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TWO GENERAL FIXED POINT THEOREMS ON THREE COMPLETE METRIC SPACES

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Abstract. Two general fixed point theorems on three complete metric spaces which generalize the results from [1] and [2] for mappings satisfying implicit relations, are proved.

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1. Introduction

The following fixed point theorem was proved by Nung [1] and Jain, Shrivastava and Fisher [2].

Theorem 1. *[2] Let (X,d),(Y,ρ) and (Z,σ) be complete metric spaces and suppose T is a mapping of X into Y, S is a mapping of Y into Z and R is a mapping of Z into X satisfying the inequalities*

$$\begin{aligned} &d(RSy, RSTx).max\{d(x, RSy), d(x, RSTx)\} \leq \\ &\quad c\sigma(Sy, STx).max\{\sigma(Sy, STx), d(x, RSTx)\} \end{aligned}$$

$$\begin{aligned} &\rho(TRz, TRSy).max\{\rho(y, TRz), \rho(y, TRSy)\} \leq \\ &\quad cd(Rz, RSy).max\{d(Rz, RSy), \rho(y, TRSy)\} \end{aligned}$$

$$\begin{aligned} &\sigma(STx, STRz).max\{\sigma(z, STx), \sigma(z, STRz)\} \leq \\ &\quad c\rho(Tx, TRz), .max\{\rho(Tx, TRz), \sigma(z, STRz)\} \end{aligned}$$

for all x in X, y in Y, z in Z, where $0 \leq c < 1$. If one of the mappings R, S, T is continuous, then RST has a unique fixed point u in X, TRS has a unique fixed point v in Y, and STR has a unique fixed point w in Z. Further,$Tu = v$, $Sv = w$ and $Rw = u$.

Theorem 2. *[1] Let (X,d), (Y,ρ) and (Z,σ) be complete metric spaces and suppose T a continuous mapping of X into Y, S is a continuous mapping of Y into Z and R is a continuous mapping of Z into X, satisfying the inequalities*

$$\begin{aligned} d(RSTx, RSy) &\leq cmax\{\rho(y, Tx), d(x, RSTx), d(x, RSy), \sigma(Sy, STx)\} \\ \rho(TRSy, TRz) &\leq cmax\{\sigma(z, Sy), \rho(y, TRSy), \rho(y, TRz), d(Rz, RSy)\} \\ \sigma(STRz, STx) &\leq cmax\{\sigma(z, STx), \sigma(z, STRz), d(x, Rz), \rho(Tx, TRz)\}, \end{aligned}$$

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for all x in X, y in Y and z in Z, where $0 \leq c < 1$. *Then RST has a unique fixed point u in X, TRS has a unique fixed point v in Y, and STR has a unique fixed point w in Z. Further,* $Tu = v, Sv = w$ *and* $Rw = u$.

Theorem 3. *[2] Let* (X, d), (Y, ρ) *and* (Z, σ) *be complete metric spaces and suppose that T is a mapping of X into Y, S is a mapping of Y into Z, and R is a mapping of Z into X satisfying the inequalities*

$$\begin{aligned} d^2(RSy, RSTx) \leq\ & cmax\{d(x, RSy).\rho(y, Tx); \rho(y, Tx).d(x, RSTx); \\ & d(x, RSTx).\sigma(Sy, STx); \sigma(Sy, STx).d(x, RSy)\} \\ \rho^2(TRz, TRSy) \leq\ & cmax\{\rho(y, TRz).\sigma(z, Sy); \sigma(z, Sy).\rho(y, TRSy); \\ & \rho(y, TRSy).d(Rz, RSy); d(Rz, RSy).\rho(y, TRz)\} \\ \sigma^2(STx, STRz) \leq\ & cmax\{\sigma(z, STx).d(x, Rz); d(x, Rz).\sigma(z, STRz); \\ & \sigma(z, STRz).\rho(Tx, TRz); \rho(Tx, TRz).\sigma(z, STx)\} \end{aligned}$$

for all x in X, y in Y and z in Z, where $0 \leq c < 1$. *If one of the mappings* R, S, T *is continuous, then RST has a unique fixed point u in X, TRS has a unique fixed point v in Y and STR has a unique fixed point w in Z. Further,* $Tu = v$, $Sv = w$ *and* $Rw = u$.

In this paper, new generalizations of Theorem 1-3 are proved for the mappings satisfying three implicit relations on three complete metric spaces.

2. Implicit relations

Let F_4 be the set of all continuous functions $F : R_+^4 \to R$ such that there exists $h \in [0, 1)$ having the following property: for every $u \geq 0$, $v \geq 0$ with
a) $F(u, v, u, 0) \leq 0$ or
b) $F(u, v, 0, u) \leq 0$
we have $u \leq hv$.

Ex. 1. $F(t_1, \ldots, t_4) = t_1 - kmax\{t_2, t_3, t_4\}$ where $k \in [0, 1)$.

Let $u > 0$ and $F(u, v, u, 0) = u - kmax\{u, v, 0\} \leq 0$. If $u \geq v$ then $u(1-k) \leq 0$, a contradiction.

Thus $u < v$ and $u \leq hv$, where $h = k \in [0, 1)$. If $F(u, v, 0, u) \leq 0$ then $u \leq hv$. If $u = 0$ then $u \leq hv$.

Ex. 2. $F(t_1, \ldots, t_4) = t_1 max\{t_3, t_4\} - ct_2 max\{t_2, t_3\}$ where $c \in [0, 1)$.

Let $u > 0$ and $F(u, v, u, 0) = umax\{u, 0\} - cvmax\{u, 0\} \leq 0$, then $u^2 - cu \leq 0$ which implies $u \leq hv$, where $h = c \in [0, 1)$. Similary, $F(u, v, 0, u) \leq 0$ implies $u \leq hv$.

Ex. 3. $F(t_1, \ldots t_4) = t_1^3 - (at_1^2 t_2 + bt_1 t_3 t_4 + ct_2 t_3 t_4)$ where $a, b, c \geq 0$ and $a < 1$.

Let $u > 0$ and $F(u, v, u, 0) = u^3 - au^2 v \leq 0$. Then $u \leq hv$ where $h = a \in [0, 1)$. If $u = 0$ then $u \leq hv$. Similary, $F(u, v, 0, u) \leq 0$ implies $u \leq hv$. Let F_5 be the set of all continuous function $F : R_+^5 \to R$ such that there exists $h \in [0,1)$ having the following property: for every $u \geq 0$, $v \geq 0$ with $a')$ $F(u,v,u,0,w) \leq 0$ or $b')$ $F(u,v,0,u,w) \leq 0$ we have $u \leq hmax\{v,w\}$.

Ex. 4. $F(t_1,\ldots,t_5) = cmax\{t_2,\ldots,t_5\}$ where $c \in [0,1)$.

Let $u > 0$ and $F(u,v,u,0,w) = u - cmax\{u,v,w\} \leq 0$. If $u \geq max\{v,w\}$ then $u(1-c) \leq 0$, a contradiction. Thus $u < hmax\{v,w\}$ where $h = c \in [0,1)$. If $u = 0$ then $u \leq hmax\{v,w\}$. Similary, if $F(u,v,0,u,w) \leq 0$ we have $u \leq hmax\{v,w\}$.

Ex. 5. $F(t_1,\ldots,t_5) = t_1^2 - cmax\{t_4t_2, t_2t_3, t_3t_5, t_5t_4\}$ where $0 \leq c < 1$.

Let $u > 0$ and $F(u,v,u,0,w) = u^2 - cmax\{0, uv, uw\} \leq 0$. If $u \geq max\{v,w\}$ then $u^2(1-c) \leq 0$, a contradiction. Then $u < max\{v,w\}$ and $u \leq hv$ where $h = c^{\frac{1}{2}} \in [0,1)$. If $u = 0$ then $u \leq hmax\{v,w\}$. Similary, if $F(u,v,0,u,w) \leq 0$ we have $u \leq hmax\{v,w\}$.

Ex. 6. $F(t_1,\ldots,t_5) = t_1^3 + t_1^2 - (at_1t_2 + bt_1t_3 + ct_1t_4 + dt_5^2)$ where $0 \leq a+b+c+d < 1$.

Let $u > 0$ and $F(u,v,u,0,w) = u^3 + u^2 - (auv + bu^2 + dw^2) \leq 0$ which implies $u^2 - (auv + bu^2 + dw^2) \leq 0$. If $u \geq max\{v,w\}$ than $u^2(1-a-b-d) \leq 0$, a contradiction. Then, $u < max\{v,w\}$ and $u \leq hmax\{v,w\}$ where $h = \sqrt{a+b+c+d} \in [0,1)$. If $u = 0$ then $u \leq hmax\{v,w\}$. Similary if $F(u,v,0,u,w) \leq 0$ then $u \leq hmax\{v,w\}$.

3. Main results

Theorem 4. *Let (X,d),(Y,ρ) and (Z,σ) be complete metric spaces and suppose T is a mapping of X into Y, S is a mapping of Y into Z and R is a mapping of Z into X, satisfying the inequalities*

$$F(d(RSy,RSTx),\sigma(Sy,STx),d(x,RSTx),d(x,RSy)) \leq 0 \tag{1}$$

$$G(\rho(TRz,TRSy),d(Rz,RSy),\rho(y,TRSy),\rho(y,TRz)) \leq 0 \tag{2}$$

$$H(\sigma(STx,STRz),\rho(Tx,TRz),\sigma(z,STRz),\sigma(z,STx)) \leq 0 \tag{3}$$

for all x in X, y in Y and z in Z, where $F,G,H \in F_4$. If one of the mappings R,S,T is continuous, then RST has a unique fixed point u in X, TRShas a unique fixed point v in Y, and STR has a unique fixed point w in Z. Further, $Tu = v$, $Sv = w$, $Rw = u$.

Proof. Let x_0 be an arbitrary point in X. Define the sequence $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ in X,Y and Z,respectively, by

$$x_n = (RST)^n x_0,\ y_n = Tx_{n-1},\ z_n = Sy_n\ \ forn = 1,2,\ldots.$$

Applying the inequality (1) for $y = y_n$ and $x = x_n$ we have

$$-F(d(x_n, x_{n+1}), \sigma(z_n, z_{n+1}), d(x_n, x_{n+1}), 0) \leq 0$$

which implies by (a) that

$$d(x_n, x_{n+1}) \leq h_1\sigma(z_n, z_{n+1}) \tag{4}$$

where $h_1 \in [0, 1)$. Applying the inequality (3) for $x = x_{n-1}$ and $z = z_n$ we have

$$H(\sigma(z_n, z_{n+1}), \rho(y_n, y_{n+1}), \sigma(z_n, z_{n+1}), 0) \leq 0$$

which implies by (a) that

$$\sigma(z_n, z_{n+1}) \leq h_3\rho(y_n, y_{n+1}) \tag{5}$$

where $h_3 \in [0, 1)$. Applying the inequality (2) for $z = z_n$ and $y = y_n$ we have

$$G(\rho(y_n, y_{n+1}), d(x_{n-1}, x_n), \rho(y_n, y_{n+1}), 0) \leq 0$$

which implies by (a) that

$$\rho(y_n, y_{n+1}) \leq h_2 d(x_{n-1}, x_n) \tag{6}$$

where $h_2 \in [0, 1)$.

Now it follows from the inequalities (4), (5) and (6) that

$$d(x_n, x_{n+1}) \leq h_1\sigma(z_n, z_{n+1}) \leq h_1h_3\rho(y_n, y_{n+1}) \leq \ldots \leq (h_1h_2h_3)^n d(x_0, x_1)$$

Since $0 \leq h_1h_2h_3 < 1, \{x_n\}, \{y_n\}, \{z_n\}$ are Cauchy sequences with the limits u, v, w in X, Y and Z, respectively. Now suppose that S is continuous. Then

$$\lim_{n\to\infty} Sy_n = \lim_{n\to\infty} z_n$$

and so

$$Sv = w \tag{7}$$

Applying the inequality (1) we now have

$$F(d(RSv, x_{n-1}), \sigma(Sv, z_n), d(x_{n-1}, x_n), d(x_{n-1}, RSv)) \leq 0.$$

Letting n tend to infinity, it follows

$$F(d(RSv, u), \sigma(Sv, w), 0, d(u, RSv)) \leq 0.$$

Using equation (7) we have

$$F(d(RSv, u), 0, 0, d(u, RSv)) \leq 0.$$

By (b) follows that $d(u, RSv) \leq h.0$ which implies by (7) that

$$u = RSv = Rw \tag{8}$$

Applying the inequality (2) we have

$$G(\rho(Tu, y_{n+1}), d(u, x_n), \rho(y_n, y_{n+1}), \rho(y_n, TRw)) \leq 0.$$

Letting n tend to infinity,it follows that

$$G(\rho(Tu, v), 0, 0, \rho(v, Tu)) \leq 0.$$

By (b) follows that

$$Tu = v. \tag{9}$$

Now it follows from the equations (7), (8) and (9)

$$\begin{aligned} TRSv &= TRw = Tu = v, \\ STRw &= STu = Sv = w, \\ RSTu &= RSv = Rw = u. \end{aligned}$$

The same results of course will hold if R or T is continuous instead of S.
We now prove the uniqueness of the fixed point u. Suppose that RST has a second fixed point u'.Then using the inequality (1) we have

$$F(d(RSTu, RSTu'), \sigma(STu', STu), d(u, RSTu), d(u, RSTu') \leq 0$$
$$F(d(u, u'), \sigma(STu, STu'), 0, d(u, u')) \leq 0.$$

By (b) we have

$$d(u, u') \leq h_1 \sigma(STu, STu'). \tag{10}$$

Further, using the inequality (3) we have successively:

$$H(\sigma(STRSTu, STu'), \rho(Tu', TRSTu), 0, \sigma(STu, STu')) \leq 0$$
$$H(\sigma(STu, STu'), \rho(Tu', Tu), 0, \sigma(STu, STu')) \leq 0.$$

By (b) we have

$$\sigma(STu, STu') \leq h_3 \rho(Tu, Tu'). \tag{11}$$

Finally, using the inequality (2), we have

$$\rho(Tu, Tu') \leq h_2 d(u, u'). \tag{12}$$

By (10), (11) and (12) we have

$$d(u, u') \leq (h_1 h_2 h_3) d(u, u')$$

which implies $u = u'$. The fixed point u of RST is therefore unique. Similary, it can be proved that v is the unique fixed point of TRS and w is the unique fixed point of STR. This completes the proof of the theorem. □

Corollary 1. *Theorem 1.*

Proof. It follows from Theorem 4 and Ex. 1. □

Theorem 5. *Let (X, d) , (Y, ρ) , (Z, σ) be complete metric spaces and suppose T is a mapping of X into Y, S is a mapping of Y into Z, and R is a mapping of Z into X, satisfying the inequalities*

(1') $\quad F(d(RSy, RSTx), \rho(y, Tx), d(x, RSTx), d(x, RSy), \sigma(Sy, STx)) \leq 0$

(2') $\quad F(\rho(TRz, TRSy), \sigma(z, Sy), \rho(y, TRSy), \rho(y, TRz), d(Rz, RSz)) \leq 0$

(3') $\quad F(\sigma(STx, STRz), d(x, Rz), \sigma(z, STRz), \sigma(z, STx), \rho(Tx, TRz)) \leq 0$

for all x in X, y in Y and z in Z where $F \in F_5$. If one of the mappings R, S, T is continuous, then RST has a unique fixed point u in X, TRS has a unique fixed point v in Y, and STR has a unique fixed point w in Z. Further, $Tu = v$, $Sv = w$ and $Rw = u$.

Proof. Let x_0 be an arbitrary point in X and define the sequence $\{x_n\}, \{y_n\}, \{z_n\}$ in X, Y and Z, respectively, as in the proof of Theorem 4. Applying the inequality $(2')$ for $z = z_{n-1}$ and $y = y_n$ we have

$$F(\rho(y_n, y_{n+1}), \sigma(z_{n-1}, z_n), \rho(y_n, y_{n+1}), 0, d(x_{n-1}, x_n)) \leq 0$$

which by (a') implies that

$$(4') \qquad \rho(y_n, y_{n+1}) \leq h max\{d(x_{n-1}, x_n), \sigma(z_{n-1}, z_n)\}.$$

Applying the inequality $(3')$ for $x = x_{n-1}$ and $z = z_n$ we have

$$F(\sigma(z_n, z_{n+1}), d(x_{n-1}, x_n), \sigma(z_n, z_{n+1}), 0, \rho(y_n, y_{n+1})) \leq 0$$

which by (a') and the inequality $(4')$ implies that

$$(5') \qquad \begin{aligned} \sigma(z_n, z_{n+1}) &\leq h max\{d(x_{n-1}, x_n), \rho(y_n, y_{n+1})\} \\ &\leq h max\{d(x_{n-1}, x_n), \sigma(z_{n-1}, z_n)\}. \end{aligned}$$

Applying the inequality $(1')$ for $y = y_n$ and $x = x_n$ we have

$$F(d(x_n, x_{n+1}), \rho(y_n, y_{n+1}), d(x_n, x_{n+1}), 0, \sigma(z_n, z_{n+1})) \leq 0$$

which by (a') and the inequality $(4')$ and $(5')$ implies that

$$(6') \qquad \begin{aligned} d(x_n, x_{n+1}) &\leq h max\{\rho(y_n, y_{n+1}), \sigma(z_n, z_{n+1})\} \\ &\leq h max\{d(x_n, x_{n-1}), \sigma(z_{n-1}, z_n)\}. \end{aligned}$$

Now it follows easily by induction on using the inequalities $(4')$, $(5')$ and $(6')$ that

$$\begin{aligned} d(x_n, x_{n+1}) &\leq h^{n-1} max\{d(x_1, x_2), \sigma(z_1, z_2)\}, \\ \rho(y_n, y_{n+1}) &\leq h^{n-1} max\{d(x_1, x_2), \sigma(z_1, z_2)\}, \\ \sigma(z_n, z_{n+1}) &\leq h^{n-1} max\{d(x_1, x_2), \sigma(z_1, z_2)\} \end{aligned}$$

Since $0 \leq h < 1$, it follows that $\{x_n\}, \{y_n\}, \{z_n\}$ are Cauchy sequences with the limits u, v and w in X, Y and Z, respectively.

Now suppose that S is continuous. Then

$$\lim_{n\to\infty} Sy_n = \lim_{n\to\infty} z_n$$

and so

$$(7') \qquad Sv = w$$

Applying the inequality $(1')$ for $y = v$ and $x = x_n$ we now have

$$F(d(RSv, x_{n+1}), \rho(v, Tx_n), d(x_n, x_{n+1}), d(x_n, RSv), \sigma(Sv, STx_n)) \leq 0.$$

Letting n tend to infinity it follows

$$F(d(RSv, u), 0, 0, d(RSv, u), 0) \leq 0$$

which by (b') implies that $d(RSv, u) = 0$ and so

$$(8') \qquad Rsv = u$$

Using the equation $(7')$ this gives us

$$(9') \qquad Rw = u$$

Using the equation $(8')$ and the inequality $(2')$ for $z = Sv$ and $y = y_n$, we have

$$F(\rho(Tu, y_{n+1}), \sigma(Sv, Sy_n), \rho(y_n, TRSy_n), \rho(y_n, TRSv), d(RSv, RSy_n)) \leq 0.$$

Letting n tend to infinity it follows

$$F(\rho(Tu, v), 0, 0, \rho(v, Tu), 0) \leq 0$$

which by (b') implies that $\rho(Tu, v) = 0$ and so

$$(10') \qquad Tu = v$$

It follows from the equations $(7')$, $(9')$ and $(10')$ that

$$TRSv = TRw = Tu = v,$$
$$STRw = STu = Sv = w,$$
$$RSTu = RSv = Rw = u.$$

The same results of course hold if R or T is continuous instead of S.

We now prove the uniqueness of the fixed point u. Suppose that RST has a second fixed point u'. Then using the inequality (1') for $y = Tu$ and $x = u'$ we have

$$F(d(u, u'), \rho(Tu, Tu'), 0, d(u, u'), \sigma(STu, STu')) \leq 0$$

which by (b') implies that

$$(11') \qquad d(u,u') \le hmax\{\rho(Tu,Tu'),\sigma(STu,STu')\}.$$

Further, using the inequality $(2')$ for $z = STu$ and $y = Tu'$ we have

$$F(\rho(Tu,Tu'),\sigma(STu,STu'),0,\rho(Tu,Tu'),d(u,u')) \le 0$$

which by (b') implies that

$$(12') \qquad \rho(Tu,Tu') \le hmax\{\sigma(STu,STu'),d(u,u')\}.$$

The inequalities $(11')$ and $(12')$ imply that

$$(13') \qquad d(u,u') \le h\sigma(STu,STu').$$

Finally, using the inequality $(3')$, we have

$$F(\sigma(STu,STu'),d(u,u'),0,\sigma(STu,STu'),\rho(Tu,Tu')) \le 0$$

which by (b') implies

$$(14') \qquad \sigma(STu,STu') \le hmax(\{d(u,u'),\rho(Tu,Tu')\}.$$

It now follows from the inequalities $(12')$, $(13')$ and $(14')$ that

$$d(u,u') \le h\sigma(STu,STu') \le h^2\sigma(STu,STu')$$

and so $u = u'$, since $h < 1$. The fixed point u of RST is therefore unique. Similary, it can be proved that v is the unique fixed point of TRS and w is the unique fixed point of STR. This completes the proof of Theorem. □

Corollary 2. *Theorem 2.*

Proof. It follows from Theorem 5 and Ex. 4. □

Corollary 3. *Theorem 3.*

Proof. It follows from Theorem 5 and Ex. 5. □

References

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