Novi Sad J. Math.
Vol. 29, No. 3, 1999, 325–330
XII Yugoslav Geometric Seminar
Novi Sad, October 8–11, 1998

DEGENERATE METRICAL (H,V)-STRUCTURE ON VECTOR BUNDLES

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Abstract

The theory of the pairs of structures is well applied in the case of the generalised Riemann spaces and Romanian geometrist had remarkable achievements in this field.

In order to solve this problem, professor R. Miron and Gh. Atanasiu have imposed the essential condition of permutability of Obata operators attached to the metrical and almost simplectic structures of the Riemann - Einsenhart space.

In this paper we will take into consideration the problem of geometric structure pairs, defined on the total space of a vector bundle from another point of view, i.e. a pair of structures is given by both a horizontal and vertical structure, these structures being called (h,v)-structures.

No such structures were studied so far. The determination of the compatible connections with such structures on E is simpler then the determination of the connections simultaneously compatible with two defined structures on E.

AMS Mathematics Subject Classification (1991): 53B40.
Key words and phrases: degenerated metric tensor, linear connection.

Let $\xi = (E,\pi,M)$ be a vector bundle over the n-dimensional manifold M, with an m-dimensional vector space as type fibre and let N be a non-linear connection on E.

Under these conditions, in any point $u \in E$, the tangent space TE_u and the vector bundle ξ can be written as $TE = HE \oplus VE$.

Let us denote with $\left(\frac{\delta}{\delta x^i}, \frac{\partial}{\partial y^a}\right)$, $i = 1,2,\dots,n$; $a = 1,2,\dots,m$, the frame adapted to the non-linear connection N in the module of vector fields $\chi(E)$ and with $(dx^i, \delta y^a)$ the correspondent coframe.

We may consider a Riemann metric $g = (g_{ij})$ in the horizontal bundle HE and in the vertical bundle VE, a d-tensorial field $A = (h_{ab})$ of $\binom{0\ 0}{0\ 2}$ type, symmetric, positively definite and of constant rank p, $1 \leq p \leq m$.

Under these conditions the tensorial field

$$G = g_{ij}(x,y)dx^i \oplus dx^j + h_{ab}(x,y)\delta y^a \oplus \delta y^b, \tag{1}$$

is symmetric, of constant rank $n+p$ on E, positively definite and on the total space E gives a degenerated Riemannian structure.

The tensorial field $G : \chi(E) \to \chi^*(E)$ associates to a vectorial field X the 1-form G_X, $G_X(Y) = G(X,Y)$ and $X \to g$, $g_X(Y) = G(X^H, Y^H)$, $X \to A_X$, $A_X(Y) = G(X^V, Y^V)$, $\forall X, Y \in \chi(E)$.

The system $h_{ac}X^c = 0$, admits $(m-p)$ linearly independent solutions which will be denoted by $\{\xi_x^a\}$, $a,b,c = 1,2,\ldots,m$, $x = p+1,\ldots,m$.

From

$$A_X\xi_x = 0,\ (h_{ac}\xi_x^c = 0),\ \forall x = p+1,\ldots,m \tag{2}$$

it follows that Ker $A = \{\xi_x\}$ represents a subdistribution E^{Vv} of the vertical distribution E^V.

If we denote by $\{\xi_x^a\}$ a orthonormed basis in relation to the metric (h_{ab}) in E^{Vv}, then the 1-form $\eta_a^x = h_{ac}\xi^c$ are linearly independent and represent the cobasis of fields $\{\xi_x\}$.

$$\eta^x(\xi_y) = \delta_y^x. \tag{3}$$

If we denote by $V = \sum\limits_{x=p+1}^{m} \eta^x \oplus \xi_x$ and $h = I_m - V$, we obtain two d-tensor fields of $\binom{0\ 1}{0\ 1}$ type, globally defined on E and they represent the projectors of the E^{Vv} and E^{Vh} subdistributions respectively. The h and v operators are the supplementary projectors on the module of vertical vector field.

In local coordinates, we have:

$$h_c^a h_b^c = h_b^a,\ v_c^a v_b^c = v_b^a,\ h_c^a v_b^a = v_c^a h_b^a = 0,\ h_b^a + v_b^a = \delta_b^a. \tag{4}$$

The matrix $B = \begin{pmatrix} h_{ab} & \eta_a^x \\ \eta_b^x & 0 \end{pmatrix}$ is symmetrical, non degenerate, having an inverse the matrix $B^{-I} = \begin{pmatrix} h^{ab} & \xi_x^a \\ \xi_x^b & 0 \end{pmatrix}$, symmetric and nondegenerate.

The following relations hold:

$$\begin{cases} h_{ac}h^{bc} + \sum_{x=1}^{m-p} \eta_a^x \xi_x^b = \delta_a^b,\ \eta_a^x \xi_y^a = \delta_y^x \\ h_{ac}\xi_x^c = h_{ca}\xi_x^c = 0 \\ \eta_c^x h^{bc} = \eta_c^x h^{cb} = 0. \qquad \forall x, y = p+1,\ldots,m. \end{cases} \tag{5}$$

Taking into account the first relation (5) we have $h^{a'b'} = M_a^{a'} M_b^{b'} h^{ab}$, so (h^{ab}) are the components of a d-tensor $\check{A} \in F\binom{0\ 2}{0\ 0}(E)$, globally defined on E.

Definition 1 *The d-connection compatible with degenerate Riemannian structure G, is a linear connection ∇ on E with the properties:*

1. *∇ preserves by parallelism the distributions N and E^V.*
2. *$\nabla_X G = 0$.*

Proposition 1 *A linear connection ∇ on E is a d-connection compatible with the structure G if and only if*

$$\nabla_X g = 0 \text{ and } \nabla_X A = 0. \tag{6}$$

Proposition 2 *If ∇^0 is a d-connection fixed on E, then the connection $\nabla_X = \nabla^0 + P_X$ is a d-connection if and only if $P_X = P_X^1 + P_X^2$, where $P_X^1 \in F\binom{0\ 1}{0\ 1}(E)$.*

We will determine the tensorial field P_X under the condition that the d-connection $\nabla_X = \nabla_X^0 + P_X$ is compatible with the degenerate metrical structure G.

The condition $(\nabla_X G)(X, Y) = 0$, $\forall X,\ Y \in \chi(E)$ is equivalent to the following relations:

$$X(g(Y^H, Z^H)) - g(\nabla_X Y^H, Z^H) - g(Y^H, \nabla_X Z^H) = 0 \tag{7}$$

$$X(A(Y^V, Z^V)) - A(\nabla_X Y^V, Z^V) - g(Y^V, \nabla_X Z^V) = 0. \tag{8}$$

The equation (7) is equivalent to:

$$g(P_X^1 Y^H, Z^H) + g(Y^H, P_X^1 Z^H) = (\nabla_X^0 g)(Y^H, Z^H) \tag{9}$$

from which we obtain:

$$P_X^1 = \frac{1}{2} g^{-1} \circ \nabla_X g + \Omega Q_X,\ Q_X \in F\begin{pmatrix}1\ 0\\1\ 0\end{pmatrix}(E) \tag{10}$$

where Ω and Ω^* are Obata's operators of metric structure g.

In local coordinates we have:

$$\begin{cases} L_{jk}^i = L_{jk}^{0i} + \frac{1}{2} g^{is} g_{js|^\circ k} \\ C_{jc}^i = C_{jc}^{0i} + \frac{1}{2} g^{is} g_{js|^\circ c} \end{cases}. \tag{11}$$

The equation (8) is equivalent to:

$$A(P^2_X Y^V, Z^V) + A(Y^V, P^2_X Z^V) = (\nabla^\circ_X A)(Y^V, Z^V). \tag{12}$$

Taking into account relations (5), then the equation (12) we get:

$$P^2(X, Y^V) - VP^2(X, Y^V) + A_{Y^V} P^2_X \bar{A} = (\nabla^\circ_X A)_{Y^V} \bar{A}. \tag{13}$$

If we consider the operators (V. Oproiu [15]):

$$\begin{cases} \Phi = \frac{1}{2}[I \oplus I + v \oplus I - A \oplus \bar{A}] \\ \Psi = \frac{1}{2}[I \oplus I - v \oplus I + A \oplus \bar{A}] \\ \Theta = \frac{1}{2}[v \oplus h] \end{cases} \tag{14}$$

we shall obtain the supplementary projectors:

$$\Lambda = \varphi - \Theta, \ \Lambda^* = \Psi + \Theta. \tag{15}$$

Thus, from (13) we have:

$$\Lambda^* P^2_X = \frac{1}{2}(\nabla^\circ_X A) \bullet \bar{A} + ((\nabla^\circ) \bullet A) \tag{16}$$

and it follows:

$$P^2_X = \frac{1}{2}(\nabla^\circ_X A) \bullet A + \Theta((\nabla^\circ_X A) \bullet \bar{A} + \Lambda Q_X), \tag{17}$$

where $Q_X \in F\binom{0\ 1}{0\ 1}(E)$ is an arbitrary tensor field.

In local coordinates we have:

$$\begin{cases} L^a_{bk} = L^{\circ a}_{bk} + \frac{1}{2}[h^{ac} h_{bc|^\circ k} + v^c_b h^{ad} h_{cd|^\circ k}] \\ C^a_{bc} = C^{\circ a}_{bc} + \frac{1}{2}[h^{ad} h_{bd|^\circ c} + v^e_b h^{ad} h_{ed|^\circ c}]. \end{cases} \tag{18}$$

Theorem 1 *There exist d-connections compatible with degenerate Riemann structure G, one of them is given by:*

$$\begin{cases} L^i_{jk} = L^{\circ i}_{jk} + \frac{1}{2} g^{is} g_{js|^\circ k} \\ L^a_{bk} = L^{\circ a}_{bk} + \frac{1}{2} h^{ac}[h_{bc|^\circ k} + v^e_b h_{ec|^\circ k}] \\ C^i_{jc} = C^{\circ i}_{jc} + \frac{1}{2} g^{is} g_{js|^\circ c} \\ C^a_{bc} = C^a_{bc} + \frac{1}{2} h^{ad}[h_{bd|^\circ c} + v^e_b h_{ed|^\circ c}] \end{cases} \tag{19}$$

where $(L^{\circ i}_{jk}, L^{\circ a}_{bk}, C^{\circ i}_{jc}, C^{\circ a}_{bc})$ *are the components of any d-connection on E.*

Theorem 2 *All d-connections compatible with the degenerate metric structures G are given by:*

$$\begin{cases} \bar{L}^i_{jk} = L^i_{jk} + \Omega^{si}_{jm} Q^m_{ks} \quad , \quad (Q^m_{ks}) \in F\binom{1\ 0}{2\ 0}(E) \\ \bar{L}^a_{bk} = L^a_{bk} + \Lambda^{da}_{be} Q^e_{kd} \quad , \quad (Q^e_{kd}) \in F\binom{0\ 1}{1\ 1}(E) \\ \bar{C}^i_{jc} = C^i_{jc} + \Omega^{si}_{jm} Q^m_{cs} \quad , \quad (Q^m_{cs}) \in F\binom{1\ 0}{1\ 1}(E) \\ \bar{C}^a_{bc} = C^a_{bc} + \Lambda^{da}_{be} Q^e_{cd} \quad , \quad (Q^e_{cd}) \in F\binom{0\ 1}{0\ 2}(E) \end{cases} \tag{20}$$

where Ω and Λ are operators of Obata type.

Remark. If rank $(h_{as}) = m$, then $v^a_b = 0$ and we obtain d-connections compatible with the metric structure G, [10], [7].

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