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A CLASSIFICATION OF 3–TYPE CURVES IN MINKOWSKI 3–SPACE E_1^3, I

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Abstract

In this paper we give a complete classification of all *planar* curves in Minkowski 3–space $\mathbb{E}_1^3$, which are of type 3. A corresponding classification of all *non–planar* curves of type 3 in the same space will be given in the part II of this paper.

1. Introduction

The notion of submanifolds of *finite type* was introduced by B. Y. Chen in [2]. A submanifold M in the Euclidean space E^n is said to be of *finite type* if each component of its position vector field x can be written as a finite sum of eigenfunctions of the Laplacian Δ of M. This means that

$$\mathrm{x} = \mathrm{x}_0 + \sum_{t=1}^{k} \mathrm{x}_t, \quad \Delta \mathrm{x}_t = \lambda_t \mathrm{x}_t, \tag{1.1}$$

where $0 = \lambda_0 < \lambda_1 < \ldots < \lambda_k$ are mutually different eigenvalues of Δ. When M is compact, the component x_0 in (1.1) is constant vector. However, when M is non–compact, the component x_0 is not necessary a constant vector. In particular, a submanifold M is said to be of *k–type* if all eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_k$ are different from zero. If one of the λ_i's is equal to zero ($i = 1, 2, \ldots, k$), M is said to be of *null k–type*.

Finite type curves in Euclidean space E^n were studied intensively in [2], [3] and [4]. The classification of all 2–type curves in E^n is given in [6].

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where $0 = \lambda_0 < \lambda_1 < \ldots < \lambda_k$ are mutually different eigenvalues of Δ. When M is compact, the component x_0 in (1.1) is constant vector. However, when M is non–compact, the component x_0 is not necessary a constant vector. In particular, a submanifold M is said to be of *k–type* if all eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_k$ are different from zero. If one of the λ_i's is equal to zero $(i = 1, 2, \ldots, k)$, M is said to be of *null k–type*.

Finite type curves in Euclidean space E^n were studied intensively in [2], [3] and [4]. The classification of all 2–type curves in E^n is given in [6].

2. Preliminaries

Minkowski 3 space E_1^3 is a manifold R^3 equipped with a metric tensor g of index 1. Define the metric tensor g with

$$g = -dx_1^2 + dx_2^2 + dx_3^2. \tag{2.1}$$

Let α be a curve in E_1^3 parameterized by an pseudo–arclength parameter s. Then the Laplace operator Δ of α is given by

$$\Delta = \pm \frac{d^2}{ds^2}. \tag{2.2}$$

Its eigenfunctions are $s, \cos(as), \sin(as), \cosh(as)$ and $\sinh(as)$. Following the definition of Chen, every *finite type* curve α in E_1^3 can be written as

$$\alpha(s) = a_0 + b_0 s + \sum_{t=1}^{k_1} (a_t \cos(p_t s) + b_t \sin(p_t s)) + \sum_{t=1}^{k_2} (c_t \cosh(q_t s) + d_t \sinh(q_t s)), \tag{2.3}$$

where $a_0, b_0, a_i, b_i, c_j, d_j \in R^3$ are constants, $i = 1, \ldots, k_1$, $j = 1, \ldots, k_2$ and $0 < p_1 < \ldots < p_{k_1}$, $0 < q_1 < \ldots < q_{k_2}$ are mutually different eigenvalues of Δ. In particular, a curve α in E_1^3 is said to be of *k–type* if there are k mutually different eigenvalues $\lambda_1, \ldots, \lambda_k$ of Δ and they are all different from zero. If one of the λ_i's $(i = 1, ..., k)$, is equal to zero, α is said to be of *null k–type*.

Recall that an arbitrary vector v in E_1^3 can have one of three causal characters: it can be *spacelike* if $g(v,v) > 0$ or $v = 0$, *timelike* if $g(v,v) < 0$, and *null* if $g(v,v) = 0$ and $v \neq 0$. The norm of a vector v is given by

$$||v|| = \sqrt{|g(v,v)|}. \tag{2.4}$$

The unit vectors, orthogonality and orthonormality are defined as in the Euclidean spaces. An arbitrary, unit–speed curve $\alpha(s)$ can localy be *spacelike, timelike* or *null* curve if respectively all of its velocity vectors $\dot{\alpha}(s)$ are spacelike, timelike or null vectors. An arbitrary plane π in E_1^3 can be spacelike plane, if $g|_\pi$ is positive definite, timelike plane, if $g|_\pi$ is nondegenerate of index 1, or isotropic (lightlike) plane, if $g|_\pi$ is degenerate.

Curves of finite type in Minkowski space–time have been investigated in [5] and also in [7], independently. The following classification theorem is obtained in [7].

where $\rho^2 - 12a\epsilon = 0$, $\rho, \epsilon, a \in R_0$;

(viii) $\alpha(s) = (\epsilon \sinh s + a \sinh 3s, \epsilon \cosh s + a \cosh 3s, \rho \sinh s)$,

where $\rho^2 - 12a\epsilon = 0$, $\rho, \epsilon, a \in R_0$.

All closed 3–type curves in Euclidean 3–space E^3 were classified by D. E. Blair in [1]. He obtained the following classification theorem.

Theorem 2.5. *A closed 3–type curve in E^3 is either a curve which lies on a quadric of revolution or a curve whose frequency ratio is $1:3:7$ and the curve belongs to a 3–parameter family of such curves or the frequency ratio is $1:3:5$ or the curve belongs to a 5–parameter family of such curves. Some curves with frequency ratio $1:3:5$ or $1:3:7$ also lie on quadrics of revolution.*

3. A classification of planar 3–type curves in E_1^3

In this part we will classify all planar 3–type curves in Minkowski 3–space E_1^3. Main results are contained in Theorems 3.1 and 3.2.

Theorem 3.1. *A planar 3–type curve, lying in an isotropic plane of E_1^3, is a null 3–type spacelike curve.*

Proof. Let α be a 3–type curve in E_1^3, parametrized by an pseudo–arclength parameter s. Then α can have one of the following forms:

$$\alpha(s) = a + bs + c\cos(ps) + d\sin(ps) + e\cosh(ts) + f\sinh(ts), \qquad \text{(i)}$$

$$\alpha(s) = a + bs + c\cos(ps) + d\sin(ps) + e\cos(ts) + f\sin(ts), \qquad \text{(ii)}$$

$$\alpha(s) = a + bs + c\cosh(ps) + d\sinh(ps) + e\cosh(ts) + f\sinh(ts), \qquad \text{(iii)}$$

$$\alpha(s) = a + b\cos(ps) + c\sin(ps) + d\cos(ts) + e\sin(ts) + f\cosh(qs) + h\sinh(qs), \qquad \text{(iv)}$$

$$\alpha(s) = a + b\cos(ps) + c\sin(ps) + d\cos(ts) + e\sin(ts) + f\cos(qs) + h\sin(qs), \qquad \text{(v)}$$

$$\alpha(s) = a + b\cosh(ps) + c\sinh(ps) + d\cosh(ts) + e\sinh(ts) + f\cos(qs) + h\sin(qs), \qquad \text{(vi)}$$

$$\alpha(s) = a + b\cosh(ps) + c\sinh(ps) + d\cosh(ts) + e\sinh(ts) + f\cosh(qs) + h\sinh(qs), \qquad \text{(vii)}$$

Theorem 2.1. *Every curve of finite type in Minkowski plane E_1^2 is of 1–type and hence an open part of an orthogonal hyperbola or an open part of a straight line.*

Curves of Chen–type 2 in E_1^3 are investigated in [8]. The following result is obtained there.

Theorem 2.2. *A planar 2–type curve, lying in an isotropic plane of E_1^3, is a null 2 type spacelike curve.*

Theorem 2.3. *Up to a rigid motions of E_1^3, a non–planar curve α in E_1^3 is a null 2–type curve if and only if α is a part of one of the following curves:*

(i) $\alpha(s) = (as, b\cos s, b\sin s), \quad a, b \in R_0, |a| \neq |b|;$

(ii) $\alpha(s) = (a\cosh s, a\sinh s, bs), \quad a, b \in R_0, |a| \neq |b|;$

(iii) $\alpha(s) = (a\sinh s, a\cosh s, bs), \quad a, b \in R_0, |a| \neq |b|;$

Theorem 2.4. *Up to a rigid motions of E_1^3, a non–planar curve α in E_1^3 is a 2–type curve with both eigenvalues different from zero if and only if α is a part of one of the following curves:*

(i) $\alpha(s) = (\rho\sin s, \epsilon\cos s + a\cos 3s, \epsilon\sin s + a\sin 3s),$

where $\rho^2 - 12a\epsilon = 0,\ a, \epsilon, \rho \in R_0;$

(ii) $\alpha(s) = (a\cosh s + \lambda b\sinh s - 4ce^{3\lambda s}, -b\cosh s - \lambda a\sinh s + 4ce^{3\lambda s},$ $2de^{\lambda s}),$

where $d^2 - 6(a-b)c = 0,\ a, b, c, d \in R_0,\ \lambda \in \{-1, 1\};$

(iii) $\alpha(s) = (ae^s + b\cosh 3s, ae^s + b\sinh 3s, ce^{-s}),$

where $c^2 + 6ab = 0,\ a, b, c \in R_0;$

(iv) $\alpha(s) = (\epsilon\cosh s + a\cosh 3s, \epsilon\sinh s + a\sinh 3s, \rho\cosh s),$

where $\rho^2 + 12a\epsilon = 0,\ a, \rho, \epsilon \in R_0;$

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Proof. Let α be a 3–type curve in E_1^3, parametrized by an pseudo–arclength parameter s. Then α can have one of the following forms:

$$\alpha(s) = a + bs + c\cos(ps) + d\sin(ps) + e\cosh(ts) + f\sinh(ts), \tag{i}$$

$$\alpha(s) = a + bs + c\cos(ps) + d\sin(ps) + e\cos(ts) + f\sin(ts), \tag{ii}$$

$$\alpha(s) = a + bs + c\cosh(ps) + d\sinh(ps) + e\cosh(ts) + f\sinh(ts), \tag{iii}$$

$$\begin{aligned}\alpha(s) =& a + b\cos(ps) + c\sin(ps) + d\cos(ts) + e\sin(ts)+\\ &+f\cosh(qs) + h\sinh(qs),\end{aligned} \tag{iv}$$

$$\begin{aligned}\alpha(s) =& a + b\cos(ps) + c\sin(ps) + d\cos(ts) + e\sin(ts)+\\ &+f\cos(qs) + h\sin(qs),\end{aligned} \tag{v}$$

$$\begin{aligned}\alpha(s) =& a + b\cosh(ps) + c\sinh(ps) + d\cosh(ts) + e\sinh(ts)+\\ &+f\cos(qs) + h\sin(qs),\end{aligned} \tag{vi}$$

$$\begin{aligned}\alpha(s) =& a + b\cosh(ps) + c\sinh(ps) + d\cosh(ts) + e\sinh(ts)+\\ &+f\cosh(qs) + h\sinh(qs),\end{aligned} \tag{vii}$$

where $a, b, c, d, e, f \in R^3$ and suppose $0 < p < t < q$.

Next, suppose that α lies in an isotropic (lightlike) plane in E_1^3, with the equation $x_1 = x_2$. Then the vectors b, c, d, e, f are of the form $b = (b_1, b_1, b_3), c = (c_1, c_1, c_3), d = (d_1, d_1, d_3), e = (e_1, e_1, e_3), f = (f_1, f_1, f_3)$, i.e. they are *spacelike* or *null* vectors. Besides, we can assume that $a = (0, 0, 0)$, up to a translation and let $g = -dx_1^2 + dx_2^2 + dx_3^2$. In the sequel, we shall consider cases (i)-(vii) separately.

Case (i). $\alpha(s) = a + bs + c\cos(ps) + d\sin(ps) + e\cosh(ts) + f\sinh(ts)$.

Since $g(\dot\alpha, \dot\alpha) = \pm 1$, using the linear independence of the functions $\sin x, \cos x, \sinh x, \cosh x$ we get the system of equations:

$$g(b,b) + \tfrac{p^2}{2}(g(c,c) + g(d,d)) + \tfrac{t^2}{2}(g(f,f) - g(e,e)) = \pm 1, \quad (1)$$

$$g(d,d) - g(c,c) = 0, \quad (2)$$

$$g(f,f) + g(e,e) = 0, \quad (3)$$

$$g(b,c) = g(b,d) = g(b,e) = g(b,f) = 0, \quad (4)$$

$$g(c,d) = g(c,e) = g(c,f) = 0, \quad (5)$$

$$g(d,e) = g(d,f) = 0, \quad (6)$$

$$g(e,f) = 0. \quad (7)$$

From the equations (1)–(7) follows that $b = (b_1, b_1, \pm 1), c = (c_1, c_1, 0)$, $d = (d_1, d_1, 0), e = (e_1, e_1, 0), f = (f_1, f_1, 0)$, so the curve α reads:

$$\begin{aligned}\alpha(s) = (&b_1 s + c_1\cos(ps) + d_1\sin(ps) + e_1\cosh(ts) + f_1\sinh(ts),\\ &b_1 s + c_1\cos(ps) + d_1\sin(ps) + e_1\cosh(ts) + f_1\sinh(ts), \pm s),\end{aligned}$$

where $b_1, c_1, d_1, e_1, f_1 \in R$, c_1 and d_1 are not both zero, e_1 and f_1 are not both zero. Consequently, α is a null 3–type spacelike curve.

Cases (ii) *and* (iii). The proof in these cases is analogous to the proof of case (i). So in these cases we get the curves:

$$\begin{aligned}\alpha(s) = (&b_1 s + c_1\cos(ps) + d_1\sin(ps) + e_1\cos(ts) + f_1\sin(ts),\\ &b_1 s + c_1\cos(ps) + d_1\sin(ps) + e_1\cos(ts) + f_1\sin(ts), \pm s),\\ \beta(s) = (&b_1 s + c_1\cosh(ps) + d_1\sinh(ps) + e_1\cosh(ts) + f_1\sinh(ts),\\ &b_1 s + c_1\cosh(ps) + d_1\sinh(ps) + e_1\cosh(ts) + f_1\sinh(ts), \pm s),\end{aligned}$$

where $b_1, c_1, d_1, e_1, f_1 \in R$, c_1 and d_1 are not both 0, e_1 and f_1 are not both 0. Consequently, α is a null 3–type spacelike curve.

Case (iv). If $0 < p < t < q$, we differ the next subcases:

$$2p = t - p < p + t < 2t < 2q \text{ and } 2p \neq t - p < p + t < 2t < 2q.$$

Therefore, we have the following subcases:

(iv.1) $2p = t - p$; (iv.2) $2p \neq t - p$;

Since $g(\dot\alpha, \dot\alpha) = \pm 1$, using the linear independence of the functions $\sin x, \cos x, \sinh x$ and $\cosh x$, in these subcases we obtain different systems of the equations.

(iv.1). In this case, the corresponding system reads:

$$\frac{p^2}{2}(g(b,b) + g(c,c)) + \frac{t^2}{2}(g(d,d) + g(e,e)) + \frac{q^2}{2}(g(h,h) + g(f,f)) = \pm 1, \tag{1}$$
$$\frac{p^2}{2}(g(c,c) - g(b,b)) + pt(g(b,d) + g(c,e)) = 0, \tag{2}$$
$$g(e,e) - g(d,d) = 0, \tag{3}$$
$$g(h,h) + g(f,f) = 0, \tag{4}$$
$$-p^2 g(b,c) + pt(g(b,e) - g(c,d)) = 0, \tag{5}$$
$$g(c,e) - g(b,d) = 0, \tag{6}$$
$$g(b,e) + g(c,d) = 0, \tag{7}$$
$$g(d,e) = 0, \tag{8}$$
$$g(f,h) = 0, \tag{9}$$
$$g(b,f) = g(b,h) = g(c,f) = g(c,h) = g(d,f) = g(d,h) = \tag{10}$$
$$= g(e,f) = g(e,h) = 0.$$

Now equations (3) and (8) imply $e = (e_1, e_1, 0)$, $d = (d_1, d_1.0)$, so $g(b,d) = = g(b,e) = g(c,d) = g(c,e) = 0$. Also equations (4) and (9) imply $f = (f_1, f_1, 0)$, $h = (h_1, h_1, 0)$, (2) and (5) imply $b = (b_1, b_1, 0)$, $c = (c_1, c_1, 0)$. All equations (2)–(10) are then satisfied, but equation (1) reads $0 = \pm 1$, thus we obtain a contradiction.

(iv.2). $2p \neq t - p$. Using a similar method as in the case (iv.1), we obtain a contradiction.

From the cases (iv.1) and (iv.2) we conclude that a curve α of the form (iv) does not exist.

Case (v). After calculation $g(\dot\alpha,\dot\alpha)=\pm1$, we get that arguments of the functions $\sin x$ and $\cos x$ are the numbers $2p$, $2t$, $2q$, $p+t$, $p+q$, $t+q$, $q-p$, $q-t$, $t-p$.

If $0<p<t<q$, we find that $2q=\max\{2p,2t,p+t,p+q,t+q,q-p,q-t,t-p\}$, so the coefficients of $\sin 2qs$ and $\cos 2qs$ are equal to 0. Therefore, $g(f,f)=g(h,h)=0$, $g(f,h)=0$, so $f=(f_1,f_1,0)$, $h=(h_1,h_1,0)$. Then $g(b,f)=g(b,h)=g(c,f)=g(c,h)=g(d,f)=g(d,h)=g(e,f)=g(e,h)=0$. Some of the numbers $2p$, $2t$, $p+t$, $p+q$, $t+q$, $q-p$, $q-t$, $t-p$ can be mutually equal. Let us look at $2t$. There are 4 possibilities for $2t$:

(v.1) $2t=p+q$; (v.2) $2t=q-t$;

(v.3) $2t=q-p$; (v.4) $2t\neq p+q,q-t,q-p$.

We shall discuss these cases separately.

(v.1). $2t=p+q$, $q-t=t-p$. In this case the coefficients of $\sin(2ts)=\sin(p+q)s$ and $\cos(2ts)=\cos(p+q)s$ must be 0, i.e. $g(d,e)=0$, $g(e,e)-g(d,d)=0$. Hence $e=(e_1,e_1,0)$, $d=(d_1,d_1,0)$. In order to find vectors b and c, look at number $2p$. There are the following possibilities:

(v.1.1) $2p=q-p$; (v.1.2) $2p=q-t$;

(v.1.3) $2p\neq q-p,q-t$.

We shalll again distinguish between all these cases.

(v.1.1). $2p=q-p$. Since the coefficients of $\sin(2ps)=\sin(q-p)s$ and $\cos(2ps)=\cos(q-p)s$ must be 0, we have

$$\begin{aligned}&-p^2g(b,c)+pq(g(b,h)-g(c,f))=0,\\&\tfrac{p^2}{2}(g(c,c)-g(b,b))+(g(b,f)+g(c,h))=0,\end{aligned}$$

whence $g(c,c)-g(b,b)=0, g(b,c)=0$, and thus $b=(b_1,b_1,0)$, $c=(c_1,c_1,0)$. Next the equation

$$\frac{p^2}{2}(g(b,b)+g(c,c))+\frac{t^2}{2}(g(d,d)+g(e,e))+\frac{q^2}{2}(g(f,f)+g(h,h))=\pm1 \quad (*)$$

becomes $0=\pm1$, which is a contradiction.

In cases (v.1.2) and (v.1.3) the equation $(*)$ also implies a contradiction.

(v.2). $2t=q-t$. Then we get $e=(e_1,e_1,0)$, $d=(d_1,d_1,0)$, because the coefficients of $\cos 2ts=\cos(q-t)s$ and $\sin 2ts=\sin(q-t)s$ must be 0. In order to find vectors b and c, we shall look at the number $2p$. Then the following subcases occure:

(v.2.1). $2p = t - p$; (v.2.2) $2p \neq t - p$;

In both cases (v.2.1) and (v.2.2), the equation $(*)$ also implies a contradiction.

(v.3). $2t = q - p$. Differing the subcases $2p = t - p$ and $2p \neq t - p$, the equation $(*)$ implies a contradiction.

(v.4). $2t \neq p + q, q - p, q - t$. Differing the subcases $2p = q - p$, $2p = q - t$, $2p = t - p$, $2p \neq q - p, q - t, t - p$, the equation $(*)$ implies a contradiction again.

So we obtain that in the cases (v.1), (v.2), (v.3) and (v.4) a curve α of the form (v) does not exist.

Cases (vi) *and* (vii). By using the same methods as in the cases (iv) and (v) respectively, it is easily seen that the curve α of forms (vi) or (vii) does not exist. This completes the proof of Theorem 3.1. ∎

Theorem 3.2. *There are no planar 3-type curve lying in an spacelike or in an timelike plane in* E^3_1.

Proof. Firstly, suppose that α is a unit-speed curve lying in an spacelike plane in E^3_1, with the equation $x_1 = 0$. As we have seen in the proof of Theorem 3.1, the curve α can have one of seven possible forms (i)–(vii), where the vectors $b = (0, b_2, b_3), c = (0, c_2, c_3), d = (0, d_2, d_3), e = (0, e_2, e_3)$ and $f = (0, f_2, f_3)$ are all *spacelike* vectors. We may take $a = (0, 0, 0)$, up to a translation and let the metric tensor g be of the form $g = -dx_1^2 + dx_2^2 + dx_3^2$. In the sequel, we shall consider cases (i)–(vii) separately.

Case (i). Since $g(\dot\alpha, \dot\alpha) = \pm 1$ and using the linear independence of the functions $\sin x, \cos x, \sinh x$ and $\cosh x$ we get the equation $g(e, e) + g(f, f) = 0$. Because e and f are spacelike vectors, it follows that $e = f = 0$. Consequently, we obtain a contradiction.

Cases (ii) *and* (iii). We differ the subcases $t = 2p$, $t = 3p$, $t \neq 2p, 3p$ and obtain a contradiction.

Case (iv). We differ the next subcases (iv.1) $t = 3p$ and (iv.2) $t \neq 3p$. In both of these subcases, we have the equation $g(f, f) + g(h, h) = 0$, so we obtain a contradiction.

Case (v). Since $g(\dot\alpha, \dot\alpha) = \pm 1$ and using the linear independence of functions $\sin x$ and $\cos x$, we get the equations:

$$g(h,h) - g(f,f) = 0, \quad (1)$$

$$g(f,h) = g(b,f) = g(b,h) = g(c,f) = g(c,h) = 0 \quad (2)$$

It follows that b and c are spacelike vectors which are orthogonal to the spacelike plane $\{f,h\}$. Thus $b = c = 0$ implies a contradiction.

Case (vi). As in the previous case (v), we obtain the same equations (1) and (2), so they imply a contradiction.

Case (vii). Since $g(\dot{\alpha},\dot{\alpha}) = \pm 1$ and since the coefficients of $\sinh(2qs)$ and $\cosh(2qs)$ must be 0, we have $g(f,f) + g(h,h) = 0$, which gives a contradiction.

Next, suppose that α is a unit–speed 3 type curve lying in an timelike plane in E_1^3, with the equation $x_3 = 0$. Again α can have one of seven possible forms (i)–(vii), as we have seen in the proof of the Theorem 3.1, where the vectors b, c, d, e and f now can be *spacelike, timelike* or *null* vectors. We may take $a = (0,0,0)$, up to a translation and take $g = -dx_1^2 + dx_2^2 + dx_3^2$. We shall consider cases (i)–(vii) separately.

Case (i). We get the same system of the equations as in the case (i) of the Theorem 3.1. If the vectors c and d are different from zero and not null vectors, then there would be two mutually orthogonal spacelike or timelike vectors in a timelike plane, which is impossible. It follows that for c and d holds $g(c,c) = g(d,d) = 0$. Since c and d are orthogonal to the timelike plane $\{e,f\}$ and they belong to it, it follows that $c = d = 0$. Consequently, we obtain a contradiction.

Cases (ii) *and* (iii). We differ subcases $t = 2p$, $t = 3p$, $t \neq 2p, 3p$ and again obtain a contradiction.

Case (iv). We differ subcases (iv.1) $t = 3p$ and (iv.2) $t \neq 3p$.

(iv.1). We get the same system of the equations as in the case (iv.1) of the Theorem 3.1. This system implies that $g(e,e) = g(d,d) = 0$ and since d and e are orthogonal to the timelike plane $\{f,h\}$, we have $d = e = 0$. Thus we obtain a contradiction.

(iv.2). Since $g(\dot{\alpha},\dot{\alpha}) = \pm 1$ and by using the linear independence of the functions $\sin x, \cos x, \sinh x$ and $\cosh x$, we obtain the equations

$$g(c,c) - g(b,b) = 0, \tag{1}$$

$$g(h,h) + g(f,f) = 0, \tag{2}$$

$$g(b,c) = g(f,h) = g(b,f) = g(b,h) = g(c,f) = g(c,h) = 0 \tag{3}$$

Then $\{f,h\}$ is a timelike plane and the vectors b and c are lying in it and are orthogonal to it. Thus we have $b = c = 0$, which means that α is not a curve of 3–type. This is a contradiction.

Case (v). After calculating $g(\dot\alpha, \dot\alpha) = \pm 1$ and using the inequality $0 < p < t < q$, we find that there are 4 subcases:

(v.1). $2t = p + q$; (v.2). $2t = q - t$;

(v.3). $2t = q - p$; (v.4). $2t \neq p + q, q - t, q - p$.

In all of these subcases we obtain the equations:

$$g(h,h) - g(f,f) = 0, \tag{1}$$

$$g(e,e) - g(d,d) = 0, \tag{2}$$

$$g(f,h) = g(d,e) = g(d,f) = g(d,h) = g(e,f) = g(e,h) = \tag{3}$$

$$= g(b,f) = g(b,h) = g(c,f) = g(c,h) = 0.$$

It follows that b, c, d, e, f, h are collinear null vectors, which implies a contradiction.

Case (vi). We differ the subcases (vi.1) $t = 3p$ and (vi.2) $t \neq 3p$.

(vi.1). Now, we obtain the equations:

$$g(h,h) - g(f,f) = 0, \tag{1}$$

$$g(e,e) + g(d,d) = 0, \tag{2}$$

$$g(d,e) = g(d,f) = g(d,h) = g(e,f) = g(e,h) = g(f,h) = 0. \tag{3}$$

Then $\{d,e\}$ is a timelike plane and the vectors f and h are orthogonal to $\{d,e\}$. Thus we have $f = h = 0$, which implies a contradiction.

(vi.2). This subcase is analogous to the subcase (vi.1) and again we obtain a contradiction.

Case (vii). As in the case (v), there are 4 subcases:

(vii.1) $2t = p + q$; (vii.2) $2t = q - t$;

(vii.3) $2t = q - p$; (vii.4) $2t \neq p + q, q - t, q - p$.

In all of these subcases we obtain the equations:

$$g(f,f) + g(h,h) = 0, \tag{1}$$

$$g(d,d) + g(e,e) = 0, \tag{2}$$

$$g(f,h) = g(d,e) = g(d,f) = g(d,h) = g(e,f) = g(e,h) = 0, \tag{3}$$

which means that $\{f,h\}$ is timelike plane and the vectors d and e are orthogonal to $\{f,h\}$. Thus $d = e = 0$, which is a contradiction.

This completes the proof of this theorem.

References

[1] D. E. Blair, A classification of 3–type curves, Soochow J. Math. 21, (1995), 145–158.

[2] B. Y. Chen, Total mean curvature and submanifolds of finite type, World Scientific, Singapore, (1984).

[3] B. Y. Chen, J. Deprez, F. Dillen, L. Verstraelen and L. Vrancken, Curves of finite Chen type, Geometry and Topology of Submanifolds, II, World Scientific, (1990), 76–110.

[4] B. Y. Chen, F. Dillen and L. Verstraelen, Finite type space curves, Soochow J.Math. 12, (1986), 1–10.

[5] H. S. Chung, D. S. Kim, K. H. Sohn, Finite type curves in the Lorentz Minkowski plane, Honam J. Math. 17 (1995), 41–47.

[6] F. Dillen, M. Petrović–Torgašev, L. Verstraelen and L. Vrancken, Classification of curves of Chen type 2, Differential Geometry, in honor of R. Rosca, K. U. Leuven, Fac. of Science, Leuven, (1991), 101–106.

[7] F. Dillen, J. V. Woestijne, L. Verstraelen, J. Walrave, Curves and ruled surfaced of finite type in Minkowski space, Geometry and Topology of Submanifolds VII, (1995), 124–127.

[8] J. Walrave, Curves and surfaces in Minkowski space, Doctoral thesis, K. U. Leuven, Fac. of Science, Leuven, (1995).