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SPECIAL ADAPTED BASIS IN OSC^3M

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Abstract

R. Miron in [8] and R. Miron and Gh. Atanasiu in [5], [6] and [7] studied the geometry of Osc^kM. Among many various problems which was solved, they introduced the adapted basis, the d-connection and gave its curvature theory.

Here the attention on $E = Osc^3M$ will be restricted. The coefficients of the nonlinear connection, $M^{(1)}$, $M^{(2)}$ and $M^{(3)}$ are determined in such a way that T_{V_3} is orthogonal to T_{V_0}, T_{V_1} and T_{V_2} with respect to the arbitrary but fixed nondegenerative metric G. The adapted basis constructed with such connections is unique.

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1 Adapted basis in $T(Osc^3M)$ and $T^*(Osc^3M)$

Let $E = Osc^3M$ be a $4n$ dimensional C^∞ manifold. In some local chart (U, φ) some point $u \in E$ has coordinates

$$(x^a, y^{1a}, y^{2a}, y^{3a}) = (y^{0a}, y^{1a}, y^{2a}, y^{3a}) = (y^{\alpha a}),$$

where $x^a = y^{0a}$ and

$$a, b, c, d, e, \ldots = 1, 2, \ldots, n, \quad \alpha, \beta, \gamma, \delta, \kappa, \ldots = 0, 1, 2, 3.$$

If in some other chart (U', φ') the point $u \in E$ has coordinates $(x^{a'}, y^{1a'}, y^{2a'}, y^{3a'})$, then in $U \cap U'$ the allowable coordinate transformation are given by:

$$\text{(a)} \quad x^{a'} = x^{a'}(x^1, x^2, \ldots, x^n) \tag{1}$$

$$(b)\quad y^{1a'} = \frac{\partial x^{a'}}{\partial x^a} y^{1a} = \frac{\partial y^{0a'}}{\partial y^{0a}} y^{1a}$$

$$(c)\quad 2y^{2a'} = \frac{\partial y^{1a'}}{\partial y^{0a}} y^{1a} + 2\frac{\partial y^{1a'}}{\partial y^{1a}} y^{2a}$$

$$(d)\quad 3y^{3a'} = \frac{\partial y^{2a'}}{\partial y^{0a}} y^{1a} + 2\frac{\partial y^{2a'}}{\partial y^{1a}} y^{2a} + 3\frac{\partial y^{2a'}}{\partial y^{2a}} y^{3a}.$$

Determination of the group of allowable coordinate transformations is the first step to construct some geometry. The second important step is the construction of the adapted basis in $T(E)$, which depends on the choice of the coefficients of the nonlinear connections, here denoted by N and M.

The following abbreviations

$$\partial_{\alpha a} = \frac{\partial}{\partial y^{\alpha a}},\ \alpha = 1,2,3,\ \text{and}\ \partial_a = \partial_{0a} = \frac{\partial}{\partial x^a} = \frac{\partial}{\partial y^{0a}} \tag{2}$$

will be used.

The natural basis $\bar{B}$ of $T(E)$ is

$$\bar{B} = \{\partial_{0a}, \partial_{1a}, \partial_{2a}, \partial_{3a}\} = \{\partial_{\alpha a}\}. \tag{3}$$

The elements of $\bar{B}$ with respect to (1) are not transformed as d-tensors. The natural basis $\bar{B}^*$ of $T^*(E)$ is

$$\bar{B}^* = \{dx^a, dy^{1a}, dy^{2a}, dy^{3a}\} = \{dy^{\alpha a}\}. \tag{4}$$

The adapted basis B^* of $T^*(E)$ is given by (as in [8])

$$B^* = \{\delta y^{0a}, \delta y^{1a}, \delta y^{2a}, \delta y^{3a}\}, \tag{5}$$

where

$$\begin{aligned}
\delta y^{0a} &= dx^a = dy^{0a} \\
\delta y^{1a} &= dy^{1a} + M^{(1)a}_{\ b} dy^{0b} \\
\delta y^{2a} &= dy^{2a} + M^{(1)a}_{\ b} dy^{1b} + M^{(2)a}_{\ b} dy^{0b} \\
\delta y^{3a} &= dy^{3a} + M^{(1)a}_{\ b} dy^{2b} + M^{(2)a}_{\ b} dy^{1b} + M^{(3)a}_{\ b} dy^{0b}.
\end{aligned} \tag{6}$$

Theorem 1.1 *The necessary and sufficient conditions that $\delta y^{\alpha a}$ are transformed as d-tensor field, i.e.*

$$\delta y^{\alpha a'} = \frac{\partial x^{a'}}{\partial x^a} \delta y^{\alpha a},\ \alpha = 0,1,2,3,$$

are the following equations:

$$\begin{aligned}
&(a)\quad M^{(1)a}_{\ \ b}\partial_a x^{a'} = M^{(1)a'}_{\ \ b'}\partial_b x^{b'} + \partial_b y^{1a'} \\
&(b)\quad M^{(2)a}_{\ \ b}\partial_a x^{a'} = M^{(2)a'}_{\ \ b'}\partial_b x^{b'} + M^{(1)a'}_{\ \ b'}\partial_b y^{1b'} + \partial_b y^{2a'} \\
&(c)\quad M^{(3)a}_{\ \ b}\partial_a x^{a'} = M^{(3)a'}_{\ \ b'}\partial_b x^{b'} + M^{(2)a'}_{\ \ b'}\partial_b y^{1b'} + M^{(1)a'}_{\ \ b'}\partial_b y^{2b'} + \partial_b y^{3a'}.
\end{aligned} \tag{7}$$

From (7) it is obvious that we can take

$$\begin{aligned}
M^{(1)a}_{\ \ b} &= M^{(1)a}_{\ \ b}(y^{0a}, y^{1a}), \\
M^{(2)a}_{\ \ b} &= M^{(2)a}_{\ \ b}(y^{0a}, y^{1a}, y^{2a}), \\
M^{(3)a}_{\ \ b} &= M^{(3)a}_{\ \ b}(y^{0a}, y^{1a}, y^{2a}, y^{3a}),
\end{aligned} \tag{8}$$

From the choice of M depends the adapted basis B^* ((5)).

Let us denote the adapted basis of $T(E)$ by B, where

$$B = \{\delta_{0a}, \delta_{1a}, \delta_{2a}, \delta_{3a}\} = \{\delta_{\alpha a}\}, \tag{9}$$

and

$$\begin{aligned}
\delta_{0a} &= \partial_{0a} - N^{(1)b}_{\ \ a}\partial_{1b} - N^{(2)b}_{\ \ a}\partial_{2b} - N^{(3)b}_{\ \ a}\partial_{3b}, \\
\delta_{1a} &= \partial_{1a} - N^{(1)b}_{\ \ a}\partial_{2b} - N^{(2)b}_{\ \ a}\partial_{3b} \\
\delta_{2a} &= \partial_{2a} - N^{(1)b}_{\ \ a}\partial_{3b} \\
\delta_{3a} &= \partial_{3a}.
\end{aligned} \tag{10}$$

Theorem 1.2 *The necessary and sufficient conditions that B ((9)) be dual to* B^* *((5)), (when* $\bar{B}$ *((3)) is dual to* $\bar{B}^*$ *((4)) i.e.*

$$< \delta_{\alpha a}\delta y^{\beta b} >= \delta^{\beta}_{\alpha}\delta^{b}_{a}$$

are the following relations:

$$\begin{aligned}
N^{(1)b}_{\ \ a} &= M^{(1)b}_{\ \ a} \\
N^{(2)b}_{\ \ a} &= M^{(2)b}_{\ \ a} - N^{(1)c}_{\ \ a}M^{(1)b}_{\ \ c} = M^{(2)b}_{\ \ a} - M^{(1)c}_{\ \ a}M^{(1)b}_{\ \ c} \\
N^{(3)b}_{\ \ a} &= M^{(3)b}_{\ \ a} - N^{(1)c}_{\ \ a}M^{(2)b}_{\ \ c} - N^{(2)c}_{\ \ a}M^{(1)b}_{\ \ c} = \\
&\quad M^{(3)b}_{\ \ a} - M^{(1)c}_{\ \ a}M^{(2)b}_{\ \ c} - M^{(2)c}_{\ \ a}M^{(1)b}_{\ \ c} + M^{(1)d}_{\ \ a}M^{(1)c}_{\ \ d}M^{(1)b}_{\ \ c}.
\end{aligned} \tag{11}$$

From (10) and (11) it follows

Theorem 1.3 *The necessary and sufficient conditions that* $\delta_{\alpha a}$ *with respect to (1) are transformed as d-tensors, i.e.*

$$\delta_{\alpha a'} = \frac{\partial x^a}{\partial x^{a'}}\delta_{\alpha a}, \ \alpha = 0, 1, 2, 3, \tag{12}$$

are the following formulae:

$$
\begin{aligned}
N^{(1)b'}_{\ a'}\partial_a x^{a'} &= N^{(1)b}_{\ a}\partial_b x^{b'} - \partial_a y^{1b'} \\
N^{(2)b'}_{\ a'}\partial_a x^{a'} &= N^{(2)b}_{\ a}\partial_b x^{b'} + N^{(1)b}_{\ a}\partial_b y^{1b'} - \partial_a y^{2b'} \\
N^{(3)b'}_{\ a'}\partial_a x^{a'} &= N^{(3)b}_{\ a}\partial_b x^{b'} + N^{(2)b}_{\ a}\partial_b y^{1b'} + N^{(1)b}_{\ a}\partial_b y^{2b'} - \partial_a y^{3b'}.
\end{aligned}
\tag{13}
$$

From (10) and (11) it follows

$$
\begin{aligned}
\partial_{3c} &= \delta_{3c} \\
\partial_{2c} &= \delta_{2c} + M^{(1)d}_{\ c}\delta_{3d} \\
\partial_{1c} &= \delta_{1c} + M^{(1)d}_{\ c}\delta_{2d} + M^{(2)d}_{\ c}\delta_{3d} \\
\partial_{0c} &= \delta_{0c} + M^{(1)d}_{\ c}\delta_{1d} + M^{(2)d}_{\ c}\delta_{2d} + M^{(3)d}_{\ c}\delta_{3d}.
\end{aligned}
\tag{14}
$$

Let us denote by T_H, T_{V_1}, T_{V_2}, T_{V_3} the subspaces of $T(E)$ spanned by

$$\{\delta_{0a}\},\ \{\delta_{1a}\},\ \{\delta_{2a}\},\ \{\delta_{3a}\}$$

respectively. Then we have

$$T(E) = T_H \oplus T_{V_1} \oplus T_{V_2} \oplus T_{V_3}.$$

2 The metric tensor and the adapted basis

The aim of this section is to construct such an adapted basis $B^* = \{\delta y^{0a}, \delta y^{1a}, \delta y^{2a}, \delta y^{3a}\}$ that T_{V_3} be orthogonal to T_{V_0}, T_{V_1} and T_{V_2} with respect to the given symmetric metric. Let us suppose that the metric tensor G in the basis $\bar{B}^*$ can be written in the form

$$G = \bar{g}_{\alpha a\ \beta b} dy^{\alpha a} \otimes dy^{\beta b}, \tag{15}$$

where $\bar{g}_{\alpha a\ \beta b}$ are given functions and the summation is going over all indices. The tensor G in the basis B^* has the form

$$G = g_{\gamma e\ \delta f}\delta y^{\gamma e} \otimes \delta y^{\delta f}. \tag{16}$$

In (15) and (16) the notations

$$\bar{g}_{\alpha a\ \beta b} = G(\partial_{\alpha a}, \partial_{\beta b}), \quad g_{\gamma e\ \delta f} = G(\delta_{\gamma e}, \delta_{\delta f})$$

were used.

To obtain the relations between $\bar{g}_{\alpha a\,\beta b}$ and $g_{\gamma e\,\delta f}$ we use (6). After some calculations we get

$$\begin{aligned} dy^{0a} &= A^{(0)a}{}_e \delta y^{0e} \\ dy^{1a} &= A^{(0)a}{}_e \delta y^{1e} + A^{(1)a}{}_e \delta y^{0e}, \\ dy^{2a} &= A^{(0)a}{}_e \delta y^{2e} + A^{(1)a}{}_e \delta y^{1e} + A^{(2)a}{}_e \delta y^{0e}, \\ dy^{3a} &= A^{(0)a}{}_e \delta y^{3e} + A^{(1)a}{}_e \delta y^{2e} + A^{(2)a}{}_e \delta y^{1e} + A^{(3)a}{}_e \delta y^{0e}, \end{aligned} \tag{17}$$

where

$$\begin{aligned} A^{(0)a}{}_e &= \delta^a_e \\ A^{(1)a}{}_e &= -M^{(1)a}{}_e \\ A^{(2)a}{}_e &= -M^{(2)a}{}_e + M^{(1)a}{}_f M^{(1)f}{}_e \\ A^{(3)a}{}_e &= -M^{(3)a}{}_e + M^{(2)a}{}_f M^{(1)f}{}_e + M^{(1)a}{}_f M^{(2)f}{}_e - M^{(1)a}{}_g M^{(1)g}{}_f M^{(1)f}{}_e. \end{aligned} \tag{18}$$

The above equations can be solved in the form

$$\begin{aligned} M^{(1)a}{}_e &= -A^{(1)a}{}_e \\ M^{(2)a}{}_e &= -A^{(2)a}{}_e + A^{(1)a}{}_f A^{(1)f}{}_e \\ M^{(3)a}{}_e &= -A^{(3)a}{}_e + A^{(2)a}{}_f A^{(1)f}{}_e + A^{(1)a}{}_f A^{(2)f}{}_e - A^{(1)a}{}_f A^{(1)f}{}_d A^{(1)d}{}_e. \end{aligned} \tag{19}$$

The comparison of (18) with (11) gives

$$A^{(\alpha)a}{}_e = -N^{(\alpha)a}{}_e, \quad \alpha = 1,2,3$$

and (19) are the solutions of (11).

Formula (17) can be written in the shorter form as follows:

$$dy^{\alpha a} = \sum_{\gamma=0}^{\alpha} A^{(\alpha-\gamma)a}{}_e \delta y^{\gamma e}, \quad dy^{\beta b} = \sum_{\delta=0}^{\beta} A^{(\beta-\delta)b}{}_f \delta y^{\delta f}. \tag{20}$$

The substitution of (20) into (15) and the comparison of such obtained relation with (16) results

$$g_{\gamma e\,\delta f} = \bar{g}_{\alpha a\,\beta b} A^{(\alpha-\gamma)a}{}_e A^{(\beta-\delta)b}{}_f, \tag{21}$$

where the summation is going from $\alpha = \gamma, \ldots, 3$ and $\beta = \delta, \ldots, 3$.

From (21) we get for instance:

$$\begin{aligned}
g_{3e\,3f} &= \bar{g}_{3a\,3b} A^{(0)a}_{\ \ e} A^{(0)b}_{\ \ f}, \\
g_{3e\,2f} &= \bar{g}_{3a\,2b} A^{(0)a}_{\ \ e} A^{(0)b}_{\ \ f} + \bar{g}_{3a\,3b} A^{(0)a}_{\ \ e} A^{(1)b}_{\ \ f}, \\
g_{3e\,1f} &= \bar{g}_{3a\,1b} A^{(0)a}_{\ \ e} A^{(0)b}_{\ \ f} + \bar{g}_{3a\,2b} A^{(0)a}_{\ \ e} A^{(1)b}_{\ \ f} + \bar{g}_{3a\,3b} A^{(0)a}_{\ \ e} A^{(2)b}_{\ \ f}, \\
g_{3e\,0b} &= \bar{g}_{3a\,0f} A^{(0)a}_{\ \ e} A^{(0)b}_{\ \ f} + \bar{g}_{3a\,1b} A^{(0)a}_{\ \ e} A^{(1)b}_{\ \ f} + \bar{g}_{3a\,2b} A^{(0)a}_{\ \ e} A^{(2)b}_{\ \ f} \\
&\quad + \bar{g}_{3a\,3b} A^{(0)a}_{\ \ e} A^{(3)b}_{\ \ f}.
\end{aligned} \tag{22}$$

The substitution of $A^{(0)a}_{\ \ e} = \delta^a_e$ from (18) into (22) yields

$$\begin{aligned}
&\text{(a)} \quad g_{3e\,3f} = \bar{g}_{3e\,3f}, \\
&\text{(b)} \quad g_{3e\,2f} = \bar{g}_{3e\,2f} + \bar{g}_{3e\,3b} A^{(1)b}_{\ \ f}, \\
&\text{(c)} \quad g_{3e\,1f} = \bar{g}_{3e\,1f} + \bar{g}_{3e\,2b} A^{(1)b}_{\ \ f} + \bar{g}_{3e\,3b} A^{(2)b}_{\ \ f}, \\
&\text{(d)} \quad g_{3e\,0f} = \bar{g}_{3e\,0f} + \bar{g}_{3e\,1b} A^{(1)b}_{\ \ f} + \bar{g}_{3e\,2b} A^{(2)b}_{\ \ f} + \bar{g}_{3e\,3b} A^{(3)b}_{\ \ f}.
\end{aligned} \tag{23}$$

Proposition 2.1 *The necessary and sufficient condition for the orthogonality of T_{V_3} and T_{V_2} with respect to the given symmetric metric tensor G (15) is*

$$A^{(1)c}_{\ \ f} = -\bar{g}^{3e\,3c} \bar{g}_{3e\,2f}, \tag{24}$$

where $[\bar{g}^{3a\,3c}]$ is the inverse matrix of $[\bar{g}_{3a\,3b}]$.

Proof. If we multiply (23) (b) with $\bar{g}^{3e\,3c}$ and take into account that $\bar{g}_{3e\,3b}\bar{g}^{3e\,3c} = \delta^c_b$ and $g_{3e\,2f} = 0$ (as the consequence of the orthogonality condition) we obtain (24).

Proposition 2.2 *The necessary and sufficient condition for the orthogonality of T_{V_3} and T_{V_1}, T_{V_3} and T_{V_2} with respect to the given symmetric metric tensor G (15) is*

$$A^{(2)c}_{\ \ f} = -\bar{g}^{3e\,3c} \bar{g}_{3e\,1f} + A^{(1)c}_{\ \ b} A^{(1)b}_{\ \ f}, \tag{25}$$

where $A^{(1)b}_{\ \ f}$ is given by (24).

Proof. If we multiply (23) (c) with $\bar{g}^{3e\,3c}$, substitute $A^{(1)b}_{\ \ f}$ from (24) and the use $g_{3e\,1f} = 0$ we obtain (25).

Proposition 2.3 *The necessary and sufficient condition for the orthogonality of T_{V_3} and T_{V_0}, T_{V_3} and T_{V_1}, T_{V_3} and T_{V_2} with respect to the given symmetric metric tensor G (15) is*

$$A^{(3)c}_{\ \ f} = -\bar{g}^{3e\,3c}\bar{g}_{3e\,0f} + A^{(2)c}_{\ \ b}A^{(1)b}_{\ \ f} + A^{(1)c}_{\ \ b}A^{(2)b}_{\ \ f} - A^{(1)c}_{\ \ d}A^{(1)d}_{\ \ b}A^{(1)b}_{\ \ f}. \quad (26)$$

Proof. The proof is obtained from (23) (d), (24), (25) and the condition $g_{3e\,0f} = 0$.

Theorem 2.1 *There is unique adapted basis $B^* = \{\delta y^{0a}, \delta y^{1a}, \delta y^{2a}, \delta y^{3a}\}$ such that T_{V_3} is orthogonal to T_{V_0}, T_{V_1} and T_{V_2} with respect to the given symmetric metric G (15).*

The nonlinear connection coefficients which determine such basis vectors prescribed by (6) are given by

$$\begin{aligned} M^{(1)c}_{\ \ f} &= \bar{g}_{3e\,2f}\bar{g}^{3e\,3c} \\ M^{(2)c}_{\ \ f} &= \bar{g}_{3e\,1f}\bar{g}^{3e\,3c} \\ M^{(3)c}_{\ \ f} &= \bar{g}_{3e\,0f}\bar{g}^{3e\,3c} \end{aligned} \quad (27)$$

Proof. (27) follows from (19), (24), (25) and (26).

Theorem 2.2 *If the adapted bases B^* of $T^*(E)$ is constructed with $M^{(1)c}_{\ \ f}$, $M^{(2)c}_{\ \ f}$ and $M^{(3)c}_{\ \ f}$ determined by (27) (i.e. when T_{V_3} is orthogonal to T_{V_0}, T_{V_1} and T_{V_2}), then the following relations:*

$$\begin{aligned} \bar{g}_{3e\,3f} &= \bar{g}_{3e\,3f}(y^{0a}, y^{1a}) \\ \bar{g}_{3e\,2f} &= \bar{g}_{3e\,2f}(y^{0a}, y^{1a}) \\ \bar{g}_{3e\,1f} &= \bar{g}_{3e\,1f}(y^{0a}, y^{1a}, y^{2a}) \\ \bar{g}_{3e\,0f} &= \bar{g}_{3e\,0f}(y^{0a}, y^{1a}, y^{2a}, y^{3a}) \end{aligned} \quad (28)$$

(for every $e, f = 1, 2, \ldots, n$) give the sufficient conditions for the coefficients of nonlinear connections to satisfy (8).

Proof. If (28) are satisfied, then from (27) follows (8).

Theorem 2.3 *T_{V_2} is orthogonal to T_{V_1} with respect to the metric G (15) the nonlinear connection coefficient satisfy (27) if*

$$\bar{g}_{2e\,1f} = -\bar{g}_{3a\,1f}A^{(1)a}_{\ \ e} - \bar{g}_{2e\,2b}A^{(1)b}_{\ \ f} - \bar{g}_{3a\,2b}A^{(1)a}_{\ \ e}A^{(1)b}_{\ \ f}. \quad (29)$$

Proof. From (21) it follows

$$g_{2e\,1f} = \bar{g}_{2a\,1b}A^{(0)a}_{\ e}A^{(0)b}_{\ f} + \bar{g}_{2a\,2b}A^{(0)a}_{\ e}A^{(1)b}_{\ f} + \bar{g}_{2a\,3b}A^{(0)a}_{\ e}A^{(2)b}_{\ f} + \bar{g}_{3a\,1b}A^{(1)a}_{\ e}A^{(0)b}_{\ f} + \bar{g}_{3a\,2b}A^{(1)a}_{\ e}A^{(1)b}_{\ f} + \bar{g}_{3a\,3b}A^{(1)a}_{\ e}A^{(2)b}_{\ f}. \tag{30}$$

If in the above equation we substitute $A^{(0)a}_{\ e} = \delta^a_e$ and use

$$\bar{g}_{3a\,3b}A^{(1)a}_{\ e}A^{(2)b}_{\ f} = -\bar{g}_{3a\,3b}\bar{g}^{3a\,3d}\bar{g}_{3d\,2e}A^{(2)b}_{\ f} = = -\bar{g}_{3b\,2e}A^{(2)b}_{\ f} = -\bar{g}_{2e\,3b}A^{(2)b}_{\ f} = -\bar{g}_{2a\,3b}A^{(0)a}_{\ e}A^{(2)b}_{\ f}.$$

we obtain that $g_{2e\,1f}$ determined by (30) is equal to zero if (29) is satisfied.
The other orthogonality conditions can be obtained in the similar way.

Theorem 2.4 *If in $T^*(E)$ the adapted basis B^* is determined by arbitrary but fixed $M^{(\alpha)c}_{\ f}$, $\alpha = 1,2,3$, which satisfy (7), then exists one and only one nondegenerated symmetric metric tensor G with the given components $\bar{g}_{0e\,0f}$, $\bar{g}_{1e\,1f}$, $\bar{g}_{2e\,2f}$, $\bar{g}_{3e\,3f}$ (in $\bar{B}^*$), such that T_{V_0}, T_{V_1}, T_{V_2}, T_{V_3} are mutually orthogonal to each other with respect to G.*

Proof. From (21) it follows (23). From the orthogonality of T_{V_3} to T_{V_0}, T_{V_1} and T_{V_2} $(g_{3e\,0f} = 0, g_{3e\,1f} = 0, g_{3e\,2f} = 0)$ from (23) we can determine $\bar{g}_{3e\,2f}$, $\bar{g}_{3e\,1f}$ and $\bar{g}_{3e\,0f}$ as functions of $\bar{g}_{3e\,3f}$ and $N^{(\alpha)c}_{\ f}$, $\alpha = 1,2,3$. As $\bar{g}_{2e\,2f}$ is given, we have

$$g_{2e\,2f} = \bar{g}_{2e\,2f} + \bar{g}_{3b\,2f}A^{(1)b}_{\ e} + \bar{g}_{2e\,3c}A^{(1)c}_{\ f} + \bar{g}_{3b\,3c}A^{(1)b}_{\ e}A^{(1)c}_{\ f} \quad (A^{(\alpha)c}_{\ f} = -N^{(\alpha)c}_{\ f}).$$

From the condition that T_{V_2} is orthogonal to T_{V_1} and T_{V_0} we get

$$\begin{aligned} g_{2e\,1f} &= \bar{g}_{2e\,1f} + \bar{g}_{2e\,2c}A^{(1)c}_{\ f} + \bar{g}_{2e\,3c}A^{(2)c}_{\ f} + \\ &\quad \bar{g}_{3b\,1f}A^{(1)b}_{\ e} + \bar{g}_{3b\,2c}A^{(1)c}_{\ e}A^{(1)c}_{\ f} + \bar{g}_{3b\,3c}A^{(1)b}_{\ c}A^{(2)c}_{\ f} = 0, \\ g_{2e\,0f} &= \bar{g}_{2e\,0f} + \bar{g}_{2e\,1c}A^{(1)c}_{\ f} + \bar{g}_{2e\,2c}A^{(2)c}_{\ f} + \bar{g}_{2e\,3c}A^{(3)c}_{\ f} + \\ &\quad \bar{g}_{3b\,0f}A^{(1)b}_{\ e} + \bar{g}_{3b\,1f}A^{(1)b}_{\ e}A^{(1)c}_{\ f} + \bar{g}_{3b\,2c}A^{(1)b}_{\ e}A^{(2)c}_{\ f} + \\ &\quad \bar{g}_{3b\,3c}A^{(1)b}_{\ e}A^{(3)c}_{\ f} = 0. \end{aligned}$$

From the first of the above equation we determine $\bar{g}_{2e\,1f}$ and from the second $\bar{g}_{2e\,0f}$. $g_{1e\,1f}$ is given by the relation

$$\begin{aligned} g_{1e\,1f} &= \bar{g}_{1e\,1f} + \bar{g}_{1e\,2c}A^{(1)c}_{\ f} + \bar{g}_{1e\,3c}A^{(2)c}_{\ f} + \\ &\quad \bar{g}_{2b\,1f}A^{(1)b}_{\ e} + \bar{g}_{2b\,2c}A^{(1)b}_{\ e}A^{(1)c}_{\ e} + \bar{g}_{2b\,3c}A^{(1)b}_{\ e}A^{(2)c}_{\ f} + \\ &\quad \bar{g}_{3b\,1f}A^{(2)b}_{\ e} + \bar{g}_{3b\,2c}A^{(2)b}_{\ e}A^{(1)c}_{\ e} + \bar{g}_{3b\,3c}A^{(2)b}_{\ e}A^{(2)c}_{\ f}. \end{aligned}$$

At the end from the condition that T_{V_1} is orthogonal to T_{V_0} and the relation

$$\begin{aligned} g_{1e\,0f} \;=\; & \bar{g}_{1e\,0f} + \bar{g}_{1e\,1f}A^{(1)c}_{\ \ f} + \bar{g}_{1e\,2c}A^{(2)c}_{\ \ f} + \bar{g}_{1e\,3c}A^{(3)c}_{\ \ f} + \\ & \bar{g}_{2b\,0f}A^{(1)b}_{\ \ f} + \bar{g}_{2b\,1c}A^{(1)b}_{\ \ e}A^{(1)c}_{\ \ f} + \bar{g}_{2b\,2c}A^{(1)b}_{\ \ e}A^{(2)c}_{\ \ f} + \\ & \bar{g}_{2b\,3c}A^{(1)b}_{\ \ e}A^{(3)c}_{\ \ f} + \bar{g}_{3b\,0f}A^{(2)b}_{\ \ e} + \bar{g}_{3b\,1c}A^{(2)b}_{\ \ e}A^{(1)c}_{\ \ f} + \\ & \bar{g}_{3b\,2c}A^{(2)b}_{\ \ e}A^{(2)c}_{\ \ f} + \bar{g}_{3b\,3c}A^{(2)b}_{\ \ e}A^{(3)c}_{\ \ f} = 0 \end{aligned}$$

we determine $\bar{g}_{1e\,0f}$.

In this way all components of the symmetric tensor G in the basis $\bar{B}^*$ are determined, under condition that T_{V_0}, T_{V_1}, T_{V_2} and T_{V_3} are mutually orthogonal with respect to G.

References

[1] Anastasiei M., Čomić I., Geometry of k-Lagrange spaces of second order, Mat. Vesnik 49(1997) 15–22.

[2] Čomić I., The curvature theory of generalized connection in Osc^2M, Balcan Journal of geometry and its application (BJGA) Vol.1. (1996) 21–29.

[3] Čomić I., Atanasiu Gh., Stoica E., The generalized connection in Osc^3M (to be published)

[4] Čomić I., Generalized connection of $T(T^2M)$, Publ. de l'Inst. Math. t.60 (74)(1996), 88–100.

[5] Miron R., Atanasiu Gh., Compendium sur les espaces Lagrange d'ordre supérieur, Seminarul de Mecanica 40, Universitatea din Timisoara, 1994.

[6] Miron R. and Atanasiu Gh., Differential Geometry of the k-Osculator Bundle, Rev. Roum. Math. Pures et Appl., Tom XLI, No 3-4 (1996) 205–236.

[7] Miron R. and Atanasiu Gh., Higher Order Lagrange Spaces, Rev. Roum. Math. Pures et Appl., Tom XLI No 3-4 (1996) 251–263.

[8] Miron R., The Geometry of Higher Order Lagrange Spaces, Applications to Mechanics and Physics, Kluwer Academic Publishers, 1997.

[9] Munteanu Gh., Atanasiu Gh., On Miron-connections in Lagrange spaces of second order, Tensor N.S. 50(1991), 241–247.

[10] Munteanu Gh., Metric almost tangent structure of second order, Bull. Math. Soc. Sci. Mat. Roumanie, 34(1), 1990, 49–54.

[11] Opris D., Fibres vectorials de Finsler et connexions associes, The Proc. of Mat. Sem. on Finsler Spaces, Brasov, (1980), 185–193.