NOVI SAD J. MATH.
VOL. 29, NO. 3, 1999, 153–158
XII Yugoslav Geometric Seminar
Novi Sad, October 8–11, 1998



# ON PROJECTING OF $\mathbb{P}^N$ ONTO $\mathbb{P}^{N-M}$ AND THE VARIETY OF PROJECTION CENTERS

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**Abstract**

The present paper is the final one in the series of papers [1] – [6] devoted to a central projection of the projective space $\mathbb{P}^n$ on its subspace $\mathbb{P}^k$. In these papers, as well as in the present one, we consider a projection with the following property: the reference points $A_i$ ($i = \overline{1, n+2}$) have images $A'_i$ which are projectively equivalent to so-called model, i.e., an ordered set of generally disposed points $T_i$ in $\mathbb{P}^k$.

It is known [1], [2] that for any $k \in \mathbb{N}$ $(0 < k < n)$ there exists a projection of such a kind which can be defined in different ways: a set of projection centers is a $(n-k)$-dimensional variety which consists of $(n-k-1)$-dimensional planes. If $k = n-1$, this set is a normal rational curve except finite number of points [3]. For $k = n-2$ and $k = n-3$ a description of the variety of projection centers can be found in [4]-[6]. Below we generalize this description for any $k < n$.

If $k = n - m$ $(m \geqslant 1)$, then a projecting can be done by means of $m$-dimensional planes which belong to a sheaf of planes with $(m-1)$-dimensional vertex.

**Theorem 1.** *If $n \leqslant 2m$ $(m \geqslant 2)$, then the set of projection centers is a subvariety $\sigma_k^{m+1}$ in the Grassmanian manifold $G(m, n+1)$. It can be parametrized by points of $k$-dimensional plane if $n = m+k$ $(0 < k \leqslant m-1)$, and it can be paprametrized by points of $m$-dimensional plane if $n = 2m$.*

**Proof.** Let $\mathbb{P}^{n-m} =< A_1, A_2, \dots A_{n-m}, A_{n-m+1} >$. Then for the projection $f : \mathbb{P}^n \to \mathbb{P}^{n-m}$ we have $f(A_j) = A_j$ for $j = \overline{1, n-m+1}$. If we choose the image $A'_{n-m+2} \in \mathbb{P}^{n-m}$ of the point $A_{n-m+2}$, we will uniquely determine the images of all points $A_j$ $(j = \overline{n-m+2, n+2})$ because the image $A'_i$ and the model $T_i$ are projectively equivalent. The projection center, that is, $(m-1)$-dimensional plane, has to intersect the lines $< A_j, A'_j >$ $(j =$

$\overline{n-m+2, n+2})$ and has not to intersect $\mathbb{P}^{n-m}$. It is known [7] that the variety of $(m-1)$-dimensional planes, which intersect $m+1$ lines in $\mathbb{P}^n$, can be represented in the Grassmanian manifold $G(m, n+1)$ by means of the intersection of Schubert cicles $\sigma_{n-m}$, and the multiplicity of intersection is $m+1$; so this intersection is $\sigma_{n-m}^{m+1}$. The latter is not empty if $(n-m)(m+1) \leqslant m(n-m+1)$, or $n \leqslant 2m$.

If $n = 2m$ then $\sigma_{n-m}^{m+1} = \sigma_m^{m+1} = \sigma_{m+1,m+1,\dots,m+1}$, where the index $m+1$ is repeated $m$ times. It means that for the projection $f : \mathbb{P}^{2m} \to \mathbb{P}^m$ each point $A'_{n-m+2} = A'_{m+2} \in \mathbb{P}^m$ determines a unique projection center, consequently, in such a case the variety of projection centers can be parametrized by points of an $m$-dimensional plane.

If $n < 2m$ $(n-m < m)$, then for any point $A'_{m+1}$ from $\mathbb{P}^{n-m}$ we can construct the variety of projection centers $\sigma_{n-m}^{m+1} = \sigma_k^{m+1}$. Thus, in such a case the set of projection centers is a $(n-m)$-dimensional variety which consists of varieties $\sigma_{n-m}^{m+1}$; it can be parametrized by points of $(n-m)$-dimensional plane.

**Theorem 2.** *If $n = 2m+1$, then the variety of projection centers can be parametrized by points of rational hypesuface $F_m^{m+1} \in \mathbb{P}^{n-m}$ whose degree is $m+1$. If $n = 2m+k$ $(k > 1)$, then the variety of projection centers can be parametrized by points of the variety which is described as follows: it is the intersection of $k$ hypercone in $\mathbb{P}^{n-m}$; their directrices are the surfaces $F_m^{m+1}$; their verteces are $(k-2)$-dimensional planes which are disposed in a special way in $\mathbb{P}^{n-m}$.*

**Proof.** If $n = 2m+1$ then we have a projection of $\mathbb{P}^{2m+1}$ onto $\mathbb{P}^{m+1}$. Let $\mathbb{P}^{m+1}$ be defined as a plane

$$< A_1, A_2, \dots, A_{m+2} >: x_{m+3} = x_{m+4} = \dots = x_{2m+2} = 0.$$

We can consider the referece points $T_i$ $(i = \overline{1, m+3})$ of $\mathbb{P}^{m+1}$; denote the coordinates of the remainder points of the model, i.e., $T_j$ $(j = \overline{m+4, 2m+3})$, as $(\alpha_{j1}, \alpha_{j2}, \dots, \alpha_{j\,m+2})$. Now we can easyly obtain the projection $f$ fatisfying the properties

$$f(A_i) = A_i, \ i = \overline{1, m+2};$$

$$f(A_{m+3}) = A'_{m+3}, \ A'_{m+3}(a_1 : a_2 : \dots : a_{m+2} : 0 : \dots : 0);$$

$$f(A_j) = A'_j, \ j = \overline{m+4, 2m+3}, \ A'_j(a_1\alpha_{j\,1} : \dots : a_{m+2}\alpha_{j\,m+2} : 0 : \dots : 0).$$

It is defined uniquely because the image $A'_i$ and the model $T_i$ $(i = \overline{1, 2m+3})$

are projectively equivalent. The matrix of the projection has a form

$$A_f = \begin{pmatrix} \rho & 0 & \dots & 0 & \rho_{m+3}a_1 & \rho_{m+4}a_1\alpha_{m+4\ 1} & \dots & \rho_{2m+2}a_1\alpha_{2m+2\ 1} \\ 0 & \rho & \dots & 0 & \rho_{m+3}a_2 & \rho_{m+4}a_2\alpha_{m+4\ 2} & \dots & \rho_{2m+2}a_2\alpha_{2m+2\ 2} \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & \rho & \rho_{m+3}a_{m+2} & \rho_{m+4}a_{m+2}\alpha_{m+4\ m+2} & \dots & \rho_{2m+2}a_{m+2}\alpha_{2m+2\ m+2} \\ 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 \end{pmatrix} \tag{1}$$

The condition $f(A_{2m+3}) = A'_{2m+3}$ leads us to a system

$$\rho_{2m+3}a_i\alpha_{2m+3\ i} = \rho + \rho_{m+3}a_i + \rho_{m+4}a_i\alpha_{m+4\ i} +$$
$$\dots + \rho_{2m+2}a_i\alpha_{2m+2\ i}, \quad i = \overline{1, m+2}. \tag{2}$$

For given $a_i$ the system contains $m+2$ equations with $m+2$ unknowns $\rho, \rho_{m+3}, \dots, \rho_{2m+3}$ which can't be equal to zero. It means that the determinant of the system (2) is zero:

$$\begin{vmatrix} 1 & a_1 & a_1\alpha_{m+4\ 1} & \dots & a_1\alpha_{2m+2\ 1} & -a_1\alpha_{2m+3\ 1} \\ 1 & a_2 & a_2\alpha_{m+4\ 2} & \dots & a_2\alpha_{2m+2\ 2} & -a_2\alpha_{2m+3\ 2} \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 1 & a_{m+2} & a_{m+2}\alpha_{m+4\ m+2} & \dots & a_{m+2}\alpha_{2m+2\ m+2} & -a_{m+2}\alpha_{2m+3\ m+2} \end{vmatrix} = 0. \tag{3}$$

Decomposing the determinant by the first row, we obtain an equation

$$A_1a_2a_3\dots a_{m+2} + A_2a_1a_3\dots a_{m+2} +$$
$$\dots + A_{m+1}a_1a_2\dots a_{m+2} + A_{m+2}a_1a_2\dots a_{m+1} = 0. \tag{4}$$

This equation determine a hypersurface of degree $m+1$ in $\mathbb{P}^{m+1}$; we denote it as $F_m^{m+1}$.

Condition (3) means that the points $(1:1:\dots:1)$, $A'_{m+3}$, $A'_{m+4}$, $\dots$, $A'_{2m+3}$, which belong to $\mathbb{P}^{m+1}$, are not disposed in a general way because they lay in an $m$-dimensional plane. Besides, condition (3) means that the point $A'_{m+3}$ can't be taken in an arbitrary way. It has to be taken on the hypersurface $F_m^{m+1}$, which is uniquely determined by the model.

It is easy to see that any point $A'_{m+3} \in F_m^{m+1}$ uniquely determines the projection center of $f$. In other words, the variety of projection centers is parametrized by points of the variety $F_m^{m+1}$.

Here are some properties of the variety $F_m^{m+1}$.

1. The points $A_i$ $(i = \overline{1, m+2})$ lay of $F_m^{m+1}$. They are singular points of multiplicity $m$.

2. The variety $F_m^{m+1}$ passes through the point $(1:1:\dots:1:0:\dots:0)$, which is a tangent point of the variety $F_m^{m+1}$ and the $m$-dimensional plane $(K_1,\dots,K_m,I)$. Here $K_j$ $(j=\overline{m+4,2m+3})$ have coordinates $(\alpha_{j1},\alpha_{j2},\dots,\alpha_{j,m-2})$ with respect to reference points $(A_1,\dots,A_{m+2},I)$ of $\mathbb{P}^{m+1}$.

3. The variety $F_m^{m+1}$ is a rational one.

In order to prove property 3, let us rewrite equation (4) in the form

$$\begin{vmatrix} A_1a_2 & -A_3a_4 & A_4a_2 & -A_5a_2 & \dots & (-1)^{m-1}A_{m+1}a_2 & (-1)^m A_2a_1 \\ 0 & a_3 & a_2 & 0 & \dots & 0 & 0 \\ 0 & 0 & a_4 & a_5 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 & \dots & a_{m+1} & a_1 \\ (-1)^{m+1}a_{m+2} & 0 & 0 & 0 & \dots & -\frac{A_{m+2}}{A_1}a_{m+1} & a_{m+2} \end{vmatrix} = 0. \tag{5}$$

We can consider (5) a condition which is necessary and sufficient for the corresponding homogeneous system of linear equations to have a non-zero solution. If we consider the variables $\lambda_0,\lambda_1,\dots,\lambda_m$ coordinates in $\mathbb{P}^m$ and solve the system for them, we obtain a birational map of $F_m^{m+1}$ into $\mathbb{P}^m$. If we solve the system for $a_i$ $(i=\overline{1,m+2}$, we obtain formulas of the inverse map. The existence of both maps implies that $\mathbb{P}^m$ and $F_m^{m+1}$ are birationally isomorphic, consequently, $F_m^{m+1}$ is rational .

Let us consider theorem 2. As $n=2m+k$, $k>1$, we have $n-m=m+k$ and the projection $f:\mathbb{P}^{2m+k}\to\mathbb{P}^{m+k}$ with the matrix

$$A_f = \begin{pmatrix} \rho & 0 & \dots & 0 & \rho_{m+k+2}a_1 & \rho_{m+k+3}a_1\alpha_{m+k+3\ 1} & \dots & \rho_{2m+k+1}a_1\alpha_{2m+k+1\ 1} \\ 0 & \rho & \dots & 0 & \rho_{m+k+2}a_2 & \rho_{m+k+3}a_2\alpha_{m+k+3\ 2} & \dots & \rho_{2m+k+1}a_2\alpha_{2m+k+1\ 2} \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & \rho & \rho_{m+k+2}a_{m+k+1} & \rho_{m+k+3}a_{m+k+1}\alpha_{m+k+3\ m+k+1} & \dots & \rho_{2m+k+1}a_{m+k+1}\alpha_{2m+k+3\ m+k+1} \\ 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 \end{pmatrix}$$

As $f(A_{n+2})=A'_{n+2}$, we have, as well as in the previous case, a sistem of linear equations

$$\rho + \rho_{m+k+2}a_i + \rho_{m+k+3}a_i\alpha_{m+k+3\,i} + \cdots + \rho_{2m+k+1}a_i\alpha_{2m+k+1\,i} - \rho_{2m+k+2}a_i\alpha_{2m+k+2\,i} = 0, \quad i = \overline{1, m+k+1}.$$

which has $m+2$ unknowns $\rho, \rho_{m+k+2}, \ldots, \rho_{2m+k+2}$. This system has non-zero solution if its rank $r$ satisfies the condition $r \leqslant m+1$. Consequently, all minors of order $m+2$ vanish:

$$\begin{vmatrix} 1 & a_1 & a_1\alpha_{m+k+3\,1} & \cdots & a_1\alpha_{2m+k+1\,1} & -a_1\alpha_{2m+k+2\,1} \\ 1 & a_2 & a_2\alpha_{m+k+3\,2} & \cdots & a_2\alpha_{2m+k+1\,2} & -a_2\alpha_{2m+k+2\,2} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & a_{m+1} & a_{m+1}\alpha_{m+k+3\,m+1} & \cdots & a_{m+1}\alpha_{2m+k+1\,m+1} & -a_{m+1}\alpha_{2m+k+2\,m+1} \\ 1 & a_{m+l} & a_{m+l}\alpha_{m+k+3\,m+l} & \cdots & a_{m+l}\alpha_{2m+k+1\,m+l} & -a_{m+l}\alpha_{2m+k+2\,m+l} \end{vmatrix} = 0, \quad l = \overline{2, k+1}. \tag{6}$$

Each of $k$ equation of the system (6) defines a hypercone $K_l^{m+1}$ in $\mathbb{P}^{m+k}$ whose degree is $m+1$. The variety $F_{ml}^{m+1}$, which lays in the $(m+1)$-dimensional plane $a_{m+2} = \cdots = \hat{a}_{m+l} = \cdots = a_{2m+k+1} = 0$, is the directix of hypercones (the symbol $\hat{a}_{m+1}$ means that there is no equation $a_{m+l} = 0$ in the system). $(k-2)$-dimensional plane in $\mathbb{P}^{m+k}$, which is defined by equations $a_1 = \cdots = a_{m+1} = a_{m+l} = a_{m+k+2} = \cdots = a_{2m+k+1} = 0$, is the vertex of the hypercone $K_l^{m+1}$.

If we choose a point $A'_{m+k+2} \in \bigcap_{l=2}^{k+1} F_{ml}^{m+1}$, we uniquely determine the projection center of $f$, therefore, the set of projection centers is parametrized by points of the variety $\bigcap_{l=2}^{k+1} F_{ml}^{m+1}$.

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