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FIRST TYPE ALMOST GEODESIC MAPPINGS OF GENERAL AFFINE CONNECTION SPACES *

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Abstract

In this paper we investigate some reciprocity conditions of the first type almost geodesic mappings of the general affine connection spaces. Also we consider the first type $(N-2)$-projective spaces and get some relations characterizing the first type $(N-2)$-projective spaces.

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1 Introduction

Let GA_N be an N-dimensional space with an affine connection L given with the aid of components L^i_{jk} in each local map V on a differentiable manifold. Generally it is $L^i_{jk} \neq L^i_{kj}$.

Generalizing conception of a geodesic mappings for Riemannian and affine spaces Sinyukov introduced [6] following notations:

The curve $l : x^h = x^h(t)$ is called the almost geodesic line if its tangential vector $\lambda^h(t) = dx^h/dt \neq 0$ satisfies the equations

$$\overline{\lambda}^h_{(2)} = \overline{a}(t)\lambda^h + \overline{b}(t)\overline{\lambda}^h_{(1)}, \quad \overline{\lambda}^h_{(1)} = \lambda^h_{||\alpha}\lambda^\alpha, \quad \overline{\lambda}^h_{(2)} = \overline{\lambda}^h_{(1)||\alpha}\lambda^\alpha,$$

where $\overline{a}(t)$ and $\overline{b}(t)$ are functions of a parameter t, and $||$ denotes the covariant derivative with respect to the connection in $\overline{A}_N$.

A mapping f of the affine space A_N onto a space $\overline{A}_N$ is called the almost geodesic mapping if any geodesic line of the space A_N turns into almost geodesic line of the space $\overline{A}_N$.

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Sinjukov [6] singled out the three types of the almost geodesic mappings, π_1, π_2, π_3 for spaces without torsion. In the present work we investigate the mappings of the type π_1 for spaces with torsion. In a differentiable manifold with nonsymmetric affine connection L^i_{jk}, for a vector exist two kinds of covariant derivative:

$$\lambda^h_{\underset{1}{|}m} = \lambda^h_{,m} + L^h_{\alpha m}\lambda^\alpha, \quad \lambda^h_{\underset{2}{|}m} = \lambda^h_{,m} + L^h_{m\alpha}\lambda^\alpha.$$

Thus, in the case of the space with nonsymmetric affine connection we can define two kinds of almost geodesic lines and two kinds of almost geodesic mappings.

In an affine space GA_N (with nonsymmetric affine connection coefficients L^i_{jk} [4]) one can define four kinds of covariant derivative [1,2]. Signify by $\underset{\theta}{|}$, $\underset{\theta}{||}$ a covariant derivative of the kind θ ($\theta = 1, ..., 4$) in GA_N and $G\overline{A}_N$ respectively.

A curve in an affine space $G\overline{A}_N$ is called *almost geodesic line of the first kind*, if its tangential vector $\lambda^h(t) = dx^h/dt \neq 0$ satisfies the equations

$$\overline{\underset{1}{\lambda}}{}^h_{(2)} = \underset{1}{\overline{a}}(t)\lambda^h + \underset{1}{\overline{b}}(t)\overline{\underset{1}{\lambda}}{}^h_{(1)}, \quad \overline{\underset{1}{\lambda}}{}^h_{(1)} = \lambda^h_{\underset{1}{||}\alpha}\lambda^\alpha, \quad \overline{\underset{1}{\lambda}}{}^h_{(2)} = \overline{\underset{1}{\lambda}}{}^h_{(1)\underset{1}{||}\alpha}\lambda^\alpha \tag{1}$$

where $\underset{1}{\overline{a}}(t)$ and $\underset{1}{\overline{b}}(t)$ are functions of a parameter t. A curve is called the second kind almost geodesic line if its tangential vector $\lambda^h(t) = dx^h/dt \neq 0$ satisfies the equations

$$\overline{\underset{2}{\lambda}}{}^h_{(2)} = \underset{2}{\overline{a}}(t)\lambda^h + \underset{2}{\overline{b}}(t)\overline{\underset{2}{\lambda}}{}^h_{(1)}, \quad \overline{\underset{2}{\lambda}}{}^h_{(1)} = \lambda^h_{\underset{2}{||}\alpha}\lambda^\alpha, \quad \overline{\underset{2}{\lambda}}{}^h_{(2)} = \overline{\underset{2}{\lambda}}{}^h_{(1)\underset{2}{||}\alpha}\lambda^\alpha \tag{2}$$

where $\underset{2}{\overline{a}}(t)$ and $\underset{2}{\overline{b}}(t)$ are functions of a parameter t.

A mapping f of the space GA_N onto a space with nonsymmetric affine connection $G\overline{A}_N$ is called *almost geodesic mapping of the first kind* if any geodesic line of the space GA_N turns into the almost geodesic line of the first kind of the space $G\overline{A}_N$. A mapping f is called *almost geodesic mapping of the second kind* $\underset{2}{\pi}$ if any geodesic line of the space GA_N turns into almost geodesic line of the second kind of the space $G\overline{A}_N$ (For spaces A_N with symmetric affine connection see [6]).

We can put

$$\overline{L}^h_{ij}(x) = L^h_{ij}(x) + P^h_{ij}(x), \tag{3}$$

where $L^h_{ij}(x)$, $\overline{L}^h_{ij}(x)$ are connection coefficients of the space GA_N and $G\overline{A}_N$ respectively ($N > 2$), and $P^h_{ij}(x)$ is a deformation tensor. Then the next theorem is valid:

Theorem 1. *The mapping f of the space GA_N onto $G\overline{A}_N$ is almost geodesic mapping of the first kind if and only if the deformation tensor $P^h_{ij}(x)$ satisfies identically the conditions*

$$(P^h_{\alpha\beta\underset{1}{|}\gamma} + P^h_{\delta\alpha}P^\delta_{\beta\gamma})\lambda^\alpha\lambda^\beta\lambda^\gamma = \underset{1}{b}P^h_{\alpha\beta}\lambda^\alpha\lambda^\beta + \underset{1}{a}\lambda^h, \tag{4}$$

where $\underset{1}{a}$ and $\underset{1}{b}$ are invariants.

Proof. By almost geodesic mapping $\underset{1}{\pi}$ geodesic line

$$\underset{1}{\lambda}{}^h_{(1)} = \lambda^h_{\underset{1}{|}\alpha}\lambda^\alpha = \rho\lambda^h \tag{5}$$

of the space GA_N turns into the almost geodesic line of the first kind of the space $G\overline{A}_N$. In this case from (1), (3) and (5) we have

$$\begin{aligned}
\underset{1}{\overline{\lambda}}{}^h_{(1)} = \lambda^h_{\underset{1}{||}\alpha}\lambda^\alpha &= \frac{d\lambda^h}{dt} + \overline{L}^h_{\alpha\beta}\lambda^\alpha\lambda^\beta \\
&= \frac{d\lambda^h}{dt} + (L^h_{\alpha\beta} + P^h_{\alpha\beta})\lambda^\alpha\lambda^\beta \\
&= \frac{d\lambda^h}{dt} + L^h_{\alpha\beta}\lambda^\alpha\lambda^\beta + P^h_{\alpha\beta}\lambda^\alpha\lambda^\beta = \underset{1}{\lambda}{}^h_{(1)} + P^h_{\alpha\beta}\lambda^\alpha\lambda^\beta,
\end{aligned}$$

i.e.

$$\underset{1}{\overline{\lambda}}{}^h_{(1)} = \underset{1}{\lambda}{}^h_{(1)} + P^h_{\alpha\beta}\lambda^\alpha\lambda^\beta. \tag{6}$$

From (5) and (6) the next relation follows

$$\underset{1}{\overline{\lambda}}{}^h_{(1)} = \rho\lambda^h + P^h_{\alpha\beta}\lambda^\alpha\lambda^\beta. \tag{7}$$

By covariant derivation of the first kind in (6) in the space $G\overline{A}_N$ we get

$$\underset{1}{\overline{\lambda}}{}^h_{(2)} = \rho'\lambda^h + \rho\underset{1}{\overline{\lambda}}{}^h_{(1)} + P^h_{\alpha\beta\underset{1}{||}\gamma}\lambda^\alpha\lambda^\beta\lambda^\gamma + P^h_{\alpha\beta}\underset{1}{\overline{\lambda}}{}^\alpha_{(1)}\lambda^\beta P^h_{\alpha\beta}\lambda^\alpha\underset{1}{\overline{\lambda}}{}^\beta_{(1)}, \tag{8}$$

wherefrom, with respect to (6), we get

$$\underset{1}{\overline{\lambda}}{}^h_{(2)} = P^h_{\alpha\beta\underset{1}{||}\gamma}\lambda^\alpha\lambda^\beta\lambda^\gamma + P^h_{(\alpha\delta)}P^\delta_{\beta\gamma}\lambda^\alpha\lambda^\beta\lambda^\gamma + \rho'\lambda^h + \rho(\rho\lambda^h + P^h_{\alpha\beta}\lambda^\alpha\lambda^\beta). \tag{9}$$

Crossing in (9) from the covariant derivative of the first kind in $G\overline{A}_N$ to the covariant derivative of the first kind in GA_N we have

$$\begin{aligned}
P^h_{ij\underset{1}{||}k} &= P^h_{ij,k} + \overline{L}^h_{\delta k}P^\delta_{ij} - \overline{L}^\delta_{ik}P^h_{\delta j} - \overline{L}^\delta_{jk}P^h_{ij} \\
&= P^h_{ij\underset{1}{|}k} + P^h_{\delta k}P^\delta_{ij} - P^\delta_{ik}P^h_{\delta j} - P^\delta_{jk}P^h_{i\delta},
\end{aligned}$$

From here in this case for (9) we have

$$\overline{\lambda}^h_{\underset{1}{}(2)} = (P^h_{\alpha\beta\underset{1}{|}\gamma} + P^h_{\delta\alpha} P^\delta_{\beta\gamma})\lambda^\alpha \lambda^\beta \lambda^\gamma + 3\rho P^h_{\alpha\beta}\lambda^\alpha \lambda^\beta + (\rho' - \rho^2)\lambda^h . \tag{10}$$

By substitution (10) and (7) in (1) we get

$$(P^h_{\alpha\beta\underset{1}{|}\gamma} + P^h_{\delta\alpha} P^\delta_{\beta\gamma})\lambda^\alpha \lambda^\beta \lambda^\gamma = (\underset{1}{\overline{b}} - 3\rho)P^h_{\alpha\beta}\lambda^\alpha \lambda^\beta + (\underset{1}{\overline{a}} + \underset{1}{\overline{b}}\rho - \rho' - \rho^2)\lambda^h .$$

We can put

$$\underset{1}{b} = \underset{1}{\overline{b}} - 3\rho, \quad \underset{1}{a} = \underset{1}{\overline{a}} + \underset{1}{\overline{b}}\rho - \rho' - \rho^2. \tag{11}$$

Then we get (4). The theorem is proved.

Analogously, for the almost geodesic mapping of the second kind we have

Theorem 2. *The mapping f of the space GA_N onto $G\overline{A}_N$ is almost geodesic mapping of the second kind if and only if the deformation tensor $P^h_{ij}(x)$ satisfies identically the conditions*

$$(P^h_{\alpha\beta\underset{2}{|}\gamma} + P^h_{\alpha\delta} P^\delta_{\beta\gamma})\lambda^\alpha \lambda^\beta \lambda^\gamma = \underset{2}{b}P^h_{\alpha\beta}\lambda^\alpha \lambda^\beta + \underset{2}{a}\lambda^h , \tag{12}$$

where $\underset{2}{a}$ and $\underset{2}{b}$ are invariants.

According to the dependence of invariants $\underset{1}{a}$ and $\underset{1}{b}$ there exist three types of almost geodesic mappings of the first kind, and according to the dependence of invariants $\underset{2}{a}$ and $\underset{2}{b}$ there exist three types of almost geodesic mappings of the second kind.

2 Almost geodesic mappings of the first type of affine spaces

In [6] Sinyukov introduced almost geodesic mapping of the first type π_1 for affine spaces without torsion by condition:

$$b = b_\gamma \lambda^\gamma .$$

Analogously, the almost geodesic mappings of the first kind is the first type $\underset{1}{\pi}_1$ if the function $\underset{1}{b}$ in the relation (4) is a linear and homogeneous form with respect to $\lambda^1, \lambda^2, ..., \lambda^N$ i.e.

$$\underset{1}{b} = \underset{1}{b}_\gamma \lambda^\gamma . \tag{13}$$

Signify by $\underset{ijk}{Sym}$ a symmetrization with respect to i, j, k. Then the following theorem is satisfied

Theorem 3. *The almost geodesic mapping of the first kind $f : GA_N \to G\overline{A}_N$ is the first type if the deformation tensor of the connection satisfies the condition*

$$\underset{ijk}{Sym}\, P^h_{ij\underset{1}{|}k} + \underset{ijk}{Sym}\, P^h_{\alpha i} P^\alpha_{jk} = \underset{ijk}{Sym}\, \underset{1}{b}_i P^h_{jk} + \underset{ijk}{Sym}\, \underset{1}{a}_{ij} \delta^h_k \tag{14}$$

where $\underset{1}{b}_i$ is a covariant vector and $\underset{1}{a}_{ij}$ a covariant tensor.

Proof. From (13) the function $\underset{1}{a}_{ij}$ in (4) must be homogeneous quadratic form with respect to $\lambda^1, \lambda^2, ..., \lambda^N$, i.e.

$$\underset{1}{a} = \underset{1}{a}_{\alpha\beta}(x) \lambda^\alpha \lambda^\beta .$$

Substituting (13) and (6) in (4) we have

$$(P^h_{\alpha\beta\underset{1}{|}\gamma} + P^h_{\delta\alpha} P^\delta_{\beta\gamma} - \underset{1}{b}_\alpha P^h_{\beta\gamma} - \underset{1}{a}_{\alpha\beta} \delta^h_\gamma) \lambda^\alpha \lambda^\beta \lambda^\gamma = 0$$

i.e.

$$\underset{ijk}{Sym}\, (P^h_{ij\underset{1}{|}k} + P^h_{\delta i} P^\delta_{jk} - \underset{1}{b}_i P^h_{jk} - \underset{1}{a}_{ij} \delta^h_k) = 0.$$

The theorem is proved.

Almost geodesic mappings of the second kind is the first type $\underset{2}{\pi}_1$ if for the function $\underset{2}{b}$ satisfied the condition:

$$\underset{2}{b} = \underset{2}{b}_\gamma \lambda^\gamma . \tag{15}$$

Analogously, for the almost geodesic mapping of the second kind the next theorem is valid

Theorem 4. *The almost geodesic mapping of the second kind $f : GA_N \to G\overline{A}_N$ is the first type if the deformation tensor of the connection satisfied*

$$\underset{ijk}{Sym}\, P^h_{ij\underset{2}{|}k} + \underset{ijk}{Sym}\, P^h_{i\alpha} P^\alpha_{jk} = \underset{ijk}{Sym}\, \underset{2}{b}_i P^h_{jk} + \underset{ijk}{Sym}\, \underset{2}{a}_{ij} \delta^h_k \tag{16}$$

where $\underset{2}{b}_i$ is a covariant vector and $\underset{2}{a}_{ij}$ is a covariant tensor.

In the case when $G\overline{A}_N$ is a flat space then GA_N is called $(N-2)$-*projective space of the first type.* In affine coordinate system $y^1, y^2, ..., y^N$

when $G\overline{A}_N$ is a flat space we have $\overline{L}^h_{ij}(y) = 0$. Then the next theorem is satisfied:

Theorem 5. *In affine coordinate system the basic equations of* $(N-2)$*-projective space of the first type with respect to the mapping* $\underset{1}{\pi}$ *are*

$$\underset{ijk}{Sym}\, L^h_{ij\underset{1}{|}k}(y) = \underset{ijk}{Sym}\, L^h_{\alpha i}(y) L^\alpha_{jk}(y) + \underset{ijk}{Sym}\, \underset{1}{b}_i(y) L^h_{jk}(y) - \underset{ijk}{Sym}\, \underset{1}{a}_{ij}(y)\delta^h_k \quad (17)$$

where $\underset{1}{b}_i(y)$ *is a covariant vector and* $\underset{1}{a}_{ij}(y)$ *a covariant tensor.*

The proof follows from the Theorem 3.

Theorem 6. *In affine coordinate system the basic equations of* $(N-2)$*-projective space of the first type with respect to the mapping* $\underset{2}{\pi}$ *are*

$$\underset{ijk}{Sym}\, L^h_{ij\underset{2}{|}k}(y) = \underset{ijk}{Sym}\, L^h_{\alpha i}(y) L^\alpha_{jk}(y) + \underset{ijk}{Sym}\, \underset{2}{b}_i(y) L^h_{jk}(y) - \underset{ijk}{Sym}\, \underset{2}{a}_{ij}(y)\delta^h_k \quad (18)$$

where $\underset{2}{b}_i(y)$ *is a covariant vector and* $\underset{2}{a}_{ij}(y)$ *a covariant tensor.*

The proof follows from the Theorem 4.

3 The property of reciprocity of almost geodesic mappings of the first type

The mapping $\underset{1}{\pi}_1 : GA_N \to G\overline{A}_N$ has *the property of reciprocity* if his inverse mapping is $\underset{1}{\pi}_1$ type too. Crossing in (14) from the covariant derivative of the first kind in GA_N to the covariant derivative of the first kind in $G\overline{A}_N$, we get

$$\underset{ijk}{Sym}\, P^h_{ij\underset{1}{||}k} + \underset{ijk}{Sym}\, P^h_{\alpha i}P^\alpha_{kj} + \underset{ijk}{Sym}\, P^h_{i\alpha}P^\alpha_{jk} = \underset{ijk}{Sym}\, \underset{1}{b}_i P^h_{jk} + \underset{ijk}{Sym}\, \underset{1}{a}_{ij}\delta^h_k. \quad (19)$$

The deformation tensor of the mapping $\underset{1}{\pi}_1^{-1} : G\overline{A}_N \to GA_N$ satisfies the condition of the form (14), i.e.

$$\underset{ijk}{Sym}\, \overline{P}^h_{ij\underset{1}{||}k} + \underset{ijk}{Sym}\, \overline{P}^h_{\alpha i}\overline{P}^\alpha_{jk} = \underset{ijk}{Sym}\, \underset{1}{\overline{b}}_i \overline{P}^h_{jk} + \underset{ijk}{Sym}\, \underset{1}{\overline{a}}_{ij}\delta^h_k. \quad (20)$$

From $\overline{P}^h_{ij} = -P^h_{ij}$ and (20) we have

$$-\underset{ijk}{Sym}\, P^h_{ij\underset{1}{||}k} + \underset{ijk}{Sym}\, P^h_{\alpha i}P^\alpha_{jk} = -\underset{ijk}{Sym}\, \underset{1}{\overline{b}}_i P^h_{jk} + \underset{ijk}{Sym}\, \underset{1}{\overline{a}}_{ij}\delta^h_k. \quad (21)$$

From (19) and (21) we get

$$\underset{ijk}{Sym}\, P^h_{\alpha i} P^\alpha_{kj} + \underset{ijk}{Sym}\, P^h_{i\alpha} P^\alpha_{jk} + \underset{ijk}{Sym}\, P^h_{\alpha i} P^\alpha_{jk} = \underset{ijk}{Sym}\, \underset{1}{d}_i P^h_{jk} + \underset{ijk}{Sym}\, \underset{1}{c}_{ij} \delta^h_k, \quad (22)$$

where

$$\underset{1}{d}_i = \underset{1}{b}_i - \underset{1}{\overline{b}}_i, \qquad \underset{1}{c}_{ij} = \underset{1}{a}_{ij} - \underset{1}{\overline{a}}_{ij}.$$

On the base of the facts given above, we get

Theorem 7. *A necessary and sufficient condition that a mapping $\underset{1}{\pi}_1 : GA_N \to G\overline{A}_N$ has the property of reciprocity, is given by* (22).

Theorem 8. *If the almost geodesic mapping $\underset{1}{\pi}_1 : GA_N \to G\overline{A}_N$ has the property of reciprocity, then a basic equations of this mapping has a form*

$$\underset{ijk}{Sym}\, P^h_{ij\underset{1}{|}k} = \underset{ijk}{Sym}\, P^h_{\alpha i} P^\alpha_{kj} + \underset{ijk}{Sym}\, P^h_{i\alpha} P^\alpha_{jk} + \underset{ijk}{Sym}\, \underset{1}{\overline{\overline{b}}}_i P^h_{jk} + \underset{ijk}{Sym}\, \underset{1}{\overline{\overline{a}}}_{ij} \delta^h_k, \quad (23)$$

where

$$\underset{1}{\overline{\overline{b}}}_i = \underset{1}{b}_i - \underset{1}{d}_i, \qquad \underset{1}{\overline{\overline{a}}}_{ij} = \underset{1}{a}_{ij} - \underset{1}{c}_{ij}.$$

The proof follows from (14) and (22).

By the same procedure are proved

Theorem 9. *A necessary and sufficient condition that a mapping $\underset{2}{\pi}_1 : GA_N \to G\overline{A}_N$ has the property of reciprocity, is given by*

$$\underset{ijk}{Sym}\, P^h_{i\alpha} P^\alpha_{kj} + \underset{ijk}{Sym}\, P^h_{i\alpha} P^\alpha_{jk} + \underset{ijk}{Sym}\, P^h_{\alpha i} P^\alpha_{jk} = \underset{ijk}{Sym}\, \underset{2}{d}_i P^h_{jk} + \underset{ijk}{Sym}\, \underset{2}{c}_{ij} \delta^h_k, \quad (24)$$

where

$$\underset{2}{d}_i = \underset{2}{b}_i - \underset{2}{\overline{b}}_i, \qquad \underset{2}{c}_{ij} = \underset{2}{a}_{ij} - \underset{2}{\overline{a}}_{ij}.$$

Theorem 10. *If the almost geodesic mapping $\underset{2}{\pi}_1 : GA_N \to G\overline{A}_N$ has the property of reciprocity, then a basic equations of this mapping has a form*

$$\underset{ijk}{Sym}\, P^h_{ij\underset{2}{|}k} = \underset{ijk}{Sym}\, P^h_{i\alpha} P^\alpha_{kj} + \underset{ijk}{Sym}\, P^h_{\alpha i} P^\alpha_{jk} + \underset{ijk}{Sym}\, \underset{1}{\overline{\overline{b}}}_i P^h_{jk} + \underset{ijk}{Sym}\, \underset{1}{\overline{\overline{a}}}_{ij} \delta^h_k, \quad (25)$$

where

$$\underset{2}{\overline{\overline{b}}}_i = \underset{2}{b}_i - \underset{2}{d}_i, \qquad \underset{2}{\overline{\overline{a}}}_{ij} = \underset{2}{a}_{ij} - \underset{2}{c}_{ij}.$$

4 Some relations for $(N-2)$-projective spaces of the first type

In the space GA_N we have five independent curvature tensors [3]:

$$\begin{aligned}
\underset{1}{R}{}^i{}_{jmn} &= L^i_{jm,n} - L^i_{jn,m} + L^\alpha_{jm}L^i_{\alpha n} - L^\alpha_{jn}L^i_{\alpha m},\\
\underset{2}{R}{}^i{}_{jmn} &= L^i_{mj,n} - L^i_{nj,m} + L^\alpha_{mj}L^i_{n\alpha} - L^\alpha_{nj}L^i_{m\alpha},\\
\underset{3}{R}{}^i{}_{jmn} &= L^i_{jm,n} - L^i_{nj,m} + L^\alpha_{jm}L^i_{n\alpha} - L^\alpha_{nj}L^i_{\alpha m} + L^\alpha_{nm}(L^i_{\alpha j} - L^i_{j\alpha}),\\
\underset{4}{R}{}^i{}_{jmn} &= L^i_{jm,n} - L^i_{nj,m} + L^\alpha_{jm}L^i_{n\alpha} - L^\alpha_{nj}L^i_{\alpha m} + L^\alpha_{mn}(L^i_{\alpha j} - L^i_{j\alpha}),\\
\underset{5}{R}{}^i{}_{jmn} &= \frac{1}{2}(L^i_{jm,n} + L^i_{mj,n} - L^i_{jn,m} - L^i_{nj,m} + L^\alpha_{jm}L^i_{\alpha n} + L^\alpha_{mj}L^i_{n\alpha}\\
&\quad - L^\alpha_{jn}L^i_{m\alpha} - L^\alpha_{nj}L^i_{\alpha m}).
\end{aligned}$$

Signify by

$$\underset{\theta}{\overline{R}}{}^i_{jmn} \ (\theta = 1, ..., 5)$$

corresponding curvature tensors of the space $G\overline{A}_N$.

For the curvature tensors $\underset{1}{R}{}^i_{jmn}$ and $\underset{1}{\overline{R}}{}^i_{jmn}$ of the spaces GA_N and $G\overline{A}_N$ is satisfied the relation (see [5,7])

$$\underset{1}{\overline{R}}{}^i_{jmn} = \underset{1}{R}{}^i{}_{jmn} + P^i_{jm\underset{1}{|}n} - P^i_{jn\underset{1}{|}m} + P^i_{\alpha n}P^\alpha_{jm} - P^i_{\alpha m}P^\alpha_{jn} + L^\alpha_{[mn]}P^i_{j\alpha}. \quad (26)$$

By symmetrization in (26) with respect to j and m and using (14), we obtain

$$\begin{aligned}
\underset{1}{R}{}^i{}_{(jm)n} &+ P^i_{(jm)\underset{1}{|}n} + P^i_{\alpha j}P^\alpha_{(mn)} + P^i_{jm\underset{1}{|}n} + P^i_{\alpha n}P^\alpha_{jm}\\
&+ P^i_{[nj]\underset{1}{|}m} + P^i_{\alpha m}P^\alpha_{[nj]} + L^\alpha_{[mn]}P^i_{j\alpha} + L^\alpha_{[jn]}P^i_{m\alpha}\\
&= \underset{1}{\overline{R}}{}^i_{(jm)n} + \underset{jmn}{Sym}\, \underset{1}{b_j}P^i_{mn} + \underset{jmn}{Sym}\, \underset{1}{a_{jm}}\delta^i_n,
\end{aligned}$$

where (jm) denotes a symmetrization and $[nj]$ is an antisymmetrization (without division). If the space GA_N is the first type $(N-2)$-projective, i.e. $G\overline{A}_N$ is a flat, then we have

$$\underset{1}{\overline{R}}{}^i_{jmn} \equiv 0$$

wherefrom

$$\begin{aligned} &P^i_{(jm)\underset{1}{|}n} + P^i_{jm\underset{1}{|}n} + P^i_{[nj]\underset{1}{|}m} + P^i_{\alpha n}P^\alpha_{(jm)} + P^i_{\alpha j}P^\alpha_{mn} \\ &\quad + P^i_{\alpha m}P^\alpha_{[nj]} + L^\alpha_{[mn]}P^i_{j\alpha} + L^\alpha_{[jn]}P^i_{m\alpha} \\ &\quad = -\underset{1}{R}{}^i{}_{(jm)n} + \underset{jmn}{Sym}\, \underset{1}{b}_j P^i_{mn} + \underset{jmn}{Sym}\, \underset{1}{a}_{jm}\delta^i_n \end{aligned} \tag{27}$$

On the base of the facts given above, we get

Theorem 11. *A space GA_N is $(N-2)$-projective of the first type with respect to the tensor $\underset{1}{\overline{R}}{}^i_{jmn}$ if there exists a tensor P^h_{ij} satisfying the equations* (27) *for any tensor $\underset{1}{a}_{ij}$ and vector $\underset{1}{b}_i$.*

For the curvature tensors $\underset{2}{R}{}^i{}_{jmn}$ and $\underset{2}{\overline{R}}{}^i_{jmn}$ of the spaces GA_N and $G\overline{A}_N$ is satisfied the relation (see [5,7])

$$\underset{2}{\overline{R}}{}^i_{jmn} = \underset{2}{R}{}^i{}_{jmn} + P^i_{mj\underset{1}{|}n} - P^i_{nj\underset{1}{|}m} + P^i_{n\alpha}P^\alpha_{mj} - P^i_{m\alpha}P^\alpha_{nj} + L^\alpha_{[nm]}P^i_{\alpha j}. \tag{28}$$

By symmetrization in (28) with respect to j and m and using (14), we obtain

$$\begin{aligned} &\underset{2}{R}{}^i{}_{(jm)n} + P^i_{(mj)\underset{1}{|}n} + P^i_{n\alpha}P^\alpha_{(mj)} + P^i_{jm\underset{1}{|}n} + P^i_{n\alpha}P^\alpha_{jm} \\ &\quad + P^i_{[mn]\underset{1}{|}j} + P^i_{j\alpha}P^\alpha_{[mn]} + L^\alpha_{[nm]}P^i_{\alpha j} + L^\alpha_{[nj]}P^i_{\alpha m} \\ &\quad = \underset{2}{\overline{R}}{}^i_{(jm)n} + \underset{jmn}{Sym}\, \underset{1}{b}_j P^i_{mn} + \underset{jmn}{Sym}\, \underset{1}{a}_{jm}\delta^i_n. \end{aligned}$$

Using

$$\underset{2}{\overline{R}}{}^i_{jmn} \equiv 0$$

we get

$$\begin{aligned} &P^i_{(mj)\underset{1}{|}n} + P^i_{jm\underset{1}{|}n} + P^i_{[mn]\underset{1}{|}j} + P^i_{n\alpha}P^\alpha_{(mj)} + P^i_{n\alpha}P^\alpha_{jm} \\ &\quad + P^i_{j\alpha}P^\alpha_{[mn]} + L^\alpha_{[nm]}P^i_{\alpha j} + L^\alpha_{[nj]}P^i_{\alpha m} \\ &\quad = -\underset{2}{R}{}^i{}_{(jm)n} + \underset{jmn}{Sym}\, \underset{1}{b}_j P^i_{mn} + \underset{jmn}{Sym}\, \underset{1}{a}_{jm}\delta^i_n. \end{aligned} \tag{29}$$

On the base of the facts given above, we get

Theorem 12. *A space GA_N is $(N-2)$-projective of the first type with respect to the tensor $\underset{2}{\overline{R}}{}^i_{jmn}$ if there exists a tensor P^h_{ij} satisfying the equations* (29) *for any tensor $\underset{1}{a}_{ij}$ and vector $\underset{1}{b}_i$.*

For curvature tensors $\underset{3}{R}{}^i_{jmn}$ and $\underset{3}{\overline{R}}{}^i_{jmn}$ of the spaces GA_N and $G\overline{A}_N$ is satisfied the relation

$$\underset{3}{\overline{R}}{}^i_{jmn} = \underset{3}{R}{}^i_{jmn} + P^i_{jm\underset{2}{|}n} - P^i_{nj\underset{1}{|}m} + P^i_{n\alpha}P^\alpha_{jm} - P^i_{\alpha m}P^\alpha_{nj} + P^\alpha_{nm}L^i_{[\alpha j]} + P^\alpha_{nm}P^i_{[\alpha j]}.$$

Using

$$\underset{3}{\overline{R}}{}^i_{jmn} \equiv 0,$$

analogously to previous cases we obtain

Theorem 13. *A space GA_N is $(N-2)$-projective of the first type with respect to the tensor $\underset{3}{\overline{R}}{}^i_{jmn}$ if there exists a tensor P^h_{ij} satisfying the equation*

$$\begin{aligned} P^i_{(jm)\underset{2}{|}n} &+ P^i_{jm\underset{1}{|}n} + P^i_{[mn]\underset{1}{|}j} + P^i_{n\alpha}P^\alpha_{(jm)} + P^i_{\alpha n}P^\alpha_{jm} + P^i_{\alpha j}P^\alpha_{[mn]} \\ &+ L^i_{[\alpha j]}P^\alpha_{nm} + L^i_{[\alpha m]}P^\alpha_{nj} + P^\alpha_{nm}P^i_{[\alpha j]} + P^\alpha_{nj}P^i_{[\alpha m]} \\ &= -\underset{3}{R}{}^i_{(jm)n} + \underset{jmn}{Sym}\, \underset{1}{b}_j P^i_{mn} + \underset{jmn}{Sym}\, \underset{1}{a}_{jm}\delta^i_n \end{aligned} \tag{30}$$

for any tensor $\underset{1}{a}_{ij}$ and vector $\underset{1}{b}_i$.

For curvature tensors $\underset{4}{R}{}^i_{jmn}$ and $\underset{4}{\overline{R}}{}^i_{jmn}$ of the spaces GA_N and $G\overline{A}_N$ is satisfied the relation (see [5,7])

$$\underset{4}{\overline{R}}{}^i_{jmn} = \underset{4}{R}{}^i_{jmn} + P^i_{jm\underset{2}{|}n} - P^i_{nj\underset{1}{|}m} + P^i_{n\alpha}P^\alpha_{jm} - P^i_{\alpha m}P^\alpha_{nj} + P^\alpha_{mn}L^i_{[\alpha j]} + P^\alpha_{mn}P^i_{[\alpha j]}.$$

Using

$$\underset{4}{\overline{R}}{}^i_{jmn} \equiv 0,$$

analogously to previous cases we have

Theorem 14. *A space GA_N is $(N-2)$-projective of the first type with respect to the tensor $\underset{4}{\overline{R}}{}^i_{jmn}$ if there exists a tensor P^h_{ij} satisfying the equation*

$$\begin{aligned} P^i_{(jm)\underset{2}{|}n} &+ P^i_{jm\underset{1}{|}n} + P^i_{[mn]\underset{1}{|}j} + P^i_{n\alpha}P^\alpha_{(jm)} + P^i_{\alpha n}P^\alpha_{jm} + P^i_{\alpha j}P^\alpha_{[mn]} \\ &+ L^i_{[\alpha j]}P^\alpha_{mn} + L^i_{[\alpha m]}P^\alpha_{jn} + P^\alpha_{mn}P^i_{[\alpha j]} + P^\alpha_{jn}P^i_{[\alpha m]} \\ &= -\underset{4}{R}{}^i_{(jm)n} + \underset{jmn}{Sym}\, \underset{1}{b}_j P^i_{mn} + \underset{jmn}{Sym}\, \underset{1}{a}_{jm}\delta^i_n \end{aligned} \tag{31}$$

for any tensor $\underset{1}{a}_{ij}$ and vector $\underset{1}{b}_i$.

In the same manner, using a covariant derivative of the third and the fourth kind, we can find an analog relation for the tensor $\underset{5}{R}{}^i_{jmn}$.

In the case of the space A_N with symmetric affine connection, the relations (27, 29, 30, 31) reduce to (see [6])

$$3(P^h_{ij;k} + P^\alpha_{ij} P^h_{\alpha k}) = -R^{\;h}_{(ij)k} + \underset{ijk}{Sym}\, \underset{1}{b}_i P^h_{jk} + \underset{ijk}{Sym}\, \underset{1}{a}_{ij} \delta^h_k \tag{32}$$

where R^h_{ijk} is curvature tensor of the space A_N.

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