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LIFTS OF THE ALMOST COMPLEX STRUCTURES TO T(OSC^2M)

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Abstract

The geometry of the k-osculator bundle (Osc^kM, π, M) has been studied by R. Miron and Gh. Atanasiu in their joint papers [5-7]. Obviously, the osculator bundle of second order, or the bundle of accelerations correspond to the case k=2, [1], [5], and Osc^1M is the tangent bundle TM of the base manifold M, [4].

In the present paper we consider the group G_{ac} of transformations of almost complex N-linear connections on Osc^2M and we determine its invariants, which are d-tensor fields. By means of these invariants, we get characterizations of the integrability of type I, II, III or IV for the almost complex d-structures on Osc^2M. All this integrability relies only on the geometry of 2-osculator bundle (Osc^2M, π, M).

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1 Preliminaries: the k-osculator bundle, d-tensor fields and N-linear connections

Let M be a real C^∞-manifold with n dimensions and (Osc^kM, π, M) its k-osculator bundle. A transformation of canonical coordinates on the (k+1)n-dimensional manifold Osc^kM, $(x^i, y^{(1)i}, ..., y^{(k)i}) \to (\tilde{x}^i, \tilde{y}^{(1)i}, ..., \tilde{y}^{(k)i})$ is given by

$$\left.\begin{array}{rcl} \tilde{x}^i &=& \tilde{x}^i(x^1,...,x^n), \qquad rank\|\dfrac{\partial\tilde{x}^i}{\partial x^j}\| = n \\ \tilde{y}^{(1)i} &=& \dfrac{\partial\tilde{x}^i}{\partial x^j}y^{(1)j} \\ 2\tilde{y}^{(2)i} &=& \dfrac{\partial\tilde{y}^{(1)j}}{\partial x^j}y^{(1)j} + 2\dfrac{\partial\tilde{y}^{(1)i}}{\partial y^{(1)j}}y^{(2)j} \\ \multicolumn{3}{l}{\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots} \\ k\tilde{y}^{(k)i} &=& \dfrac{\partial\tilde{y}^{(k-1)i}}{\partial x^j}y^{(1)j} + 2\dfrac{\partial\tilde{y}^{(k-1)i}}{\partial y^{(1)j}}y^{(2)j} + ... + k\dfrac{\partial\tilde{y}^{(k-1)i}}{\partial y^{(k-1)j}}y^{(k)j} \end{array}\right\} \qquad (1)$$

If N is a nonlinear connection on E and J is the k-tangent structure [3] J:$\mathcal{X}$(E)→ $\mathcal{X}$(E) given by:

$$J(\frac{\partial}{\partial x^i}) = \frac{\partial}{\partial y^{(1)i}} \quad , \quad J(\frac{\partial}{\partial y^{(1)i}}) = \frac{\partial}{\partial y^{(2)i}} \quad ,..., J(\frac{\partial}{\partial y^{(k-1)i}}) = \frac{\partial}{\partial y^{(k)i}} \quad ,$$
$$J(\frac{\partial}{\partial y^{(k)i}}) = 0,$$

then $N_0 = N$, $N_1 = J(N_0)$,..., $N_{k-1} = J(N_{k-2})$ are k distributions, everyone having a finite dimension n.

Hence, the tangent space of E in the point $u = (x, y^{(1)}, ..., y^{(k)}) \in E$ is given by the direct sum of the vector spaces:

$$T_uE = N_0(u) \oplus N_1(u) \oplus ... \oplus N_{k-1}(u) \oplus V_k(u) \quad , \quad \forall u \in E \qquad (2)$$

An adapted basis to (2) is given by

$$\left\{\frac{\delta}{\delta x^i}, \frac{\delta}{\delta y^{(1)i}}, ..., \frac{\delta}{\delta y^{(k-1)i}}, \frac{\delta}{\delta y^{(k)i}}\right\} \quad , (i = 1, ..., n) \qquad (3)$$

where

$$\frac{\delta}{\delta x^i} = \frac{\partial}{\partial x^i} - \underset{(1)}{N^j{}_i}\frac{\partial}{\partial y^{(1)j}} - ... - \underset{(k)}{N^j{}_i}\frac{\partial}{\partial y^{(k)j}} \quad , \qquad (4)$$

and

$$\frac{\delta}{\delta y^{(1)i}} = \frac{\partial}{\partial y^{(1)i}} - \underset{(1)}{N}{}^m_i \frac{\partial}{\partial y^{(2)m}} - \ldots - \underset{(k-1)}{N}{}^m_i \frac{\partial}{\partial y^{(k)m}} \;;$$

$$\cdots\cdots \quad \cdot\cdot \quad \cdots\cdots\cdots\cdots$$

$$\frac{\delta}{\delta y^{(k-1)i}} = \frac{\partial}{\partial y^{(k-1)i}} - \underset{(1)}{N}{}^m_i \frac{\partial}{\partial y^{(k)m}}$$

Then $\underset{(1)}{N}{}^i_j$,..., $\underset{(k)}{N}{}^i_j$ are the coefficients of the nonlinear connection N.

The transformation of coordinates (1) implies:

$$\frac{\delta}{\delta x^i} = \frac{\partial \tilde{x}^j}{\partial x^i}\frac{\delta}{\delta \tilde{x}^j} \quad , \quad \frac{\delta}{\delta y^{(\alpha)i}} = \frac{\partial \tilde{x}^j}{\partial x^i}\frac{\delta}{\delta \tilde{y}^{(\alpha)j}} \quad , \quad (\alpha = 1, ..., k)$$

If we consider the projectors $h, v_1, ..., v_k$ determined by (2) and denote $v_\alpha X = X^{v_\alpha} (\alpha = 1, ..., k)$ we can uniquely write

$$X = X^H + X^{v_1} + ... + X^{v_k} \quad , \quad \forall X \in \mathcal{X}(E) \tag{5}$$

Thus we have

$$X^H = X^{(0)i}\frac{\delta}{\delta x^i} \quad , \quad X^{v_\alpha} = X^{(\alpha)i}\frac{\delta}{\delta y^{(\alpha)i}} \quad , \quad (\alpha = 1, .., k, \frac{\delta}{\delta y^{(k)i}} = \frac{\partial}{\partial y^{(k)i}})$$

The coordinates $X^{(\alpha)i}$, $(\alpha = 0, 1, ..., k)$ change under (1) as follows:

$$\tilde{X}^{(\alpha)i} = \frac{\partial \tilde{x}^i}{\partial x^j} X^{(\alpha)j} \quad , \quad (\alpha = 0, 1, ..., k).$$

Each of them is called a distinguished vector field, shortly a d-vector field.

Let us consider the dual basis of (3):

$$\left\{dx^i, \delta y^{(1)i}, ..., \delta y^{(k)i}\right\} \quad , \quad (i = 1, ..., n)$$

Then for a field of 1-form ω on E we can write:

$$\omega = \omega^{H} + \omega^{v_1} + ... + \omega^{v_k} \quad , \tag{6}$$

where

$$\omega^{H} = \omega_i^{(0)} dx^i \quad , \quad \omega^{v_\alpha} = \omega_i^{(\alpha)} \delta y^{(\alpha)i}$$

and with respect to (1) we have:

$$\omega_i^{v_\alpha} = \frac{\partial \tilde{x}^j}{\partial x^i} \tilde{\omega}_j^{(\alpha)} \quad , \quad (\alpha = 0, 1, ..., k)$$

Now,we can define a distinguished tensor field on E of type (r,s) (shortly a d-tensor field) as an element $T \in \mathcal{T}_s^r(E)$ with the property:

$$T(\underset{1}{X}, ..., \underset{s}{X}, \overset{1}{\omega}, ..., \overset{r}{\omega}) = T(\underset{1}{X^H}, ..., \underset{s}{X^{v_2}}, \overset{1}{\omega^H}, ..., \overset{r}{\omega^{v_2}}) \tag{7}$$

$$\forall \underset{1}{X}, ..., \underset{s}{X} \in \mathcal{X}(E) \; , \quad \forall \overset{1}{\omega}, ..., \overset{r}{\omega} \in \mathcal{X}^*(E)$$

Then in adapted basis we obtain:

$$T = T_{j_1,...,j_s}^{i_1,...,i_r}(x, y^{(1)}, ..., y^{(k)}) \frac{\delta}{\delta x^{i_1}} \otimes .. \otimes \frac{\partial}{\partial y^{(k)i_r}} \otimes dx^{j_1} \otimes .. \otimes \delta y^{(k)j_s}$$

and with respect to (1), we get:

$$\tilde{T}_{j_1,...,j_s}^{i_1,...,i_r} = \frac{\partial \tilde{x}^{i_1}}{\partial x^{m_1}} ... \frac{\partial \tilde{x}^{i_r}}{\partial x^{m_r}} \frac{\partial x^{q_1}}{\partial \tilde{x}^{j_1}} ... \frac{\partial x^{q_s}}{\partial \tilde{x}^{j_s}} T_{q_1,...,q_r}^{m_1,...,m_s}$$

We define a -linear connection on E as a linear connection D on E which preserves by parallelism the horizontal distribution N and wich is compatible with the structure J (i.e. $D_x J = 0 \quad , \quad \forall X \in \mathcal{X}(E)$).

In the adapted basis (3) it is sufficient to give:

$$D_{\frac{\delta}{\delta x^j}} \frac{\delta}{\delta y^{(\alpha)i}} = L^m_{ij} \frac{\delta}{\delta y^{(\alpha)m}} \quad , \quad D_{\frac{\delta}{\delta y^{(\beta)j}}} \frac{\delta}{\delta y^{(\alpha)i}} = \underset{(\beta)}{C}{}^m_{ij} \frac{\delta}{\delta y^{(\alpha)m}} \tag{8}$$

$$(\alpha = 0, 1, ..., k \quad , \quad \beta = 1, ..., k \quad , \quad y^{(0)i} = x^i)$$

in order to obtain all the coefficients $D\Gamma(N) = (L^i_{jm}, \ \underset{(\alpha)}{C}{}^i{}_{jm})$, $(\alpha = 1, ..., k)$ of a N-linear connection D.

With respect to (1) we have for the coefficients $\underset{(\alpha)}{C}{}^i{}_{jm}\ (x, y^{(1)}, ..., y^{(k)})$ the transformation (7) of the d-tensor field of type (1,2) and for the coefficients $L^i_{jm}(x, y^{(1)}, ..., y^{(k)})$ the transformation law of an object of connection:

$$\tilde{L}^i_{pq} \frac{\partial \tilde{x}^p}{\partial x^r} \frac{\partial \tilde{x}^q}{\partial x^s} = L^m_{rs} \frac{\partial \tilde{x}^i}{\partial x^m} - \frac{\partial^2 \tilde{x}^i}{\partial x^r \partial x^s} \tag{9}$$

The h-covariant derivative noted with $|$ and the v_α-**covariant** derivative noted with $\overset{(\alpha)}{|}$ $(\alpha = 1, ..., k)$ in the algebra of the d-tensor fields act, for example, for a d-tensor field $K^i_j(x, y^{(1)}, ..., y^{(k)})$ of the type (1,1) as:

$$\left.\begin{aligned} K^i_{j|m} &= \frac{\delta K^i_j}{\delta x^m} + L^i_{rm} K^r_j - L^s_{jm} K^i_s \\ K^i{}_{j \overset{(\alpha)}{|} m} &= \frac{\delta K^i_j}{\delta y^{(\alpha)m}} + \underset{(\alpha)}{C}{}^i{}_{rm} K^r_j - \underset{(\alpha)}{C}{}^s{}_{jm} K^i_s \, , \ (\alpha = 1, ..., k) \end{aligned}\right\} \tag{10}$$

If $D\Gamma(N) = (L^i_{jm}, \ \underset{(\alpha)}{C}{}^m_{ij})$, $(\alpha = 1, ..., k)$ are the local components of a N-linear connection D on E, then we denote the d-tensor fields of torsion by: $T^r{}_{pq}$, $\underset{(0)}{\overset{(\alpha)}{R}}{}^r{}_{pq}$, $\underset{(\alpha)}{C}{}^r{}_{pq}$, $\underset{(\alpha)}{\overset{(\beta)}{P}}{}^r{}_{pq}$, $\underset{(\alpha)(\beta)}{\overset{(\gamma)}{P}}{}^r{}_{pq}$, $\underset{(\alpha)}{S}{}^r{}_{pq}$, $\underset{(\alpha)}{\overset{(\beta-\alpha)}{R}}{}^r{}_{pq}$ and

the d-tensor fields of curvature by $R_r{}^m{}_{pq}$, $\underset{(\alpha)}{P}{}_r{}^m{}_{pq}$, $\underset{(\alpha)(\beta)}{P}{}_r{}^m{}_{pq}$, $\underset{(\alpha)}{S}{}_r{}^m{}_{pq}$

$(\alpha, \beta, \gamma = 1, .., k)$.

We consider an almost complex structure F on E:

$$F \circ F = -I \tag{11}$$

Its integrability tensor field $\tilde{N}$ is given by:

$$\tilde{N}(X,Y) = [FX, FY] - F[FX, Y] - F[X, FY] - [X, Y] \quad \forall X, Y \in \mathcal{X}(E) \tag{12}$$

In the adapted basis to (2), F can be represented by:

$$\begin{aligned} F \;=\; & \overset{\circ\circ}{F}{}^i_j \frac{\delta}{\delta x^i} \otimes dx^j + \sum_{\alpha=1}^{k} \overset{\circ\alpha}{F}{}^i_j \frac{\delta}{\delta x^i} \otimes \delta y^{(\alpha)j} + \\ & + \sum_{\alpha=1}^{k} \overset{\alpha\circ}{F}{}^i_j \frac{\delta}{\delta y^{(\alpha)i}} \otimes dx^j + \sum_{\alpha=1}^{k} \sum_{\beta=1}^{k} \overset{\alpha\beta}{F}{}^i_j \frac{\delta}{\delta y^{(\alpha)i}} \otimes \delta y^{(\beta)j} \end{aligned} \tag{13}$$

In this expression $\overset{\circ\circ}{F}{}^i_j$, $\overset{\circ\alpha}{F}{}^i_j$, $\overset{\alpha\circ}{F}{}^i_j$, $\overset{\alpha\beta}{F}{}^i_j$ $(\alpha, \beta = 1, ..., k)$ are the d-tensor fields on E.

Then, for $\beta = 1, ..., k$ we have:

$$F\left(\frac{\delta}{\delta x^j}\right) = \overset{\circ\circ}{F}{}^i_j \frac{\delta}{\delta x^i} + \sum_{\alpha=1}^{k} \overset{\alpha\circ}{F}{}^i_j \frac{\delta}{\delta y^{(\alpha)i}} \quad , \quad F\left(\frac{\delta}{\delta y^{(\beta)j}}\right) = \overset{\circ\beta}{F}{}^i_j \frac{\delta}{\delta x^i} + \sum_{\alpha=1}^{k} \overset{\alpha\beta}{F}{}^i_j \frac{\delta}{\delta y^{(\beta)i}} \tag{14}$$

and the condition (11) is equivalent to:

$$\left.\begin{array}{ll} \overset{\circ\circ}{F}{}^r_i \overset{\circ\circ}{F}{}^j_r + \sum\limits_{\beta=1}^{k} \overset{\circ\beta}{F}{}^r_i \overset{\beta\circ}{F}{}^j_r = -\delta^j_i & \overset{\gamma\circ}{F}{}^r_i \overset{\circ\alpha}{F}{}^j_r + \sum\limits_{\beta=1}^{k} \overset{\gamma\beta}{F}{}^r_i \overset{\beta\alpha}{F}{}^j_r = 0 \\ \overset{\alpha\circ}{F}{}^r_i \overset{\circ\gamma}{F}{}^j_r + \sum\limits_{\beta=1}^{k} \overset{\alpha\beta}{F}{}^r_i \overset{\beta\gamma}{F}{}^j_r = 0 & \overset{\alpha\circ}{F}{}^r_i \overset{\circ\alpha}{F}{}^j_r + \sum\limits_{\beta=1}^{k} \overset{\alpha\beta}{F}{}^r_i \overset{\beta\alpha}{F}{}^j_r = -\delta^j_i \\ \forall \gamma = 0, 1, ..., k \quad , \quad \forall \alpha = 1, ..., k \quad , \quad \gamma < \alpha & \end{array}\right\} \tag{15}$$

The components of $\tilde{N}$ with respect to the adapted basis can be easily obtained. F is a complex structure on E if and only if $\tilde{N} = 0$.

2 Almost complex d-structures

Let us consider a manifold M having the dimension n=2m.

Definition 2.1 *A d-tensor field $f^i_j(x, y^{(1)}, ..., y^{(k)})$ of type (1,1) is called an almost complex d-structure on $E = Osc^k M$ if it satisfies the property:*

$$f^r_i f^j_r = -\delta^j_i \tag{16}$$

Obviously we have $rank\|f\| = 2m$.

Then the d-tensor fields:

$$\underset{1}{Q}{}^{pq}_{ij} = \frac{1}{2}(\delta^p_i \delta^q_j - f^p_i f^q_j) \quad , \quad \underset{2}{Q}{}^{pq}_{ij} = \frac{1}{2}(\delta^p_i \delta^q_j + f^p_i f^q_j) \tag{17}$$

are called the Obata operators of the d-structure f.

They have the properties:

$$\underset{1}{Q} + \underset{2}{Q} = I \,, \quad \underset{1}{Q}\,\underset{1}{Q} = \underset{1}{Q} \,, \quad \underset{2}{Q}\,\underset{2}{Q} = \underset{2}{Q} \,, \quad \underset{1}{Q}\,\underset{2}{Q} = \underset{2}{Q}\,\underset{1}{Q} = 0 \tag{18}$$

Definition 2.2 *A N-linear connection $D\Gamma(N)$ is called an almost complex N-linear connection with respect to the almost complex d-structure f, if the h- and v_α-covariant derivatives of f vanish:*

$$f^i_{j|m} \quad , \quad \overset{(\alpha)}{f^i_{j\,|\,m}} = 0 \quad , \quad (\alpha = 1, ..., k). \tag{19}$$

Theorem 2.1 *a. Obata tensor fields $\underset{1}{Q}{}^{pq}_{ij}$ and $\underset{2}{Q}{}^{pq}_{ij}$ are covariant constant with respect to any almost complex N-linear connection $D\Gamma(N)$.*

b. The d-tensor fields

$$\underset{2}{Q}{}^{ir}_{sj} R_r{}^s{}_{pq} \,, \quad \underset{2}{Q}{}^{ir}_{sj} \underset{(\alpha)}{P}{}_r{}^s{}_{pq} \,, \quad \underset{2}{Q}{}^{ir}_{sj} \underset{(\alpha\beta)}{P}{}_r{}^s{}_{pq} \,, \quad \underset{2}{Q}{}^{ir}_{sj} \underset{(\alpha)}{S}{}_r{}^s{}_{pq} \,,$$

$(\alpha, \beta = 1, ..., k)$ and their h- and v_α-covariant derivative of every order vanish for every $D\Gamma(N)$ with the property (19).

Theorem 2.2 *If on E there exists a N-linear connection*

$$D\overset{\circ}{\Gamma}(N) = (\overset{\circ}{L}, \underset{(\alpha)}{\overset{\circ}{C}}) \quad (\alpha = 1, ..., k),$$

then there exist the almost complex N-linear connections on E with respect to the d-structure f. One of these is:

$$\left.\begin{aligned} L^m_{ij} &= \overset{\circ}{L}{}^m_{ij} + \tfrac{1}{2} f^r_i f^m_{r\overset{\circ}{|}j}, \\ \underset{(\alpha)}{C}{}^m_{ij} &= \underset{(\alpha)}{\overset{\circ}{C}}{}^m_{ij} + \tfrac{1}{2} f^r_i f^m_{r\overset{\circ}{\underset{(\alpha)}{|}}j}, \quad (\alpha = 1, ..., k) \end{aligned}\right\} \tag{20}$$

where $\overset{\circ}{|}$ *and* $\overset{(\alpha)}{\overset{\circ}{|}}$ *denote the h- and* v_α*-covariant derivatives with respect to* $D\overset{\circ}{\Gamma}(N)$.

Theorem 2.3 *The set of all almost complex N-linear connections* $D\Gamma(N)$ *is given by:*

$$\left.\begin{aligned} L^m_{ij} &= \overset{\circ}{L}{}^m_{ij} + \tfrac{1}{2} f^r_i f^m_{r\overset{\circ}{|}j} + \underset{1}{Q}{}^{rm}_{is} Y^s_{rj} \\ \underset{(\alpha)}{C}{}^m_{ij} &= \underset{(\alpha)}{\overset{\circ}{C}}{}^m_{ij} + \tfrac{1}{2} f^r_i f^m_{r\overset{\circ}{\underset{(\alpha)}{|}}j} + \underset{1}{Q}{}^{rm}_{is} \underset{(\alpha)}{Z}{}^s_{rj}, \quad (\alpha = 1, ..., k) \end{aligned}\right\} \tag{21}$$

where Y^m_{ij} *and* $\underset{(\alpha)}{Z}{}^m_{ij}$, $(\alpha = 1, ..., k)$ *are arbitrary d-tensor fields.*

3 The group of transformations of almost complex N-linear connection in the bundle of acceleration

Let us consider the transformations $D\Gamma(N) \to D\bar{\Gamma}(N)$ of almost complex N-linear connections. Owing to the Theorem 2.3 they are given by:

$$\bar{L}^m_{ij} = L^m_{ij} + \underset{1}{Q}{}^{rm}_{is} Y^s_{rj}, \qquad \underset{(\alpha)}{\bar{C}}{}^m_{ij} = \underset{(\alpha)}{C}{}^m_{ij} + \underset{1}{Q}{}^{rm}_{is} \underset{(\alpha)}{Z}{}^s_{rj}, \tag{22}$$

$$(\alpha = 1, ..., k)$$

Theorem 3.1 *The set of all transformations (22) with the mapping product as law of composition form an Abelian group G_{ac}, which is isomorphic with the additive group of d-tensor fields (* $\underset{1}{Q}{}^{rm}_{is} Y^s_{rj}$, $\underset{1}{Q}{}^{rm}_{is} \underset{(1)}{Z}{}^s_{rj}$, ..., $\underset{1}{Q}{}^{rm}_{is} \underset{(k)}{Z}{}^s_{rj}$ *).*

Now, we investigate the case k=2, i.e. the case of the bundle of accelerations, Osc^2M.

We shall pay attention to the invariants of the group G_{ac} for $E = Osc^2M$. By direct calculation we have:

Theorem 3.2 *The following d-tensor fields are invariants of the group G_{ac} in the case of the bundle of accelerations:*

$$\tag{23}$$

$$\overset{10}{T}{}^m_{ij} = 4\underset{(2)}{Q}{}^{pm}_{iq}\underset{(1)}{Q}{}^{rs}_{pj}T^q_{rs}$$

$$\underset{(0)}{\overset{11}{R}}{}^m_{ij} = \underset{(0)}{\overset{(1)}{R}}{}^m_{ij} - f^r_i f^s_j \underset{(0)}{\overset{(1)}{R}}{}^m_{rs} + ab f^m_q (f^r_i \underset{(0)}{\overset{(1)}{R}}{}^q_{rj} + f^s_j \underset{(0)}{\overset{(1)}{R}}{}^q_{is})$$

$$\underset{(0)}{\overset{12}{R}}{}^m_{ij} = \underset{(0)}{\overset{(2)}{R}}{}^m_{ij} - f^r_i f^s_j \underset{(0)}{\overset{(2)}{R}}{}^m_{rs} + ac f^m_q (f^r_i \underset{(0)}{\overset{(2)}{R}}{}^q_{rj} + f^s_j \underset{(0)}{\overset{(2)}{R}}{}^q_{is})$$

$$\underset{(1)}{\overset{10}{C}}{}^m_{ij} = \underset{(1)}{C}{}^m_{ij} - ab f^r_i f^s_j \underset{(1)}{C}{}^m_{rs} + f^m_q (f^r_i \underset{(1)}{C}{}^q_{rj} + ab f^s_j \underset{(1)}{C}{}^q_{is})$$

$$\underset{(1)}{\overset{11}{P}}{}^m_{ij} = \underset{(1)}{\overset{(1)}{P}}{}^m_{ij} - ab f^r_i f^s_j \underset{(1)}{\overset{(1)}{P}}{}^m_{rs} + f^m_q (ab f^r_i \underset{(1)}{\overset{(1)}{P}}{}^q_{rj} + f^s_j \underset{(1)}{\overset{(1)}{P}}{}^q_{is})$$

$$\underset{(1)}{\overset{12}{P}}{}^m_{ij} = \underset{(1)}{\overset{(2)}{P}}{}^m_{ij} - ab f^r_i f^s_j \underset{(1)}{\overset{(2)}{P}}{}^m_{rs} + c f^m_q (a f^r_i \underset{(1)}{\overset{(2)}{P}}{}^q_{rj} + b f^s_j \underset{(1)}{\overset{(2)}{P}}{}^q_{is})$$

$$\underset{(2)}{\overset{10}{C^m_{ij}}} = \underset{(2)}{C^m_{ij}} - ac f^r_i f^s_j \underset{(2)}{C^m_{rs}} + f^m_q \left(f^r_i \underset{(2)}{C^q_{rj}} + ac f^s_j \underset{(2)}{C^q_{is}}\right)$$

$$\underset{(2)}{\overset{11}{P^m_{ij}}} = \underset{(2)}{\overset{(1)}{P^m_{ij}}} - ac f^r_i f^s_j \underset{(2)}{\overset{(1)}{P^m_{rs}}} + b f^m_q \left(a f^r_i \underset{(2)}{\overset{(1)}{P^q_{rj}}} + c f^s_j \underset{(2)}{\overset{(1)}{P^q_{is}}}\right)$$

$$\underset{(0)}{\overset{12}{R^m_{ij}}} = \underset{(0)}{\overset{(2)}{R^m_{ij}}} - f^r_i f^s_j \underset{(0)}{\overset{(2)}{R^m_{rs}}} + ac f^m_q \left(f^r_i \underset{(0)}{\overset{(2)}{R^q_{rj}}} + f^s_j \underset{(0)}{\overset{(2)}{R^q_{is}}}\right)$$

$$\underset{(1)}{\overset{11}{S^m_{ij}}} = 4 \underset{(2)}{Q^{pm}_{iq}} \underset{(1)}{Q^{rs}_{pj}} \underset{(1)}{S^q_{rs}}$$

$$\underset{(1)}{\overset{12}{S^m_{ij}}} = \underset{(1)}{\overset{(2)}{S^m_{ij}}} - f^r_i f^s_j \underset{(1)}{\overset{(2)}{S^m_{rs}}} + bc f^m_q \left(f^r_i \underset{(1)}{\overset{(2)}{S^q_{rj}}} + f^s_j \underset{(1)}{\overset{(2)}{S^q_{is}}}\right)$$

$$\underset{(2)}{\overset{11}{C^m_{ij}}} = \underset{(2)}{C^m_{ij}} - bc f^r_i f^s_j \underset{(2)}{C^m_{rs}} + f^m_q \left(f^r_i \underset{(2)}{C^q_{rj}} + bc f^s_j \underset{(2)}{C^q_{is}}\right)$$

$$\underset{(2)}{\overset{12}{I^m_{ij}}} = \underset{(2)}{\overset{(1)}{P^m_{ij}}} - \underset{(1)}{C^m_{ji}} + bc f^r_i f^s_j \left(\underset{(2)}{\overset{(1)}{P^m_{rs}}} - \underset{(1)}{C^m_{sr}}\right) +$$
$$+ f^m_q \left[bc f^r_i \left(\underset{(2)}{\overset{(1)}{P^q_{rj}}} - \underset{(1)}{C^q_{jr}}\right) + f^s_j \left(\underset{(2)}{\overset{(1)}{P^q_{is}}} - \underset{(1)}{C^q_{si}}\right)\right]$$

$$\underset{(1)}{\overset{12}{S^m_{ij}}} = 4 \underset{(2)}{Q^{pm}_{iq}} \underset{(2)}{Q^{rs}_{pj}} \underset{(1)}{S^q_{rs}}$$

where $a^2 = b^2 = c^2 = 1$;

(24)

$$\overset{20}{T}{}^m_{ij} = T^m_{ij} - f^m_q\big(f^r_i \overset{(2)}{\underset{(2)}{P}}{}^q_{jr} - f^s_j \overset{(2)}{\underset{(2)}{P}}{}^q_{is}\big)$$

$$\overset{21}{\underset{(0)}{R}}{}^m_{ij} = \overset{(1)}{\underset{(0)}{R}}{}^m_{ij} - bc f^m_q\big(f^r_i \overset{(1)}{\underset{(2)}{P}}{}^q_{jr} - f^s_j \overset{(1)}{\underset{(2)}{P}}{}^q_{is}\big)$$

$$\overset{22}{\underset{(0)}{R}}{}^m_{ij} = \overset{(2)}{\underset{(0)}{R}}{}^m_{ij} + c^2 f^r_i f^s_j \underset{(2)}{S}{}^m_{sr} - c^2 f^m_q\big(f^r_i \underset{(2)}{C}{}^q_{jr} - f^s_j \underset{(2)}{C}{}^q_{is}\big)$$

$$\overset{20}{\underset{(1)}{C}}{}^m_{ij} = \underset{(1)}{C}{}^m_{ij} - f^m_q\Big[f^r_i\big(\overset{(1)}{\underset{(2)}{P}}{}^q_{jr} - \underset{(1)}{C}{}^q_{rj}\big) - ab f^s_j \overset{(2)}{\underset{(1)}{P}}{}^q_{is}\Big]$$

$$\overset{21}{\underset{(1)}{P}}{}^m_{ij} = \overset{(1)}{\underset{(1)}{P}}{}^m_{ij} + bc f^r_i f^s_j \underset{(2)}{C}{}^m_{sr} - f^m_q\big(bc f^r_i \underset{(2)}{C}{}^q_{jr} - f^s_j \overset{(1)}{\underset{(1)}{P}}{}^q_{is}\big)$$

$$\overset{22}{\underset{(1)}{P}}{}^m_{ij} = \overset{(1)}{\underset{(2)}{P}}{}^m_{ij} + bc f^r_i f^s_j \big(\overset{(1)}{\underset{(2)}{P}}{}^m_{sr} - \underset{(1)}{C}{}^m_{rs}\big) + bc f^m_q f^s_j \underset{(1)}{C}{}^q_{is}$$

$$\overset{20}{\underset{(2)}{C}}{}^m_{ij} = \underset{(2)}{C}{}^m_{ij} + f^r_i f^s_j \underset{(2)}{C}{}^m_{sr} - f^m_q\big(f^r_i \underset{(2)}{S}{}^q_{jr} - a^2 f^s_j \overset{(2)}{\underset{(0)}{R}}{}^q_{is}\big)$$

$$\overset{21}{\underset{(2)}{P}}{}^m_{ij} = \overset{(1)}{\underset{(2)}{P}}{}^m_{ij} + f^r_i f^s_j \overset{(1)}{\underset{(2)}{P}}{}^m_{sr} + ab f^m_q f^s_j \overset{(2)}{\underset{(0)}{R}}{}^q_{is}$$

$$\overset{22}{\underset{(2)}{P}}{}^m_{ij} = \overset{(2)}{\underset{(2)}{P}}{}^m_{ij} + f^r_i f^s_j \overset{(2)}{\underset{(2)}{P}}{}^m_{sr} + f^m_q f^s_j T^q_{is}$$

$$\underset{(1)}{\overset{21}{S^m_{ij}}} = f^m_q\left(f^r_i \underset{(2)}{\overset{(1)}{S}}{}^q{}_{rj} + f^s_j \underset{(2)}{\overset{(1)}{S}}{}^q{}_{is}\right)$$

$$\underset{(1)}{\overset{22}{S^m_{ij}}} = \underset{(2)}{\overset{(1)}{S^m_{ij}}} - f^r_i f^s_j \underset{(2)}{\overset{(1)}{S}}{}^m{}_{rs}$$

$$\overset{20}{I^m_{ij}} = ab f^r_i f^s_j \underset{(1)}{C}{}^m{}_{sr} + a f^m_q\left[b f^r_i \left(\underset{(2)}{\overset{(1)}{P}}{}^q{}_{rj} - \underset{(1)}{C}{}^q{}_{jr}\right) - a f^j_s \underset{(1)}{\overset{(2)}{P}}{}^q{}_{si}\right]$$

$$\overset{21}{I^m_{ij}} = \underset{(2)}{C}{}^m{}_{ij} + ab f^r_i f^s_j \underset{(1)}{\overset{(1)}{P}}{}^m{}_{sr} + f^m_q\left(f^r_i \underset{(2)}{C}{}^q{}_{rj} - ab f^s_j \underset{(1)}{\overset{(1)}{P}}{}^q{}_{si}\right)$$

$$\overset{22}{I^m_{ij}} = -\left(\underset{(2)}{\overset{(1)}{P}}{}^m{}_{ij} - C^m{}_{ij}\right) - ab f^r_i f^s_j \underset{(1)}{\overset{(2)}{P}}{}^m{}_{sr} + f^m_q f^s_j \underset{(1)}{C}{}^q{}_{si}$$

$$\underset{(2)}{\overset{20}{S^m_{ij}}} = f^r_i f^s_j T^m{}_{rs} - f^m_q\left(f^r_i \underset{(2)}{\overset{(2)}{P}}{}^m{}_{rj} - f^s_j \underset{(2)}{\overset{(2)}{P}}{}^q{}_{si}\right)$$

$$\underset{(2)}{\overset{21}{S^m_{ij}}} = a f^r_i f^s_j \underset{(0)}{\overset{(1)}{R}}{}^m{}_{rs} - b f^m_q\left(f^r_i \underset{(2)}{\overset{(1)}{P}}{}^q{}_{rj} - f^s_j \underset{(2)}{\overset{(1)}{P}}{}^q{}_{si}\right)$$

$$\underset{(2)}{\overset{22}{S^m_{ij}}} = \underset{(2)}{S^m_{ij}} - a^2 f^r_i f^s_j \underset{(0)}{\overset{(2)}{R}}{}^m{}_{rs} + f^m_q\left(f^r_i \underset{(2)}{C}{}^q{}_{rj} - f^s_j \underset{(2)}{C}{}^q{}_{si}\right)$$

where $ac = 1$, $b^2 = 1$;

(25)

$$\underset{(0)}{\overset{31}{R^m_{ij}}} = \underset{(0)}{\overset{(1)}{R^m{}_{ij}}} - f^r_i f^s_j \underset{(0)}{\overset{(1)}{R^m{}_{rs}}} + ab f^m_q \big(f^r_i \underset{(0)}{\overset{(2)}{R^q{}_{rj}}} + f^s_j \underset{(0)}{\overset{(2)}{R^q{}_{is}}} \big)$$

$$\underset{(0)}{\overset{32}{R^m_{ij}}} = \underset{(0)}{\overset{(2)}{R^m{}_{ij}}} - f^r_i f^s_j \underset{(1)}{\overset{(2)}{R^m{}_{rs}}} + ac f^m_q \big(f^r_i \underset{(0)}{\overset{(1)}{R^q{}_{rj}}} + f^s_j \underset{(0)}{\overset{(2)}{R^q{}_{is}}} \big)$$

$$\underset{(1)}{\overset{30}{C^m_{ij}}} = \underset{(1)}{C^m{}_{ij}} - ac f^r_i f^s_j \underset{(2)}{C^m{}_{rs}} + f^m_q \big(f^r_i \underset{(1)}{C^q{}_{rj}} + ac f^s_j \underset{(2)}{C^q{}_{is}} \big)$$

$$\underset{(1)}{\overset{31}{P^m_{ij}}} = \underset{(1)}{\overset{(1)}{P^m{}_{ij}}} - ac f^r_i f^s_j \underset{(2)}{C^m{}_{rs}} + f^m_q \big(ab f^r_i \underset{(1)}{\overset{(2)}{P^q{}_{rj}}} + f^s_j \underset{(2)}{\overset{(2)}{P^q{}_{is}}} \big)$$

$$\underset{(1)}{\overset{32}{P^m_{ij}}} = \underset{(1)}{\overset{(2)}{P^m{}_{ij}}} - ac f^r_i f^s_j \underset{(2)}{\overset{(2)}{P^m{}_{rs}}} + c f^m_q \big(a f^r_i \underset{(1)}{\overset{(1)}{P^q{}_{rj}}} + f^s_j \underset{(2)}{\overset{(1)}{P^q{}_{is}}} \big)$$

$$\underset{(2)}{\overset{30}{C^m_{ij}}} = \underset{(2)}{C^m{}_{ij}} - ab f^r_i f^s_j \underset{(1)}{C^m{}_{rs}} + f^m_q \big(f^r_i \underset{(2)}{C^q{}_{rj}} + ab f^s_j \underset{(1)}{C^q{}_{is}} \big)$$

$$\underset{(2)}{\overset{31}{P^m_{ij}}} = \underset{(2)}{\overset{(1)}{P^m{}_{ij}}} - ab f^r_i f^s_j \underset{(1)}{\overset{(1)}{P^m{}_{rs}}} + b f^m_q \big(a f^r_i \underset{(2)}{\overset{(2)}{P^m{}_{rj}}} + b f^s_j \underset{(1)}{\overset{(2)}{P^q{}_{is}}} \big)$$

$$\underset{(2)}{\overset{32}{P^m_{ij}}} = \underset{(2)}{\overset{(2)}{P^m{}_{ij}}} - ab f^r_i f^s_j \underset{(1)}{\overset{(2)}{P^m{}_{rs}}} + f^m_q \big(ac f^r_i \underset{(1)}{\overset{(2)}{P^q{}_{rj}}} + f^s_j \underset{(1)}{\overset{(1)}{P^q{}_{is}}} \big)$$

$$\underset{(1)}{\overset{31}{S^m_{ij}}} = \underset{(2)}{S^m{}_{ij}} + f^m_q \big[f^r_i \big(\underset{(2)}{\overset{(1)}{P^q{}_{rj}}} - \underset{(1)}{C^q{}_{jr}} \big) - f^s_j \big(\underset{(2)}{\overset{(1)}{P^q{}_{si}}} - \underset{(1)}{C^q{}_{is}} \big) \big]$$

$$\underset{(1)}{\overset{32}{S^m_{ij}}} = f^r_i f^s_j \underset{(2)}{S^m{}_{rs}} - f^m_q \big(f^r_i \underset{(2)}{C^q{}_{jr}} + f^s_j \underset{(2)}{C^q{}_{is}} \big)$$

$$\overset{31}{I}{}^m_{ij} = \underset{(2)}{C}{}^m_{ij} + f^r_i f^s_j \underset{(2)}{C}{}^m_{sr} + f^m_q (f^r_i \underset{(2)}{S}{}^q{}_{jr} - b^2 f^s_j \underset{(2)}{\overset{(1)}{S}}{}^q{}_{is})$$

$$\overset{32}{I}{}^m_{ij} = (\underset{(2)}{\overset{(1)}{P}}{}^m_{ji} - \underset{(1)}{C}{}^m_{ij}) + f^r_i f^s_j (\underset{(2)}{\overset{(1)}{P}}{}^m_{rs} - \underset{(1)}{C}{}^m_{sr}) + f^m_q f^s_j \underset{(1)}{S}{}^q{}_{is}$$

$$\underset{(2)}{\overset{31}{S}}{}^m_{ij} = f^r_i f^s_j \underset{(1)}{S}{}^m{}_{rs} - f^m_q [f^r_i (\underset{(2)}{\overset{(1)}{P}}{}^m_{jr} - \underset{(1)}{C}{}^q{}_{rj}) - f^s_j (\underset{(2)}{\overset{(1)}{P}}{}^q_{is} - \underset{(1)}{C}{}^q{}_{si})]$$

$$\underset{(2)}{\overset{32}{S}}{}^m_{ij} = \underset{(2)}{S}{}^m_{ji} + b^2 f^r_i f^s_j \underset{(2)}{\overset{(1)}{S}}{}^m_{rs} - f^m_q (f^r_i \underset{(2)}{C}{}^q{}_{rj} - f^s_j \underset{(2)}{C}{}^q{}_{si})$$

where $a^2 = 1 \quad , \quad bc = 1;$

$$(26)$$

$$\overset{40}{T}{}^m_{ij} = T^m_{ij} - f^m_q (f^r_i \underset{(1)}{\overset{(1)}{P}}{}^q{}_{jr} - f^s_j \underset{(1)}{\overset{(1)}{P}}{}^q{}_{is})$$

$$\underset{(0)}{\overset{41}{R}}{}^m_{ij} = \underset{(0)}{\overset{(1)}{R}}{}^m{}_{ij} - b^2 f^r_i f^s_j \underset{(1)}{S}{}^m{}_{rs} - b^2 f^m_q (f^r_i \underset{(1)}{C}{}^q{}_{jr} - f^s_j \underset{(1)}{C}{}^q{}_{is})$$

$$\underset{(0)}{\overset{42}{R}}{}^m_{ij} = \underset{(0)}{\overset{(2)}{R}}{}^m{}_{ij} - b^2 f^r_i f^s_j \underset{(2)}{\overset{(1)}{S}}{}^m{}_{rs} - bc f^m_q (f^r_i \underset{(1)}{\overset{(2)}{P}}{}^q{}_{jr} - f^s_j \underset{(1)}{\overset{(2)}{P}}{}^q{}_{is})$$

$$\underset{(1)}{\overset{40}{C}}{}^m_{ij} = \underset{(1)}{C}{}^m_{ij} + f^r_i f^s_j \underset{(1)}{C}{}^m{}_{sr} + f^m_q (f^r_i \underset{(1)}{S}{}^m{}_{rj} + a^2 f^s_j \underset{(0)}{\overset{(1)}{R}}{}^q{}_{is})$$

$$\underset{(1)}{\overset{41}{P}}{}^m_{ij} = \underset{(1)}{\overset{(1)}{P}}{}^m_{ij} + f^r_i f^s_j \underset{(1)}{\overset{(1)}{P}}{}^m_{sr} + f^m_q f^s_j T^q{}_{is}$$

$$\overset{42}{\underset{(1)}{P^m_{ij}}} = \overset{(2)}{\underset{(1)}{P^m_{ij}}} + f^r_i f^s_j \overset{(2)}{\underset{(1)}{P^m_{rs}}} + c f^m_q \big(b f^r_i \overset{(1)}{\underset{(2)}{S^q_{rj}}} + a f^s_j \overset{(2)}{\underset{(0)}{R^q_{is}}}\big)$$

$$\overset{40}{\underset{(2)}{C^m_{ij}}} = \underset{(2)}{C^m_{ij}} + f^m_q \big(f^r_i \underset{(2)}{C^q_{rj}} + ac f^s_j \overset{(1)}{\underset{(2)}{P^q_{is}}}\big)$$

$$\overset{41}{\underset{(2)}{P^m_{ij}}} = \overset{(1)}{\underset{(2)}{P^m_{ij}}} - bc f^r_i f^s_j \underset{(2)}{C^m_{rs}} + bc f^m_q f^s_j \underset{(2)}{C^q_{rj}}$$

$$\overset{42}{\underset{(2)}{P^m_{ij}}} = bc f^r_i f^s_j \big(\overset{(1)}{\underset{(2)}{P^m_{sr}}} - \underset{(1)}{C^m_{rs}}\big) - f^m_q \Big[bc f^r_i \big(\overset{(1)}{\underset{(2)}{P^q_{jr}}} - \underset{(1)}{C^q_{rj}}\big) - f^s_j \overset{(2)}{\underset{(2)}{P^q_{si}}}\Big]$$

$$\overset{40}{\underset{(1)}{S^m_{ij}}} = -f^r_i f^s_j T^m_{rs} + f^m_q \big(f^r_i \overset{(1)}{\underset{(1)}{P^q_{rj}}} - f^s_j \overset{(1)}{\underset{(1)}{P^q_{si}}}\big)$$

$$\overset{41}{\underset{(1)}{S^m_{ij}}} = \underset{(2)}{S^m_{ij}} - a^2 f^r_i f^s_j \overset{(1)}{\underset{(0)}{R^m_{rs}}} + f^m_q \big(f^r_i \underset{(1)}{C^q_{rj}} - f^s_j \underset{(1)}{C^q_{si}}\big)$$

$$\overset{42}{\underset{(1)}{S^m_{ij}}} = \overset{(1)}{\underset{(2)}{S^m_{ij}}} - a^2 f^r_i f^s_j \overset{(2)}{\underset{(0)}{R^m_{rs}}} + ac f^m_q \big(f^r_i \overset{(2)}{\underset{(1)}{P^q_{rj}}} - f^s_j \overset{(2)}{\underset{(1)}{P^q_{si}}}\big)$$

$$\overset{40}{I^m_{ij}} = c f^r_i f^s_j \underset{(2)}{C^m_{rs}} - f^m_q \big(a f^r_i (\overset{(1)}{\underset{(2)}{P^q_{rj}}} + c f^s_j \underset{(2)}{C^q_{si}}\big)$$

$$\overset{41}{I^m_{ij}} = \underset{(2)}{C^m_{ij}} - ac f^r_i f^s_j \overset{(1)}{\underset{(2)}{P^m_{rs}}} + f^m_q f^r_i \underset{(2)}{C^q_{rj}}$$

$$\overset{42}{I}{}^{m}_{ij\cdot} = (\underset{(2)}{\overset{(1)}{P}}{}^m_{ij} - \underset{(1)}{C}{}^m_{ij}) - acf^r_i f^s_j \underset{(2)}{\overset{(2)}{P}}{}^m_{rs} + f^m_q[acf^r_i \underset{(2)}{\overset{(2)}{P}}{}^q_{rj} + f^s_j(\underset{(2)}{\overset{(1)}{P}}{}^q_{si} - \underset{(1)}{C}{}^q_{is})]$$

where $ab = 1$, $c^2 = 1$;

4 The integrability of an almost complex d-structure in the bundle of accelerations

Using the ideas from Irena Čomić's recent papers [2], [3], an almost complex d-structure $f^i_j(x, y^{(1)}, y^{(2)})$ on the bundle of accelerations Osc^2M can be lifted to an almost complex structure F on $T(Osc^2M)$ in the following manners:

$$F^I = af^i_j\frac{\delta}{\delta x^i} \otimes dx^j + bf^i_j\frac{\delta}{\delta y^{(1)i}} \otimes \delta y^{(1)j} + cf^i_j\frac{\partial}{\partial y^{(2)i}} \otimes \delta y^{(2)j} \tag{27}$$

where $a^2 = b^2 = c^2 = 1$,

$$F^{II} = af^i_j\frac{\delta}{\delta x^i} \otimes \delta y^{(2)j} + bf^i_j\frac{\delta}{\delta y^{(1)i}} \otimes \delta y^{(1)j} + cf^i_j\frac{\partial}{\partial y^{(2)i}} \otimes dx^i \tag{28}$$

where $ac = 1$, $b^2 = 1$,

$$F^{III} = af^i_j\frac{\delta}{\delta x^i} \otimes dx^j + bf^i_j\frac{\delta}{\delta y^{(1)i}} \otimes \delta y^{(2)j} + cf^i_j\frac{\partial}{\partial y^{(2)i}} \otimes \delta y^{(1)j} \tag{29}$$

where $a^2 = 1$, $bc = 1$, and

$$F^{IV} = af^i_j\frac{\delta}{\delta x^i} \otimes \delta y^{(1)j} + bf^i_j\frac{\delta}{\delta y^{(1)i}} \otimes dx^j + cf^i_j\frac{\partial}{\partial y^{(2)i}} \otimes \delta y^{(2)j} \tag{30}$$

where $ab = 1$, $c^2 = 1$

Then the values of the distinguished components of F, from (13) and $k = 2$, are given in Table 1.

Table 1: Distinguished components of F

F	$\overset{00}{F}$	$\overset{01}{F}$	$\overset{02}{F}$	$\overset{10}{F}$	$\overset{11}{F}$	$\overset{12}{F}$	$\overset{20}{F}$	$\overset{21}{F}$	$\overset{22}{F}$
F^I	af^i_j	0	0	0	bf^i_j	0	0	0	cf^i_j
F^{II}	0	0	af^i_j	0	bf^i_j	0	cf^i_j	0	0
F^{III}	af^i_j	0	0	0	0	bf^i_j	0	cf^i_j	0
F^{IV}	0	af^i_j	0	bf^i_j	0	0	0	0	cf^i_j

Definition 4.1 *An almost complex d-structure f on the bundle of accelerations is called integrable of the type I, II, III or IV with respect to the nonlinear connection N, if the corresponding lifted structures F^I, F^{II}, F^{III} or F^{IV} are integrable.*

We characterise these cases of integrability using only the invariants of the group G_{ac}.

Theorem 4.1 *The almost complex d-structure $f^i_j(x, y^{(1)}, y^{(2)})$ is integrable of the type I, II, III or IV if and only if the invariants of the group G_{ac} have the values given in Table 2.*

Proof. I:

The almost complex d-structure f is integrable of type I if and only if $\tilde{N}(X,Y) = 0 \quad , \quad \forall X, Y \in \mathcal{X}(Osc^2M)$. But $\tilde{N}(X,Y) = 0$ is equivalent to the equations:

$$\tilde{N}(\frac{\delta}{\delta x^i}, \frac{\delta}{\delta x^j}) = 0, \tilde{N}(\frac{\delta}{\delta x^i}, \frac{\delta}{\delta y^{(1)j}}) = 0, \tilde{N}(\frac{\delta}{\delta x^i}, \frac{\partial}{\partial y^{(2)j}}) = 0,$$

$$\tilde{N}(\frac{\delta}{\delta y^{(1)i}}, \frac{\delta}{\delta y^{(1)j}}) = 0, \tilde{N}(\frac{\delta}{\delta y^{(1)i}}, \frac{\partial}{\partial y^{(2)j}}) = 0, \tilde{N}(\frac{\partial}{\partial y^{(2)i}}, \frac{\partial}{\partial y^{(2)j}}) = 0,$$

which is equivalent to:

$$\overset{10}{T} = 0 \quad \underset{(1)}{\overset{10}{C}} = 0 \quad \underset{(2)}{\overset{10}{C}} = 0$$

$$\underset{(0)}{\overset{11}{R}} = 0 \quad \underset{(1)}{\overset{11}{P}} = 0 \quad \underset{(2)}{\overset{11}{P}} = 0 \quad \underset{(1)}{\overset{11}{S}} = 0 \quad \underset{(2)}{\overset{11}{C}} = 0$$

Table 2: Invariants of G_{ac}

Type	Characterisation
I	$\overset{10}{T}=0 \quad \underset{(1)}{\overset{10}{C}}=0 \quad \underset{(2)}{\overset{10}{C}}=0$
	$\underset{(0)}{\overset{11}{R}}=0 \quad \underset{(1)}{\overset{11}{P}}=0 \quad \underset{(2)}{\overset{11}{P}}=0 \quad \underset{(1)}{\overset{11}{S}}=0 \quad \underset{(2)}{\overset{11}{C}}=0$
	$\underset{(0)}{\overset{12}{R}}=0 \quad \underset{(1)}{\overset{12}{P}}=0 \quad \underset{(2)}{\overset{12}{P}}=0 \quad \underset{(1)}{\overset{12}{S}}=0 \quad \overset{12}{I}=0 \quad \underset{(2)}{\overset{12}{S}}=0$
II	$\overset{20}{T}=0 \quad \underset{(1)}{\overset{20}{C}}=0 \quad \underset{(2)}{\overset{20}{C}}=0 \quad \overset{20}{I}=0 \quad \underset{(2)}{\overset{20}{S}}=0$
	$\underset{(0)}{\overset{21}{R}}=0 \quad \underset{(1)}{\overset{21}{P}}=0 \quad \underset{(2)}{\overset{21}{P}}=0 \quad \underset{(1)}{\overset{21}{S}}=0 \quad \overset{21}{I}=0 \quad \underset{(2)}{\overset{21}{S}}=0$
	$\underset{(0)}{\overset{22}{R}}=0 \quad \underset{(1)}{\overset{22}{P}}=0 \quad \underset{(2)}{\overset{22}{P}}=0 \quad \underset{(1)}{\overset{22}{S}}=0 \quad \overset{22}{I}=0 \quad \underset{(2)}{\overset{22}{S}}=0$
III	$\underset{(1)}{\overset{30}{C}}=0 \quad \underset{(2)}{\overset{30}{C}}=0$
	$\underset{(0)}{\overset{31}{R}}=0 \quad \underset{(1)}{\overset{31}{P}}=0 \quad \underset{(2)}{\overset{31}{P}}=0 \quad \underset{(1)}{\overset{31}{S}}=0 \quad \overset{31}{I}=0 \quad \underset{(2)}{\overset{31}{S}}=0$
	$\underset{(0)}{\overset{32}{R}}=0 \quad \underset{(1)}{\overset{32}{P}}=0 \quad \underset{(2)}{\overset{32}{P}}=0 \quad \underset{(1)}{\overset{32}{S}}=0 \quad \overset{32}{I}=0 \quad \underset{(2)}{\overset{32}{S}}=0$
IV	$\overset{40}{T}=0 \quad \underset{(1)}{\overset{40}{C}}=0 \quad \underset{(2)}{\overset{40}{C}}=0 \quad \underset{(1)}{\overset{40}{S}}=0 \quad \overset{40}{I}=0$
	$\underset{(0)}{\overset{41}{R}}=0 \quad \underset{(1)}{\overset{41}{P}}=0 \quad \underset{(2)}{\overset{41}{P}}=0 \quad \underset{(1)}{\overset{41}{S}}=0 \quad \overset{41}{I}=0$
	$\underset{(0)}{\overset{42}{R}}=0 \quad \underset{(1)}{\overset{42}{P}}=0 \quad \underset{(2)}{\overset{42}{P}}=0 \quad \underset{(1)}{\overset{42}{S}}=0 \quad \overset{42}{I}=0$

$$\overset{12}{\underset{(0)}{R}} = 0 \quad \overset{12}{\underset{(1)}{P}} = 0 \quad \overset{12}{\underset{(2)}{P}} = 0 \quad \overset{12}{\underset{(1)}{S}} = 0 \quad \overset{12}{I} = 0 \quad \overset{12}{\underset{(2)}{S}} = 0$$

The proof of II, III, or IV follow the same pattern.

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