It is important to note that the restriction of the notion of a tournament only to *finite* structures does not affect $\mathcal{T}$ at all. The proof of the following fact is the first appearance of equations satisfied by tournaments in any crucial role here. That role will be developed in more detail in Section 6.

Proposition 2.3. *The variety $\mathcal{T}$ is generated by all finite tournaments.*

Proof. By Birkhoff's Theorem on the equivalence of varieties and equational classes, it is enough to prove that an identity satisfied by all finite tournaments is satisfied by all tournaments without any restriction on their cardinality. So let $p(x_1, \ldots, x_n) \approx q(x_1, \ldots, x_n)$ be such an identity and let $\mathbf{T}$ be any tournament. If $a_1, \ldots, a_n \in T$ are arbitrary, then for $M = \{a_1, \ldots, a_n\}$, $\mathbf{T}|_M$ is a finite subtournament of $\mathbf{T}$, thus it satisfies the given equation. This, in particular, means that

$$p^{\mathbf{T}|_M}(a_1, \ldots, a_n) = q^{\mathbf{T}|_M}(a_1, \ldots, a_n),$$

hence

$$p^{\mathbf{T}}(a_1, \ldots, a_n) = q^{\mathbf{T}}(a_1, \ldots, a_n),$$

proving that $\mathbf{T} \models p \approx q$. □

From the above proof, one can deduce

Corollary 2.4. $\mathcal{T} \models p(x_1, \ldots, x_n) \approx q(x_1, \ldots, x_n)$ *if and only if the identity $p \approx q$ is satisfied by all n-element tournaments.* □

For the reasons explained above, from here on the word *tournament* will cover only finite graphs and algebras. For tournaments having arbitrary cardinality, we shall use the expression *generalized tournaments.*

Many examples from universal algebra illustrate the fact that understanding the nature of simple and subdirectly irreducible algebras is of a great importance in the study of varieties which they belong to. This justifies the significance of the research on the structure and properties of *simple tournaments*, having the diagonal and the full relation as their only congruences, i.e. the simplest possible congruence lattices. The main charaterization theorem of simple tournaments is as follows.

Theorem 2.5. [24] *Let $\mathbf{T}$ be a tournament. Then the following statements are equivalent:*

(1) $\mathbf{T}$ *is simple.*

(2) If a tournament $\mathbf{S}$ *cannot be embedded into* $\mathbf{T}$*, then* $|Hom(\mathbf{T},\mathbf{S})| = |S|$.

(3) If $|T| \geq 3$*, then for any proper subset* $M \subset T$ *with* $|M| \geq 2$ *there exists* $x \in T \setminus M$ *such that* $x \to y$ *and* $z \to x$ *for some* $y, z \in M$. □

Erdős, Fried, Hajnal and Milner in [6] define a *convex subset* of a tournament $\mathbf{T}$ as a subset $M \subseteq T$ such that for all $x \in T \setminus M$ either $x \to y$ for all $y \in M$ or for all $y \in M$, $y \to x$. Now if we define $C(\mathbf{T})$ as in [6] to be the set of all convex subsets M of $\mathbf{T}$ with $M \neq T$ and $|M| \geq 2$, the condition (3) from the above theorem is obviously equivalent to $C(\mathbf{T}) = \emptyset$. Concerning this family of convex subsets, we have two interesting results.

Theorem 2.6. [6] *Suppose* $\mathbf{T} = \langle T, \to \rangle$ *is not a transitive generalized tournament, that is,* $\to$ *is not a linear order on* T*. Then there exists a linear order* $\hookrightarrow$ *on* T *such that* $C(\mathbf{T}) \subset C(\mathbf{T}')$ *for* $\mathbf{T}' = \langle T, \hookrightarrow \rangle$. □

Theorem 2.7. [6] *For any generalized tournament* $\mathbf{T}$*,* $C(\mathbf{T})$ *has the Bernstein property, i.e. there is a set* B *such that* $B \cap M \neq \emptyset$ *and* $B \not\subseteq M$ *for all* $M \in C(\mathbf{T})$. □

Equipped with these two, Erdős, Fried, Hajnal and Milner prove

Theorem 2.8. [6] *Any generalized tournament with at least three elements has a two-point simple extension.* □

In general, this is not true for one-point extensions. But later it turned out that the only exceptions are the three-element tournaments and transitive tournaments with an odd number of vertices. This was proved first for tournaments by Moon [23] and extended for generalized tournaments by Erdős, Hajnal and Milner in [7].

Note that the previous theorem implies that there exist simple generalized tournaments of any cardinality except 4. It can be effectivelly checked that there is no 4-element simple tournament.

The connection between convex subsets and simple tournaments becomes even more transparent after the following

Lemma 2.9. *Let* **T** *be a tournament and* $M \in C(\mathbf{T})$ *then* $\theta_M = M^2 \cup \Delta_T$ *is a congruence of* **T**. □

But the really interesting thing is that all principal congruences of tournaments are of the above form.

Proposition 2.10. *If* θ *is a principal congruence of a nontrivial tournament* **T**, *then there exists* $M \subseteq T$ *such that* $\theta = \theta_M$. □

The latter proposition now allows to easily characterize atoms of congruence lattices of tournaments.

Proposition 2.11. [24] *Let* **T** *be a tournament. The atoms of* **Con(T)** *are exactly the congruences of* **T** *of the form* θ_M, *where* $M \in C(\mathbf{T})$ *and* $\mathbf{T}|_M$ *is nontrivial simple subtournament of* **T**. □

It is clear that the factor-tournament $\mathbf{T}/\theta$ is a simple tournament, whenever θ is a maximal congruence of **T**, i.e. a coatom in **Con(T)**. Thus any tournament has a nontrivial simple homomorphic image. On the other hand, the above result shows that simple tournaments turn out to be related with minimal congruences on tournaments as well.

Recall that a *Hamiltonian tournament* is a tournament with a Hamiltonian cycle, that is, a cycle containing all elements of the tournament. The importance of Hamiltonian tournaments from the algebraic pont of view lies in two facts. First, it is well-known that every tournament has a Hamiltonian path (by Rédei's Theorem) and so every tournament has a one-point Hamiltonian extension. Therefore, all Hamiltonian tournaments generate the variety $\mathcal{T}$. The other reason why Hamiltonian tournaments are interesting in the current context is the following

Theorem 2.12. [28] *All simple tournaments with at least three elements are Hamiltonian.* □

The converse of this theorem is false: a five-element counterexample can be found in Subotić [28]. Nevertheless, the previous conclusion can now be strengthened in the sense that $\mathcal{T}$ is generated by simple Hamiltonian tournaments only. Another nice feature of Hamiltonian tournaments is given below.

Proposition 2.13. [24] *If* $\mathbf{T}$ *is a Hamiltonian tournament, then* $\mathbf{Con}(\mathbf{T})$ *has a unique coatom.* □

Using this proposition and

Proposition 2.14. [24] *A tournament* $\mathbf{T}$ *is Hamiltonian if and only if it has a Hamiltonian homomorphic image.* □

we can characterize Hamiltonian tournaments by purely algebraic terms.

Theorem 2.15. *A tournament* $\mathbf{T}$ *is Hamiltonian if and only if* $\mathbf{Con}(\mathbf{T})$ *has a unique coatom, which is a congruence of index at least 3.*

Proof. ($\Rightarrow$) By Proposition 2.13, $\mathbf{T}$ has a unique maximal congruence θ_0. By Proposition 2.2, the factor-tournament $\mathbf{T}/\theta_0$ is Hamiltonian, so it must have at least three elements.

($\Leftarrow$) Let θ_0 be (the unique) maximal congruence of $\mathbf{T}$ of index ≥ 3. Then $\mathbf{T}/\theta_0$ is a simple tournament having at least three elements, which is Hamiltonian by Theorem 2.12. The conclusion now follows by Proposition 2.14, since $\mathbf{T}/\theta_0$ is a homomorphic image of $\mathbf{T}$. □

However, one question concerning Hamiltonian tournaments, having a rather model-theoretical character, remains open. Roughly speaking, we are interested whether the property of tournaments of being Hamiltonian can be expressed by a first-order formula, either in relational, or in algebraic settings. It is not hard to see that for each n, the property "if a tournament has n elements, then it is Hamiltonian" can be first-order codified by a formula. If we let n to run over the set of natural numbers, the obtained set of formulæ will certainly "filter" the Hamiltonian tournaments as its finite models. But we are looking for *just one* formula Φ capturing the Hamiltonian property.

Problem 1. *Is there a first-order formula* Φ *such that a (finite) tournament* $\mathbf{T}$ *is Hamiltonian if and only if* $\mathbf{T} \models \Phi$*?*

Such formula (if it exists at all) clearly cannot be universal, since universal formulas are preserved under taking submodels (and it is quite easy to construct a Hamiltonian tournament having a non-Hamiltonian subtournament). The posed problem asks, in other words, whether there exists a finitely axiomatizable class of tournaments having Hamiltonian tournaments as the set of its finite members. We conjecture the negative answer. To prove it, it would be sufficient, for example, to construct an ultraproduct of non-Hamiltonian tournaments which is a Hamiltonian tournament. Here we quote a related result.

Theorem 2.16. [6] *The class of all simple generalized tournaments is not closed for ultraproducts.* □

We end this section by mentioning two more results on simple tournaments. A first one is a deep result of Müller, Nešetřil and Pelant [24], which is essentially of a graph-theoretical nature. However, its proof heavily employs the notion of tournament congruences and so it stands as a beautiful example of an application of algebra in graph theory.

Theorem 2.17. [24] *Let* $\mathbf{T} = \langle T, \rightarrow \rangle$ *be a Hamiltonian tournament such that* $|T| \neq 4$. *Then there exists a simple tournament* $\mathbf{T}' = \langle T, \hookrightarrow \rangle$ *which is degree-equivalent to* $\mathbf{T}$. □

Finally, one can ask about the number of simple tournaments with n elements. Is a simple tournament a rare phenomenon in the multitude of all possible ones? Not at all! Namely, some calculations that are concerned with maximal congruences of tournaments show that *almost all* tournaments are simple.

Theorem 2.18. [6, 24] *Let* S_n *and* T_n *denote the number of all simple tournaments and all tournaments with* n *elements, respectively. Then*

$$\lim_{n \to \infty} \frac{S_n}{T_n} = 1. \quad \square$$

In [6], the above result was obtained by calculations on convex subsets of an arbitrary tournament.

3. Congruence lattices of tournaments

In the previous section we already exhibited several properties of congruences of tournaments and their congruence lattices. Here we give a short account on the work done by Müller, Nešetřil and Pelant in [24]. They gave a characterization theorem which describes finite lattices representable as congruence lattices of tournaments. This characterization relies on the description of the partially ordered set formed by join-irreducible congruences. If $\mathbf{T}$ is a tournament, the set of join-irreducible elements of $\mathbf{Con}(\mathbf{T})$ is denoted by $I_\vee(\mathbf{T})$.

One additional notion we shall need is the one of *cyclic decomposition* of a tournament. Its definition is based on the following

Proposition 3.1. *Let* $\mathbf{T}$ *be a tournament and let* $\mathbf{T}_n$ *denote the n-element chain. Let* φ *be a epimorphism from* $\mathbf{T}$ *onto* $\mathbf{T}_n$ *such that there is no epimorphism from* $\mathbf{T}$ *onto* $\mathbf{T}_m$ *for* $m > n$. *Then* φ *is determined uniquely and* $\mathbf{T}|_{\varphi^{-1}(x)}$ *is a Hamiltonian tournament for every* $x \in T$. □

Now the partition $\{\varphi^{-1}(x) : x \in T\}$ from the above proposition is called the cyclic decomposition of $\mathbf{T}$. Note that the cyclic decomposition is a congruence of $\mathbf{T}$ as being the kernel of the homomorphism φ. Its importance lies in the fact that it supplies a useful criterion whether a tournament congruence is join-irreducible.

Proposition 3.2. [24] *Let* $\mathbf{T}$ *be a tournament. Then* $\theta \in I_\vee(\mathbf{T})$ *if and only if* $\theta = \theta_M (= M^2 \cup \Delta_T)$ *for some* $M \subseteq T$, $|M| \geq 2$ *such that the cyclic decomposition of* $\mathbf{T}|_M$ *has at most two elements.* □

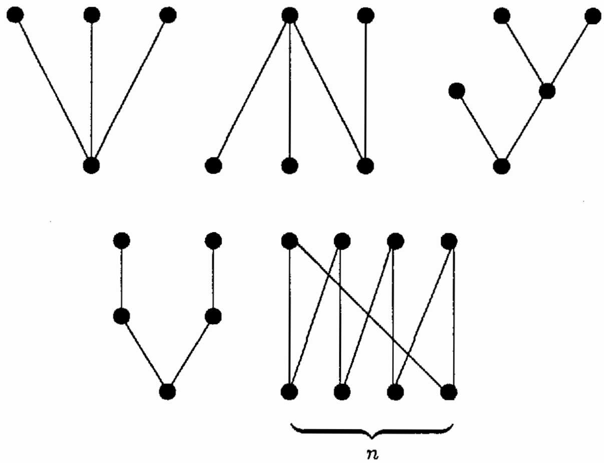
Hence, we can identify join-irreducibles in $\mathbf{Con}(\mathbf{T})$ with the corresponding nontrivial congruence classes. The following lemma is one of the key ingredients for the characterization of posets of join-irreducible tournament congruences, but it is also interesting itself.

Lemma 3.3. [24] *Let* $\theta_1, \theta_2, \theta_3$ *be three mutually distinct join-irreducible congruences of a tournament* $\mathbf{T}$ *and* M, N, P *the corresponding nontrivial congruence classes. If* $P \subseteq M$ *and* $P \subseteq N$ *(that is, if* $\theta_3 \leq \theta_1$ *and* $\theta_3 \leq \theta_2$*), then* $\mathbf{T}|_{M\cup N}$ *has the cyclic decomposition* C_1, C_2, C_3 *such that* $C_2 = P$, $C_1 \cup C_2 = M$ *and* $C_1 \cup C_3 = N$. □

Now we can state one of the three main results of this section.

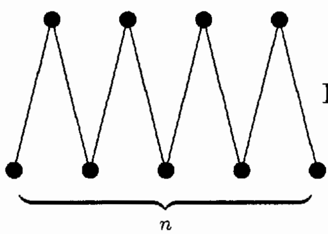
Theorem 3.4. [24] *Let* $\mathbf{P} = \langle P, \leq \rangle$ *be a finite partially orederd set. There exists a tournament* $\mathbf{T}$ *such that* $\mathbf{P} \cong \langle I_\vee(\mathbf{T}), \leq \rangle$ *if and only if the following two conditions hold:*

(1) The diagram of $\mathbf{P}$ *does not contain the following patterns:*



for all $n \geq 2$.

(2) The partial order induced on the set of maximal elements of $\mathbf{P}$ *and the elements they cover is for some* n *isomorphic either to the partial order* $\mathbf{P}_n$ *given below, or to the partial order which arises from it by deletion of some of its lower elements.*



$\mathbf{P}_n$

□

Further on, one could prove that if $\theta, \theta_1, \ldots, \theta_n$ are join-irreducible congruences of a tournament $\mathbf{T}$ such that $\theta \leq \theta_1 \vee \ldots \vee \theta_n$, then $\theta \leq \theta_i$ for some $1 \leq i \leq n$. By a well-known theorem of Birkhoff, this means that $\mathbf{Con}(\mathbf{T})$ is a distributive lattice. Since distributive lattices are completely determined by the poset of its join-irreducibles, one obtains the following theorem immediately.

Theorem 3.5. [24] *Let* $\mathbf{L}$ *be a finite lattice. There exists a tournament* $\mathbf{T}$ *such that* $\mathbf{L} \cong \mathbf{Con}(\mathbf{T})$ *if and only if* $\mathbf{L}$ *is distributive and the poset of its join-irreducibles satisfies the conditions (1) and (2) of Theorem 3.4.* □

Finite distributive lattices satisfying the conditions of the previous theorem will be called *admissible.*

However, although tournaments have distributive congruence lattices, the variety $\mathcal{T}$ they generate is *not* congruence-distributive. To prove this statement, once more we are going to make use of equations satisfied by tournaments.

Lemma 3.6. *Let* $t(x, y)$ *be a groupoid term in which both of the variables* x, y *appear. Then* $\mathcal{T} \models t(x, y) \approx xy$.

Proof. By induction on the complexity of the term $t(x, y)$. Recall that tournaments satisfy the commutative and the idempotent law, as well as

$$x(xy) \approx xy,$$

which can be easily checked.

If $t(x, y) = xy$ or $t(x, y) = yx$, there is nothing to prove. So let $t(x, y)$ be a term with $n + 1$ operation symbols and assume that the lemma holds for all terms having $\leq n$ operation symbols. By commutativity, it suffices to consider three cases:

(1) $t(x, y) = xp(x, y)$, where $p(x, y)$ contains both x and y. Then

$$t(x, y) \approx x(xy) \approx xy$$

holds in $\mathcal{T}$.

(2) $t(x, y) = yp(x, y)$, where $p(x, y)$ contains both x and y. In this case, $\mathcal{T}$ satisfies the identities

$$t(x, y) \approx y(xy) \approx y(yx) \approx yx \approx xy.$$

(3) Finally, if $t(x,y) = p(x,y)q(x,y)$ for some terms $p(x,y), q(x,y)$ containing both variables x, y, then

$$t(x,y) \approx (xy)^2 \approx xy$$

is satisfied in $\mathcal{T}$, thus the induction is completed. □

From here, using the well-known result of Freese and Nation on congruence lattices of semilattices, it is easy to derive

Proposition 3.7. [28] *The variety $\mathcal{T}$ satisfies only regular identities. Hence, it contains the variety $\mathcal{SL}$ of all semilattices and consequently, the class of congruence lattices of its members does not satisfy a nontrivial lattice identity.* □

Concluding this section, whose topic was highly related to simple tornaments and vice versa, we should mention one more unsolved problem. Namely, while the simple tournaments are relatively well explored, nothing is known about subdirectly irreducible tournaments. More generally, one can ask about subdirectly irreducible members of $\mathcal{T}$.

Problem 2. *Characterize all subdirectly irreducible members of the variety $\mathcal{T}$. Are they all tournaments?*

This problem is important for a better understanding of the structure of $\mathcal{T}$, because subdirectly irreducible algebras are building blocks of any variety.

4. Automorphisms of tournaments

Suppose that a finite group $\mathbf{G}$ is isomorphic to $\mathbf{Aut}(\mathbf{T})$ for some tournament $\mathbf{T}$. The first observation that can be made is that the order of $\mathbf{G}$ must be odd, because all its members have odd orders. Really, if we assume the contrary, i.e. that there exists an automorphism φ of $\mathbf{T}$ of order $2k$, then $\psi = \varphi^k$ is a nontrivial automorphism of order 2. If $x \in T$ is not a fixed point of ψ, then $\psi(x) = y$ implies $\psi(y) = x$ and vice versa. Since $x \neq y$, it has to be either $x \to y$, or $y \to x$. But the first case implies $\psi(x) \to \psi(y)$, that is, $y \to x$, which is a contradiction; the other case is handled analogously. Therefore, $|Aut(\mathbf{T})|$ cannot be an even number.

On the other hand, every group of odd order can be represented as an automorphism group of a tournament, see Moon [22]. Moreover, the following result says that we can even restrict ourselves only to simple tournaments.

Proposition 4.1. [24] *For any group* $\mathbf{G}$ *of odd order there exist a simple tournament* $\mathbf{T}$ *such that* $\mathbf{Aut}(\mathbf{T}) \cong \mathbf{G}$. □

This assertion is proved using the *Cayley technique*. Namely, given an odd group $\mathbf{G}$, $G = \{g_1, \ldots, g_{2n+1}\}$ one can fix a linear order $\leq$ of its elements (for example, $g_i \leq g_j$ if and only if $i \leq j$) and consider a minimal system of its generators, say $H = \{g_1, \ldots, g_k\}$. The central role in the proof is played by the model $\mathbf{R_G} = \langle G, \rho_0, \rho_1, \ldots, \rho_k \rangle$, where $\rho_i = \lambda_{g_i} \cup \Delta_G$ for $1 \leq i \leq k$ (λ_g is the left translation corresponding to $g \in G$) and

$$\rho_0 = \{\langle g, h\rangle : \ h^{-1}g < g^{-1}h\} \cup \Delta_G.$$

It can be shown that $\mathbf{Aut}(\mathbf{R_G}) \cong \mathbf{G}$ and that all endomorphisms of $\mathbf{R_G}$ which are not automorphisms are constant mappings.

It is worth noting that in the proof of the above proposition a special type of simple tournaments appear. These are *the rigid tournaments*, whose only endomorphisms are the constants and the identity mapping. See [3] for a detailed treatment of rigid tournaments. We shall return to rigid tournaments once more at a later stage of this section.

We now turn our attention to the problems of simultaneous realization of automorphism groups and congruence lattices of tournaments. The well-known result of Lampe shows that these two structures are independent for universal algebras in general. Müller, Nešetřil and Pelant prove that the same is true for tournaments. In fact, even the previous proposition is a result of this kind: one can always construct a tournament with a given odd group of automorphisms and trivial congruence lattice. Now the same conclusion holds if we require the automorphism group to be trivial and the congruence lattice to be arbitrary (but, of course, admissible).

Proposition 4.2. [24] *Let* $\mathbf{L}$ *be an admissible lattice. There exists an asymmetric tournament* $\mathbf{T}$ *(a tournament having the identity mapping as its only automorphism) such that* $\mathbf{Con}(\mathbf{T}) \cong \mathbf{L}$. □

Towards obtaining the general result, the following *doubling construction* was used. If $\mathbf{T}$ is a given tournament, make two disjoint copies of $\mathbf{T}$ over the sets of vertices $T \times \{0\}$ and $T \times \{1\}$, respectively. The arrows between the copies are defined so that $\langle x, 0\rangle \to \langle y, 1\rangle$ if $x, y \in T$, $x \neq y$ and $\langle x, 1\rangle \to \langle x, 0\rangle$ for all $x \in T$. The tournament which is the result of this construction we denote by $2\mathbf{T}$.

Proposition 4.3. [24] *Let* $\mathbf{G}$ *be an odd group. There exists a tournament* $\mathbf{T}$ *such that* $\mathbf{Aut}(\mathbf{T}) \cong \mathbf{Aut}(2\mathbf{T}) \cong \mathbf{G}$ *and that both* $\mathbf{T}$ *and* $2\mathbf{T}$ *are simple tournaments.* □

We arrived at the main result which deals with automorphisms of tournaments.

Theorem 4.4. [24] *Let* $\mathbf{L}$ *be an admissible lattice and* $\mathbf{G}$ *a group of odd order. There exists a tournament* $\mathbf{T}$ *such that* $\mathbf{Con}(\mathbf{T}) \cong \mathbf{L}$ *and* $\mathbf{Aut}(\mathbf{T}) \cong \mathbf{G}$. □

The above theorem can be strenghtened in such a way that the considered tournament has a specified subtournament. This shows that the congruence lattice and the automorphism group of a tournament do not force any forbidden parts whatsoever.

Theorem 4.5. [24] *Let* $\mathbf{L}$ *be an admissible lattice,* $\mathbf{G}$ *a group of odd order and* $\mathbf{S}$ *a tournament. There exists a tournament* $\mathbf{T}$ *such that* $\mathbf{S} \leq \mathbf{T}$, $\mathbf{Con}(\mathbf{T}) \cong \mathbf{L}$ *and* $\mathbf{Aut}(\mathbf{T}) \cong \mathbf{G}$. □

The presented results fully answer the question of representability of finite groups by automorphism groups of tournaments. However, the analogous problem for endomorphism monoids seems to be much harder.

Problem 3. *Characterize all finite monoids isomorphic to* $\mathbf{End}(\mathbf{T})$ *for some tournament* $\mathbf{T}$.

Finally, we shall mention two more results concerning automorphisms of tournaments. We say that a sequence of integers d *forces* the property P of tournaments if any tournament with a degree sequence (score vector) d has the property P. It turned out that very few automorphism groups of tournaments are forced by degree sequences.

Theorem 4.6. [24] *The only automorphism groups of tournaments which may be forced by degree sequences are the trivial group and finite direct products of cyclic groups* $\mathbf{Z}_3$ *and* $\mathbf{Z}_5$. *In addition, the sequences forcing the trivial automorphism group of a tournament (i.e. that a tournament is asymmetric) are exactly the sequences having no three equal elements.* □

As in Section 2, one can raise a question about the number of all asymmetric and, in particular, rigid tournaments. The last theorem of this section shows that these properties hold for almost all tournaments.

Theorem 4.7. [24] *Let* T_n *denote the number of all tournaments with* n *elements and let* A_n *and* R_n *be respectively the numbers of all* n*-element asymmetric and rigid tournaments. Then*

$$\lim_{n\to\infty}\frac{R_n}{T_n}=1 \text{ and consequently, } \lim_{n\to\infty}\frac{A_n}{T_n}=1. \quad \square$$

5. Decidability problems

Whenever a variety is given, different algorithmic problems may arise about its members and formulæ they satisfy. Among the most interesting ones are, for example, the decision problem for the equational and the elementary theory, the membership problem and word problems. These questions are usually formulated in terms of exsistence of various algorithms, but, accepting Church's Thesis and applying the arithmetization process proposed by Gödel, this is just the same as asking whether some sets of algebras and formulæ are recursive. For example, if one is concerned with the set of pairs of terms

$$Eq(\mathcal{V})=\{\langle p,q\rangle :\ \mathcal{V}\models p\approx q\},$$

where $\mathcal{V}$ is a variety, the *decidability problem* for equations of $\mathcal{V}$ is to determine whether the above set is recursive or not. The same question for the set $\mathcal{V}_{fin}$ of all finite members of $\mathcal{V}$ is the *membership problem* for $\mathcal{V}$, etc.

Let Ω_n denote the set of all tournaments with $\{1,2\ldots,n\}$ as the set of their vertices. Clearly, Ω_n is always a finite set and each n-element tournament is isomorphic to some member of Ω_n. Because of the observation made back in Corollary 2.4, the following result we have for free.

Proposition 5.1. [4] *The equational theory of* $\mathcal{T}$ *is decidable.* □

The required algorithm which decides an equation in $\mathcal{T}$ consists of consecutive checkings of the given equation for all possible interpretations of variables in all tournaments from Ω_n.

Most of the other positive results in this field about the variety $\mathcal{T}$ follow from the assertion below. Namely, the fortunate occurence is that it is possible to give a closer localization of finitely generated free algebras of $\mathcal{T}$. But the importance of the following statement is much more emphasized in its corollaries.

Proposition 5.2. [4] *If $\mathbf{F}_{\mathcal{T}}(n)$ denotes the free algebra of $\mathcal{T}$ on n free generators, then $\mathbf{F}_{\mathcal{T}}(n) \in \mathsf{ISP}_{fin}(\Omega_n)$. Moreover, if $\Omega_n = \{\mathbf{T}_1, \dots, \mathbf{T}_k\}$, then $\mathbf{F}_{\mathcal{T}}(n)$ can be embedded into the finite direct product $\prod_{i=1}^{k} \mathbf{T}_i^{n^n}$.* □

Since this yields that all finitely generated free algebras of $\mathcal{T}$ are finite, we immediately obtain

Corollary 5.3. [4] *$\mathcal{T}$ is a locally finite variety.* □

as well as the solution of the membership problem:

Corollary 5.4. [4] *There exists an algorithm which for every finite groupoid* $\mathbf{G}$ *decides whether* $\mathbf{G} \in \mathcal{T}$. □

In [24], Müller, Nešetřil and Pelant prove that one cannot obtain $\mathcal{T}$ by taking only finitely many tournaments. As a consequence of Proposition 5.2, we can strengthen this result.

Corollary 5.5. [4] *$\mathcal{T}$ is not finitely generated.* □

Maybe the most important consequence of Proposition 5.2 is that it admits to effectively construct any finitely presented algebra in $\mathcal{T}$. By Corollary 5.3, we know that $\mathbf{P}_{\mathcal{T}}(G, R)$ must be a finite groupoid, provided the presentation $\langle G, R \rangle$ is finite. Moreover, if $|G| = n$, then $\mathbf{P}_{\mathcal{T}}(G, R) \in \mathsf{HS}(\prod_{i=1}^{k} \mathbf{T}_i^{n^n})$, where $\Omega_n = \{\mathbf{T}_1, \dots, \mathbf{T}_n\}$. Therefore, it remains to algorithmically determine which one of the quotients of subalgebras of $\prod_{i=1}^{k} \mathbf{T}_i^{n^n}$ is isomorphic to $\mathbf{P}_{\mathcal{T}}(G, R)$. But for each of these "candidates" (note there are only finitely many of them) one can check whether they are n-generated and if so, whether they satisfy the relations from R for some sequence of n generators. In this way, precisely all homomorphic images of $\mathbf{P}_{\mathcal{T}}(G, R)$ are produced and now it remains to pick the largest among them. Thus we just proved

Corollary 5.6. [4] *The word problem for $\mathcal{T}$ is uniformly solvable.* □

As the uniform solvability of the word problem is for any variety known to be equivalent to the decidability problem for quasi-identities, we have

Corollary 5.7. *The theory of quasi-identities of $\mathcal{T}$ is decidable.* □

Finally, it remains to discuss the decidability of the elementary theory, that is, the set of all first-order formulæ satisfied by $\mathcal{T}$. The negative result stated below is very much an expexcted one.

Theorem 5.8. [4] *The elementary theory of the variety $\mathcal{T}$ is hereditarily undecidable.* □

This result is obtained by a heavy use of interpretation schemes of first-order languages and interpretations of models and theories. The necessary technique is described in the monograph of McKenzie and Valeriote [20] and we shall make no further reference here about interpretations. Let us only say that the fact that a theory Γ_0 in a finite language is *interpretable* in a theory Γ if there exists a class of models having Γ_0 as its elementary theory and if every structure from this class can be in a certain sense "encoded" by a model of Γ. McKenzie and Valeriote proved that in this case if Γ_0 is finitely axiomatizable and decidable, then Γ is decidable too, and if Γ_0 is *hereditarily undecidable* (i.e. if any axiomatic extension of Γ_0 is undecidable), so is Γ.

In the present situation, the authors in [4] interpreted the theory of finite *undirected* graphs (which is well-known to be hereditarily undecidable) in $Th(\mathcal{T})$. The key formula $R(x_1, x_2, y)$ which allowed to carry out the program above was the follwing one:

$$(x_1 \not\approx x_2) \wedge (\exists u)(\exists v)(u \not\approx v \wedge u \not\approx y \wedge yu \approx yv \approx y \wedge x_1 u \approx x_1 \wedge x_2 v \approx x_2 \wedge$$

$$\wedge((x_1x_2 \approx x_1 \ \wedge \ uv \approx u) \ \vee \ (x_1x_2 \approx x_2 \ \wedge \ uv \approx v))).$$

Intuitively, this formula is concerned with the realization of a certain 5-element subtournament, which is used to describe a single undirected edge.

Now a finite graph $\mathbf{G} = (G, \rho)$ with n elements, $G = \{a_1, a_2, \ldots, a_n\}$ is interpreted by a $(2n+1)$-element tournament $\mathbf{T}$, defined over the set $T = \{b_1, \ldots, b_n, c_1, \ldots, c_n, d\}$ in such a way that the vertices b_i encode the edges which are present in $\mathbf{G}$, vertices c_i serve to fix a linear order on vertices of the original graph, while d is a special vertex which connects these two and plays a special role in the interpretation scheme. Namely, in this setting, $\mathbf{T} \models R[b_i, b_j, d]$ turns out to be equivalent to $\langle a_i, a_j \rangle \in \rho$ and this equivalence will suffice to complete the proof.

6. Tournaments are not finitely based

The aim of this section is to review results about equations satisfied by all tournaments, i.e. about the equational theory of $\mathcal{T}$. We have already seen that tournaments satisfy the following identities:

$$x^2 \approx x, \tag{1}$$

$$xy \approx yx, \tag{2}$$

$$x(xy) \approx xy, \tag{3}$$

while the associative law does not hold in $\mathcal{T}$. Moreover, we know that tournaments can satisfy only regular identities. Because of the latter fact, we have not only that the class of congruence lattices of $\mathcal{T}$ does not satisfy a nontrivial lattice identity, but that $\mathcal{T}$ is not a congruence-permutable variety, using the well-known criterion of Mal'cev, see [2].

Unfortunately, one cannot expect to find more "nice" identities in $Eq(\mathcal{T})$. The following theorem partially explains why is that so.

Theorem 6.1. [28] *Let* $\mathbf{T}$ *be a tournament. The following conditions are equivalent:*

(1) $\mathbf{T}$ *is a semilattice.*

(2) $\mathbf{T}$ *is a medial groupoid, i.e. it satisfies the equation*

$$(xy)(zt) \approx (xz)(yt).$$

(3) **T** *is a distributive groupoid, i.e. it satisfies the equation*

$$(xy)z \approx (xz)(yz).$$

(4) **T** *satisfies a balanced identity (that is, a regular identity in which every variable has exactly one occurence on each side of the identity) which is not a consequence of the commutative law.*

(5) **T** *is an acyclic tournament, i.e. a chain.* □

In the proof of this theorem, of essential importance were several characterizations of semilattices and medial groupoids which are due to J. Dudek.

Most of the research in this field was chiefly motivated by a simple and natural question posed in Müller, Nešetřil and Pelant [24] which asked whether $\mathcal{T}$ can be defined by a finite set of equations, i.e. whether $Eq(\mathcal{T})$ is finitely based. Namely, if Γ is any set of equations, we say that a set of equations $\Sigma \subseteq \Gamma$ is an *(equational) base* of Γ if every element of Γ can be formally deduced from Σ using the rules of equational logic (see [2]). In case that Γ is the equational theory of a variety or an algebra, we simply call Σ *the base* of that variety or algebra, respectively.

Müller, Nešetřil and Pelant note that equations (1)–(3) form a base for all identities of $\mathcal{T}$ in two variables. Also, they consider the following groupoid terms

$$A_{1,k} = (\ldots((x_1x_2)x_3)\ldots)x_k, \quad A_{n,k} = (\ldots((A_{n-1,k}x_1)x_2)\ldots)x_k \ \ (n \geq 2)$$

and identities $A_{n!+n,k} \approx A_{n,k}$. These identities are for all $k \geq 1$ satisfied by all tournaments with at most n elements, but for each n one can find a tournament with more than n elements in which this equation is false for a suitable k. On the other hand, it can be proved that the equations

$$A_{n!+n,n} \approx A_{n,n} \tag{4}$$

are satisfied in $\mathcal{T}$ for all $n \geq 1$. There exist infinitely many of them which are independent, but nevertheless, Subotić [28] proved that they do not form a base for $\mathcal{T}$ (even with laws (1) and (2) adjoined) by considering the following groupoid:

	0	a	b	c
0	0	0	0	0
a	0	a	b	0
b	0	b	b	a
c	0	0	a	c

This groupoid is idempotent and commutative and satisfies all equations (4). However,

$$b(bc) = ba = b \neq a = bc,$$

thus it does not satisfy (3). The situtation remains the same when one adds the identity (3) as an axiom. Namely, the illustrious *Park groupoid* (one of the best known nonfinitely based algebras) given by the following table:

	0	a	b	c
0	0	0	0	0
a	0	a	b	0
b	0	b	b	c
c	0	0	c	c

satisfies all identities (1)–(4) but does not satisfy

$$x(yz) \approx x((xz)(yz)),$$

which holds in all tournaments (recall that it is enough to check it only for all 3-element tournaments) and hence it holds in $\mathcal{T}$. Therefore, we have

Proposition 6.2. [28] *The identity $x(yz) \approx x((xz)(yz))$ is not deducible from (1)–(4).* □

The question is whether the process of adding new identites to (1)–(4) can be finished in finitely many steps if we want to obtain a base for $\mathcal{T}$. Thus we have reached our next open problem.

Problem 4. *Is there a finite set of identities which together with (1)–(4) form an equational base for $\mathcal{T}$?*

In order to serve with some "inspiration", we give a list of some identities of $\mathcal{T}$ in three and four variables.

Proposition 6.3. [28] *The variety $\mathcal{T}$ satisfies the following identities:*

$$\begin{aligned} (x(yz))z &\approx x((xy)z), \\ x((x(yz))z) &\approx (x((xy)z))z, \\ (xy)(xz) &\approx x((xy)(xz)), \end{aligned}$$

$$\begin{aligned} x(yz) &\approx ((((xy)z)y)z)x, \\ ((xy)z)x &\approx ((((xy)z)y)x)z, \\ ((xy)z)y &\approx ((((xy)z)x)y)z, \\ (x(yz))((xy)z) &\approx ((x(yz))((xy)z))(xz), \\ (xz)(x(yz)) &\approx (xz)((xy)z), \\ (((xy)z)t)z &\approx (((xy)t)z)t. \quad \square \end{aligned}$$

But one can also consider different classes identities *not* holding in $\mathcal{T}$. The following is an illustration of results of this kind.

Proposition 6.4. [28] *Let $t(x, y, z)$ be a groupoid term in which all parentheses are grouped to the left. Then $\mathcal{T} \not\models (xy)(xz) \approx t(x, y, z)$.* $\square$

Another interesting property of tournament identities is the following one.

Proposition 6.5. [28] *Assume $\mathcal{T} \models p(x_1, \ldots, x_n) \approx q(x_1, \ldots, x_n)$. Then $\mathcal{T}$ satisfies any identity which one obtains by deletion of some variable from $p \approx q$.* $\square$

The above proposition is proved by considering all tournaments with $n-1$ elements enlarged with a new vertex which is beaten by all other vertices. The result now falls out by applying Corollary 2.4.

After all this work done, a slight suspicion arised on whether the answer to the previously formulated finite base problem for $\mathcal{T}$ might be negative, i.e. that all attempts to find a finite set of equations axiomatizing $\mathcal{T}$ are absolutely hopeless. In a recent paper [16], Ježek, Marković, Maróti and McKenzie proved that this is indeed the case. If $\mathcal{V}$ is any variety, let $\mathcal{V}^{(n)}$ denote the variety defined by all equations holding in $\mathcal{V}$ with at most n variables. In this way we obtain a chain of varieties

$$\mathcal{V}^{(1)} \geq \mathcal{V}^{(2)} \geq \ldots \geq \mathcal{V}^{(n)} \geq \ldots \geq \mathcal{V}.$$

For example, $\mathcal{T}^{(1)}$ is just the variety of all idempotent groupoids and $\mathcal{T}^{(2)}$ is the variety of all groupoids satisfying equations (1)–(3). Now it is well-known from universal algebra that a locally finite variety $\mathcal{V}$ of finite type (and $\mathcal{T}$ certainly meets this requirements) is finitely based if and only if $\mathcal{V} = \mathcal{V}^{(n)}$ for some $n \geq 1$, i.e. if and only if the above chain is finite.

What Ježek, Marković, Maróti and McKenzie have done is nothing else than proving the infinity of this chain if $\mathcal{V}$ is $\mathcal{T}$. In more detail, their result is as follows.

Theorem 6.6. [16] *For every $n \geq 3$ there exists a groupoid $\mathbf{G}_n$ with $n+2$ elements such that $\mathbf{G}_n \in \mathcal{T}^{(n)}$, but $\mathbf{G}_n \notin \mathcal{T}^{(n+1)}$.* □

Consequently, we obtain

Corollary 6.7. [16] *The variety $\mathcal{T}$ has no finite base for its equations.* □

Roughly speaking, the plan of the proof was the following. After defining an idempotent and commutative multiplication on a set with $n+2$ elements in a suitable way, one should construct first a special equation in $n+1$ variables (it has a very complex structure and it is defined in a recurrent way; we shall not quote it here). This equation is shown to be true in $\mathcal{T}$, mainly by a repeated application of our identity (3). On the other hand, this equation turns out to be false in $\mathbf{G}_n$ and since this groupoid is constructed so that it is generated by $n+1$ elements, one concludes that $\mathbf{G}_n \notin \mathcal{T}^{(n+1)}$. The final step is to prove that $\mathbf{G}_n \in \mathcal{T}^{(n)}$ by demonstrating that all n-generated subgroupoids of $\mathbf{G}_n$ belong to $\mathcal{T}$. This is done by considering three cases, among which the hardest requires to prove that a certain subgroupoid of $\mathbf{G}_n$ is a subdirect product of two of its quotients, which appear to be tournaments themselves.

So we know that there can be no finite list of axioms for equations of tournaments: such list must inevitably be infinite. And still, we do not have *any* equational base for $\mathcal{T}$. This is a problem itself, and at present it seems to be a hard one. Of course, the following greatly intersects with the previous Problem 4.

Problem 5. *Find an equational base for $\mathcal{T}$.*

There are several examples in the literature of nonfinitely based equational theories which are exactly equational parts of some finitely axiomatized implicational theories. It seems chalenging to determine whether this is the case with tournaments.

Problem 6. *Is there a finite set Σ of quasi-identities such that $Eq(T)$ is precisely the set of all equations which are logical consequences of Σ?*

But what about equational bases for single tournaments? Since all two-element algebras are finitely based (Lyndon, 1951) and since the three-element chain is a semilattice and thus also finitely based as being a commutative semigroup (Perkins, 1969), the first really interesting tournament is *the triangle*, the three-element cycle. The corresponding groupoid, given by the table

	a	b	c
a	a	a	c
b	a	b	b
c	c	b	c

is often denoted by **3**. It turned out that **3** is finitely based too. This result was never published, but J. Berman proves it in his letter to R. E. Park [1] by showing that the variety $\mathsf{HSP}(\mathbf{3})$ has DPC (definable principal congruences). Moreover, Berman shows that every finite member $\mathbf{A}$ of $\mathsf{HSP}(\mathbf{3})$ can be partitioned into subalgebras isomorphic to direct powers of **3** and that the partition so induced forms a congruence θ such that $\mathbf{A}/\theta$ is a semilattice. From this, Berman calculates the free spectrum of **3** and obtains

$$f_n(\mathbf{3}) = \sum_{k=1}^{n} \binom{n}{k} 3^{3^{k-1}-2^k+1}.$$

However, no axiom system for $\mathsf{HSP}(\mathbf{3})$ is known yet.

Problem 7. *Find a finite equational base for* **3**.

Clearly, such a base must involve equations with at least four variables, because it is very easy to prove that each equation with at most three variables holding in **3** is true for all tournaments.

In contrast to the previous considerations, almost every tournament has a two-point extension which can be made by a slight change into an inherently nonfinitely based groupoid. As we can see below, the trasformation is quite simple.

Proposition 6.8. *Let* $\mathbf{T}$ *be a tournament which is not a chain and* $\mathbf{G_T}$ *a groupoid which one obtains from* $\mathbf{T}$ *by adjoining a zero element 0 and an* *identity element 1 and setting $x^2 = 0$ for all $x \in T \cup \{0\}$. Then $\mathbf{G_T}$ is inherently nonfinitely based.*

Proof. The result follows immediately as a consequence of Proposition 3.7 and the famous result of McNulty and Shallon [21], according to which every nonassociative, commutative groupoid satisfying no absorptive identity and having a zero and an identity is inherently nonfinitely based. □

In other words, the construction described above consists just of adding two new vertices 0 and 1 such that 0 beats, while 1 is beaten by all other vertices and putting all zeroes on the pricipal diagonal, except, of course, $1^2 = 1$. Hence, the tournament **3** considered above can be transformed into the following inherently nonfinitely based groupoid **5**:

	0	a	b	c	1
0	0	0	0	0	0
a	0	0	a	c	a
b	0	a	0	b	b
c	0	c	b	0	c
1	0	a	b	c	1

Problem 8. *Does there exist a nonfinitely based tournament?*

Finally, the study of the lattice of subvarieties of $\mathcal{T}$ might be of some interest. Corollary 5.7 immediately yields the following

Proposition 6.9. [28] *The lattice $\mathbf{L}(\mathcal{T})$ of all subvarieties of $\mathcal{T}$ contains a countably infinite chain.* □

The cardinality of $\mathbf{L}(\mathcal{T})$ is not known. Thus we have

Problem 9. *Is the lattice $\mathbf{L}(\mathcal{T})$ countably or uncountably infinite?*

Of course, a positive answer to Problem 8 would automatically solve the one above.

Subotić [28] describes the bottom of $\mathbf{L}(\mathcal{T})$. Let $\mathbf{4}$ be a 4-element tournament whose groupoid is the following:

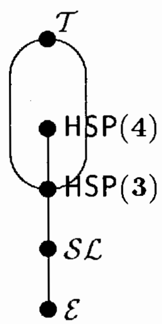
	a	b	c	d
a	a	a	a	d
b	a	b	b	d
c	a	b	c	c
d	d	d	c	d

This tournament is subdirectly irreducible. Since the two-element semilattice, $\mathbf{3}$ and $\mathbf{3}^0$ (the triangle with a zero adjoined) are proved by J. Berman in [1] to be the only subdirectly irreducible members of $\mathsf{HSP}(\mathbf{3})$, $\mathbf{4}$ does not belong to $\mathsf{HSP}(\mathbf{3})$ and, moreover, $\mathbf{4}$ is the only 4-element tournament that falls outside $\mathsf{HSP}(\mathbf{3})$. Therefore,

$$\mathsf{HSP}(\mathbf{3}) < \mathsf{HSP}(\mathbf{4}).$$

But one can prove even more: namely, $\mathsf{HSP}(\mathbf{4})$ exactly covers $\mathsf{HSP}(\mathbf{3})$, while $\mathbf{3}$ belongs to all nontrivial subvarieties of $\mathcal{T}$ which are different from the variety $\mathcal{SL}$ of all semilattices. Summing up, we have

Theorem 6.10. [28] *Let $\mathcal{E}$ be the variety of trivial groupoids and $\mathcal{SL}$ the variety of semilattices. The lattice $\mathbf{L}(\mathcal{T})$ has the following form:*



□

Problem 10. *Explore the structure of the lattice $\mathbf{L}(\mathcal{T})$. For example, describe all subvarieties of $\mathcal{T}$ covering $\mathsf{HSP}(\mathbf{3})$. Are there any coatoms in this lattice (maximal subvarieties of $\mathcal{T}$)?*

7. Representation by two binary operations

As it was already mentioned in the introduction, for a given tournament $\mathbf{T} = \langle T, \to \rangle$ we define an algebra $\mathbf{A_T} = \langle T, \vee, \wedge \rangle$ such that if $x \to y$, then

$$\begin{aligned} x \vee y &= y \vee x = x, \\ x \wedge y &= y \wedge x = y. \end{aligned}$$

The algebra $\mathbf{A_T}$ constructed in this way is not, of course, a lattice, because both $\vee$ and $\wedge$ fail to be associative, unless $\mathbf{T}$ is a chain. However, the idempotent, the commutative and the absorptive law are satisfied, as well as an additional pair of identities which can be considered as a weak form of associative laws. The algebras satisfying these conditions are called *T-lattices* in Fried [8], *trellis* in Skala [27], while we are going to adopt the terminology from Fried and Grätzer and speak about *weakly associative lattices.*

To be quite precise, a weakly associative lattice (*WA-lattice*) is an algebra $\langle L, \vee, \wedge \rangle$ satisfying the following identities:

$$\begin{aligned} &(5) \qquad & x \vee x \approx x, \qquad & x \wedge x \approx x, \\ &(6) \qquad & x \vee y \approx y \vee x, \qquad & x \wedge y \approx y \wedge x, \\ &(7) \qquad & x \vee (x \wedge y) \approx x, \qquad & x \wedge (x \vee y) \approx x, \\ &(8) \qquad & ((x \wedge z) \vee (y \wedge z)) \vee z \approx z, \qquad & ((x \vee z) \wedge (y \vee z)) \wedge z \approx z. \end{aligned}$$

Observe that the first identity (and similarly the second identity) of (8) can be written as

$$((x \wedge z) \vee (y \wedge z)) \vee z \approx (x \wedge z) \vee ((y \wedge z) \vee z),$$

which makes the term "weakly associative" much clearer. A nice feature of WA-lattices is the following

Lemma 7.1. [9] *The variety of all WA-lattices is congruence-distributive.*

Proof. Note that $p(x, y, z) = ((x \wedge y) \vee (y \wedge z)) \vee (z \wedge x)$ is a majority term for WA-lattices, i.e. they satisfy the equations

$$p(x, x, y) \approx p(x, y, x) \approx p(y, x, x) \approx x,$$

implying the lemma by the well-known result of Jónsson [18]. □

Of course, for each tournament $\mathbf{T}$, the algebra $\mathbf{A_T}$ is a WA-lattice. Let $\mathcal{T}_2$ denote the variety generated by all such algebras. Grätzer and Lasker [11] give an infinite axiom system for $\mathcal{T}_2$. Let

$$\begin{aligned} p_n &= ((x \vee z_1) \wedge z_2) \vee \ldots z_n, \\ q_n &= (((x \wedge y) \vee z_1) \wedge z_2) \vee \ldots z_n, \\ r_n &= ((y \vee t_1) \wedge t_2) \vee \ldots t_n, \\ s_n &= (((x \wedge y) \vee t_1) \wedge t_2) \vee \ldots t_n, \\ a_n &= p_n \wedge q_n, \quad b_n = p_n \vee q_n, \\ c_n &= r_n \wedge s_n, \quad d_n = r_n \vee q_n. \end{aligned}$$

Theorem 7.2. [11] *Identities (5)–(8) along with*

$$(((u \vee a_n) \wedge b_n) \vee c_n) \wedge d_n \approx (((v \vee a_n) \wedge b_n) \vee c_n) \wedge d_n$$

for all $n \geq 1$, form an equational base of $\mathcal{T}_2$. □

However, the following remains open.

Problem 11. *Is $\mathcal{T}_2$ finitely based?*

Concerning $\mathcal{T}_2$, Fried and Grätzer [9] show two further properties.

Proposition 7.3. [9] $\mathcal{T}_2 \models x \wedge (y \vee z) \approx ((x \wedge y) \vee (x \wedge z)) \wedge (y \vee z)$. □

Proposition 7.4. [9] *Let $\mathbf{L}$ be a WA-lattice satisfying the identity from Proposition 7.3. Then for all $a, b, c \in L$ if $a \vee b = a \vee c$ and $a \wedge b = a \wedge c$ then $b = c$.* □

But the truth is that the whole variety $\mathcal{T}_2$ attracted much less attention in the literature than one distinguished subvariety of it. This is the variety generated by the three element cycle, which is in this approach often denoted by $\mathbf{Z}$. To make the notation shorter, we write $\mathcal{Z}$ for $\mathsf{HSP}(\mathbf{Z})$. It can be proved an analogous description for finite members of $\mathcal{Z}$ as the one for $\mathsf{HSP}(\mathbf{3})$ and the only (though important) difference is that the $\mathcal{Z}$ has only two subdirectly irreducibles: the two-element lattice and $\mathbf{Z}$ itself. Having this, it is not hard to derive that $\mathcal{Z}$ covers the variety of all distributive lattices $\mathcal{D}$.

The most significant advantage of studying tournaments as algebras with two binary operations is that *any* finite algebra obtained from a tournament is finitely based, because of Lemma 7.1 and the celebrated Baker's finite base theorem. The gross part of the research in this field was mainly motivated by determining the base for $\mathcal{Z}$. Fried and Grätzer [9] came to this through considering properties of principal congruences in $\mathcal{Z}$ and their result was later even sharpened by Grätzer, Kisielewicz and Wolk [10]. Our aim in the rest of this section is to give an account on their efforts. Also, we review some structural theorems from [9] about the finite members of $\mathcal{Z}$. It will be interesting to compare them with the corresponding properties of $\mathsf{HSP}(\mathbf{3})$.

First of all, let

$$q = x_2 \wedge x_5,\ r = x_2 \vee x_4,\ s = (x_3 \vee x_5) \wedge x_4$$

and

$$p(x_1, x_2, x_3, x_4, x_5) = (((x_1 \vee q) \vee r) \wedge s) \wedge x_5.$$

Except that, in any WA-lattice (and so in any member of $\mathcal{Z}$) one can define a reflexive and antisymmetric relation $\leq$ such that $x \leq y$ if and only if $x \vee y = y$ (if and only if $x \wedge y = x$). Now we can state the first cornerstone theorem of [9] as follows.

Theorem 7.5. [9] *Let $\mathbf{A} \in \mathcal{Z}$ and $a, b, c, d \in A$ such that $a \leq b$ and $c \leq d$. Then the following conditions are equivalent:*

(1) $\langle c, d\rangle \in \Theta(a, b)$.

(2) $p^{\mathbf{A}}(a, a, b, c, d) = c$ *and* $p^{\mathbf{A}}(b, a, b, c, d) = d$.

(3) $(a \vee c) \vee b = (a \vee d) \vee b$ *and* $a \wedge (c \wedge b) = a \wedge (d \wedge b)$. □

Since the above theorem yields that $\langle c, d\rangle \in \Theta(a, b)$ can be decided in the subalgebra of $\mathbf{A}$ generated by $\{a, b, c, d\}$, by the result of Day [5] we obtain

Corollary 7.6. [9] *$\mathcal{Z}$ has the congruence extension property (CEP).* □

Using this, Fried and Grätzer prove

Corollary 7.7. [9] *$\mathcal{Z}$ has the amalgamation property (AP).* □

Of crucial importance in the search for the equational base for $\mathcal{Z}$ was the following

Theorem 7.8. [9] *Let $\mathcal{W}$ be a variety of WA-lattices in which for any $\mathbf{A} \in \mathcal{W}$ and $a, b, c, d \in A$ such that $a \leq b$ and $c \leq d$ the condition (1) of Theorem 7.5 implies the condition (3) of the same theorem. Then $\mathcal{W} \leq \mathcal{Z}$.* □

Towards discovering the above theorem, Fried and Grätzer were primarily leaded by similar characterization results for distributive lattices. This theorem turned out to be the key ingredient for the determination of a finite axiom system for $\mathcal{Z}$ as it will be presented in the few lines below. The main idea is that it is possible to find identities which prove that arbitrary principal congruences in $\mathcal{Z}$ are, in some sense, transitive extensions of those described by Theorem 7.5.

Let

$$q_1(x_1, x_2, x_3, x_4) = p(x_1, x_1, x_1 \vee x_2, x_3, x_3 \vee x_4)$$

and

$$q_2(x_1, x_2, x_3, x_4) = p(x_1 \vee x_2, x_1, x_1 \vee x_2, x_3, x_3 \vee x_4),$$

where p denotes the same as above. Now consider the following system of equations:

$$
\begin{array}{lrcl}
(9) & (x \vee (y \vee z)) \vee z & \approx & ((x \vee y) \vee (x \wedge z)) \vee (x \vee z), \\
(10) & q_1 \wedge x_5 & \approx & q_1(x_1, x_2, q_1 \wedge x_5, q_1 \wedge x_5), \\
(11) & q_1 \wedge x_5 & \approx & q_1(x_1, x_2, q_1 \wedge x_5, q_2 \wedge x_5), \\
(12) & q_2 \wedge x_5 & \approx & q_1(x_1, x_2, q_2 \wedge x_5, q_1 \wedge x_5), \\
(13) & q_2 \wedge x_5 & \approx & q_1(x_1, x_2, q_2 \wedge x_5, q_2 \wedge x_5), \\
(14) & q_1 \vee x_5 & \approx & q_1(x_1, x_2, q_1 \vee x_5, q_1 \vee x_5), \\
(15) & q_1 \vee x_5 & \approx & q_1(x_1, x_2, q_1 \vee x_5, q_2 \vee x_5), \\
(16) & q_2 \vee x_5 & \approx & q_1(x_1, x_2, q_2 \vee x_5, q_1 \vee x_5), \\
(17) & q_2 \vee x_5 & \approx & q_1(x_1, x_2, q_2 \vee x_5, q_2 \vee x_5), \\
(18) & q_1 \vee x_5 & \approx & q_2(x_1, x_2, q_1 \vee x_5, q_1 \vee x_5),
\end{array}
$$

$$
\begin{array}{lrcl}
(19) & (q_1 \vee x_5) \vee (q_2 \vee x_5) & \approx & q_2(x_1, x_2, q_1 \vee x_5, (q_1 \vee x_5) \vee (q_2 \vee x_5)), \\
(20) & q_2 \vee x_5 & \approx & q_2(x_1, x_2, q_2 \vee x_5, q_2 \vee x_5), \\
(21) & q_1 \wedge x_5 & \approx & q_2(x_1, x_2, q_1 \wedge x_5, q_1 \wedge x_5), \\
(22) & (q_1 \wedge x_5) \vee (q_2 \wedge x_5) & \approx & q_2(x_1, x_2, q_1 \wedge x_5, (q_1 \wedge x_5) \vee (q_2 \wedge x_5)), \\
(23) & q_2 \wedge x_5 & \approx & q_2(x_1, x_2, q_2 \wedge x_5, q_2 \wedge x_5), \\
(24) & x_1 \wedge (q_1 \wedge (x_1 \vee x_2)) & \approx & x_1 \wedge (q_2 \wedge (x_1 \vee x_2)), \\
(25) & (x_1 \vee q_1) \wedge (x_1 \vee x_2) & \approx & (x_1 \vee q_2) \wedge (x_1 \vee x_2).
\end{array}
$$

Theorem 7.9. [9] *The variety $\mathcal{Z}$ is determined by identities (5)–(25).* □

Note that in (5)–(25) no more than five variables were used. Hence, we have

Corollary 7.10. [9] $\mathbf{A} \in \mathcal{Z}$ *if and only if every subalgebra of* $\mathbf{A}$ *generated by five elements belongs to* $\mathcal{Z}$. □

For similar reasons as before, $\mathcal{Z}$ cannot be defined by identities with only three variables and hence $\mathbf{A} \in \mathcal{Z}$ cannot be decided in subalgebras of $\mathbf{A}$ generated by three elements. But for a long time it was an open problem whether four will do (and consequenly, whether $\mathcal{Z}$ can be defined by equations with at most four variables). The positive answer was discovered by Grätzer, Kisielewicz and Wolk [10]. Their equational base for $\mathcal{Z}$ is an improvement of Theorem 7.9 not only regarding the number of variables, but also in respect to the number of defining identities.

Let

$$u(x, y, z) = (x \wedge y) \wedge ((x \vee y) \wedge z)$$

and let $\tilde{u}(x, y, z)$ denotes the dual term of t (obtained by replacing $\vee$ by $\wedge$ and vice versa). Also, define

$$
\begin{aligned}
v(x, y, z, t) &= u(x, y, z) \vee \tilde{u}(x, y, t), \\
v'(x, y, z, t) &= u(x, y, z) \vee u(x, y, t), \\
w(x, y, z, t) &= u(x, y, z) \wedge \tilde{u}(x, y, t).
\end{aligned}
$$

Consider the following identities:

$$
\begin{array}{lrcl}
(26) & u(x, y, z \vee t) & \approx & (v(x, y, z, t) \wedge v(x, y, t, z)) \wedge v'(x, y, z, t), \\
(27) & u(x, y, z \wedge t) & \approx & w(x, y, z, t) \wedge w(x, y, t, z),
\end{array}
$$

and also their duals (28) and (29), respectively.

Theorem 7.11. [10] *Identities (1)–(4), (26)–(29) define the variety $\mathcal{Z}$.* □

Corollary 7.12. [10] $\mathbf{A} \in \mathcal{Z}$ *if and only if every subalgebra of* $\mathbf{A}$ *generated by four elements belongs to* $\mathcal{Z}$. □

Since any two identities of an arbitrary idempotent variety can be replaced by one, it is possible to reduce the equational base for $\mathcal{Z}$ to three identities. But a result of Padmanabhan [26] shows that a further reduction to two equations can be performed. The existential character of these conclusions motivates to pose the following

Problem 12. *Construct explicitely a base for $\mathcal{Z}$ consisting of two identities. Is there a single identity defining $\mathcal{Z}$? If yes, give an example of such identity.*

As promised, we end this survey by quoting three results which are concerned with the structure of finite members of $\mathcal{Z}$. We have already seen that the way that finite members of $\mathsf{HSP}(\mathbf{3})$ are built is pretty well understood. But the really nice thing is that finite algebras in the variety $\mathcal{Z}$ show even more regularities. We can see below that their description sounds indeed simple.

Theorem 7.13. [9] *Every finite algebra* $\mathbf{A} \in \mathcal{Z}$ *has a representation of the form*

$$\mathbf{A} \cong \mathbf{D} \times \mathbf{Z}^n,$$

where the integer $n \geq 0$ and the finite distributive lattice $\mathbf{D}$ *are unique (the latter up to an isomorphism). In fact,* $\mathbf{D}$ *is the maximal distributive lattice which is a homomorphic image of* $\mathbf{A}$. □

Because of congruence-distributivity,

$$\mathbf{Con}(\mathbf{D} \times \mathbf{Z}^n) \cong \mathbf{Con}(\mathbf{D}) \times \mathbf{Con}(\mathbf{Z})^n \cong \mathbf{Con}(\mathbf{D}) \times \mathbf{C}_2^n,$$

where $\mathbf{C}_2$ denotes the two-element chain. Since congruence lattices of finite distributive lattices are Boolean, we immediately obtain

Corollary 7.14. [9] *If* $\mathbf{A} \in \mathcal{Z}$ *is a finite algebra, then* $\mathbf{Con}(\mathbf{A})$ *is a Boolean lattice.* □

Applying Theorem 7.13 to finitely generated free algebras in $\mathcal{Z}$, one easily derives

Theorem 7.15. [9] *Let* $\mathbf{F}_n(\mathcal{D})$ *and* $\mathbf{F}_n(\mathcal{Z})$ *denote the free distributive lattice and the free algebra in* $\mathcal{Z}$ *on* n *free generators, respectively. Then*

$$\mathbf{F}_n(\mathcal{Z}) \cong \mathbf{F}_n(\mathcal{D}) \times \mathbf{Z}^{k_n},$$

where $k_n = 3^{n-1} - 2^n + 1$. *Thus* $f_n(\mathbf{Z}) = 3^{k_n} D_n$, *where* D_n *denotes the* n^{th} *Dedekind number.* □

The second part of the above theorem has obviously a high level of similarity to the corresponding result from Berman's letter [1]. But in sharp contrast to the free spectrum of **3**, which was expressed in closed form, the problem of determining the free spectrum of **Z** reduces to the calculation of D_n, a problem frequently referred to as *the Dedekind problem*, which, posed back in 1900, tends to become one of the most long-lasting classical problems in the contemporary mathematics in general.

Appendix. List of open problems

Problem 1. *Is there a first-order formula* Φ *such that a tournament* $\mathbf{T}$ *is Hamiltonian if and only if* $\mathbf{T} \models \Phi$*?*

Problem 2. *Characterize all subdirectly irreducible members of the variety* $\mathcal{T}$. *Are they all tournaments?*

Problem 3. *Characterize all finite monoids isomorphic to* $\mathbf{End}(\mathbf{T})$ *for some tournament* $\mathbf{T}$.

Problem 4. *Is there a finite set of identities which together with* $x^2 \approx x$, $xy \approx yx$, $x(xy) \approx xy$ *and* $A_{n!+n,n} \approx A_{n,n}$ *for all* $n \geq 1$, *form an equational base for* $\mathcal{T}$*?*

Problem 5. *Find an equational base for* $\mathcal{T}$.

Problem 6. *Is there a finite set* Σ *of quasi-identities such that* $Eq(\mathcal{T})$ *is precisely the set of all equations which are logical consequences of* Σ*?*

Problem 7. *Find a finite equational base for* **3**.

Problem 8. *Does there exist a nonfinitely based tournament?*

Problem 9. *Is the lattice* $\mathbf{L}(\mathcal{T})$ *countably or uncountably infinite?*

Problem 10. *Explore the structure of the lattice* $\mathbf{L}(\mathcal{T})$. *For example, describe all subvarieties of* $\mathcal{T}$ *covering* $\mathsf{HSP}(\mathbf{3})$. *Are there any coatoms in this lattice (maximal subvarieties of* $\mathcal{T}$*)?*

Problem 11. *Is* $\mathcal{T}_2$ *finitely based?*

Problem 12. *Construct explicitely a base for* $\mathcal{Z}$ *consisting of two identities. Is there a single identity defining* $\mathcal{Z}$*? If yes, give an example of such identity.*

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