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A NOTE ON NONLINEAR ORDINARY DIFFERENTIAL EQUATIONS CONTAINING DERIVATIVES OF THE DELTA FUNCTION

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Abstract. We consider a class of nonlinear ODEs perturbed by the delta function or its derivatives. Actually, we consider equations in which we approximate the delta function with Friedrich's mollifiers, since there are no classical solutions in general. We are interested in cases when families of solutions converge to some classical function. The obtained limit is an analogue to a certain delta wave, a notion used in PDE's theory.

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1. Introduction

Ordinary differential equations with some terms belonging to the space of Borel measures are considered in the series of Persson's papers, see [3] or [4], for example. He introduced a concept of solving such equations in some measure space by proving that a measure valued solution can be reconstructed from a net of solutions obtained by approximating a measure with a net of smooth functions. Persson's work and related problems were the motivations for the papers [1], [2] and [5]. Papers [1] and [5] had also a strong motivation in physics, a second-order system of equations arising in the general theory of relativity. Paper [2] is devoted to the first order equations (and some systems) and derivatives of the delta function. The idea was to obtain as large as possible class of equations which has a limit of solutions obtained by approximating delta distribution and its derivatives by nets of smooth functions.

The following equation

$$y'(t) = f(t, y(t)) + \alpha\delta^{(s)}(t), y(-1) = y_0 \tag{1}$$

is considered in [2]. The authors supposed that $f \in C^1([-1, T] \times \mathbb{R})$ and that there was a classical solution to

$$y'(t) = f(t, y(t))$$

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for every initial data at the point $t=-1$ (at least for $-1 \leq t \leq T$, $T>0$) and for every initial data at the point $t=0$ (at least for $0 \leq t \leq T$, $T>0$).

We shall briefly describe the results of [2].

The procedure of solving equation (1) begins with the substitution of the delta function by a net of mollifiers $\phi_\varepsilon(x) = \dfrac{1}{\varepsilon}\phi\left(\dfrac{x}{\varepsilon}\right)$, $\phi \geq 0$, $\int \phi = 1$. Instead of equation (1), the family of equations

$$y_\varepsilon'(t) = f(t, y_\varepsilon(t)) + \alpha\phi_\varepsilon^{(s)}(t), y(-1) = y_0,\ \varepsilon \in (0,1), \tag{2}$$

should be solved. Then, we are interested whether there exists a limit of the solutions $y_\varepsilon(t)$ as $\varepsilon \to 0$? And, if the limit exists, is it independent on a net of mollifiers?

Suppose that $\phi \in C_0^\infty(\mathbb{R})$, $\operatorname{supp}\phi = [-a,b]$, $a,b \geq 0$. Then the support of ϕ_ε and ϕ_ε' is the interval $[-a\varepsilon, b\varepsilon]$, $\varepsilon \in (0,1)$. Because of that, one has to take care that the solution $y_\varepsilon(t)$ does not blow up in this interval. Outside of this interval there is a classical solution (by the assumption). One also needs the existence of the $\lim_{\varepsilon\to 0} y_\varepsilon(b\varepsilon)$ which has to be the initial value of the classical problem after zero. In [2], one can find the following theorem.

Theorem 1. *Let f be a sublinear function of order r with respect to y uniformly on compact intervals with respect to t.*
a) If $s < 1/r$, or s is arbitrary if f is bounded, then the solution to

$$y_\varepsilon'(t) = f(t, y_\varepsilon(t)) + \alpha\phi_\varepsilon^{(s)}(t),\ y_\varepsilon(-1) = y_0 \tag{3}$$

converges to $\bar{y}(t) + \alpha\delta^{(s-1)}(t)$, where $\bar{y}(t)$, $t \in [-1,T]$ is a classical solution to

$$y'(t) = f(t, y(t)),\ y(-1) = y_0.$$

b) Let f be globally Lipschitz with respect to y uniformly on compact intervals with respect to t and let

$$\lim_{y\to\infty} \frac{f(y)}{y} = M_+,\ \lim_{y\to-\infty} \frac{f(y)}{y} = M_-.$$

Then the solution to (3) for $s=1$ converges to $y(t) = \bar{y}(t) + \alpha\delta(t)$, where

$$\bar{y}(t) = \begin{cases} \bar{y}_1(t), & t \in [-1,0] \\ \bar{y}_2(t), & t \in [0,\infty) \end{cases},$$

$\bar{y}_1$ is a classical solution to

$$y'(t) = f(t, y(t)),\ y(-1) = y_0,\ t \in [-1,0],$$

and $\bar{y}_2$ is a classical solution to

$$y'(t) = f(t, y(t)),\ y(0) = \bar{y}(0) + \beta,\ \beta \in \mathbb{R},\ t \in [0,\infty).$$

The constant β depends on $\int_{-1}^{1}\phi_+(t)dt$ and $\int_{-1}^{1}\phi_-(t)dt$, where ϕ_+ and ϕ_- are functions given by

$$\phi_+(t) = \max\{\phi(t), 0\} \text{ and } \phi_-(t) = -\min\{\phi(t), 0\}.$$

If $\phi(t) \geq 0$, $t \in \mathbb{R}$, then β does not depend on a mollifier ϕ. In this paper ϕ will be non-negative.

Let $s = 1$. The assumption that f is globally Lipschitz is not necessary for the existence of the limit of generalized solutions. The aim of this paper is to analyze functions which are not globally Lipschitz but the limit of the net of solutions exists.

We will consider the cases

(a) $f = -Cy|y|^p$ where $C = \text{const} > 0$, $p > 1$ and $\alpha > 0$;

(b) f satisfies conditions (7) and (8) given below and $f(t, 0) = 0$;

(c) f satisfies conditions of (b) and f depends only on y.

In cases (a) and (c) we will prove the existence of the limit of solutions, while in case (b) we will only show that the net of solutions is bounded.

2. Results

Lemma 1. *There exists a limit of the solutions to*

(4) $y_\varepsilon'(t) = -Cy_\varepsilon|y_\varepsilon|^p + \alpha\phi_\varepsilon'(t),\ y(-1) = y_0,\ p > 1,\ C = \text{const} > 0,\ \alpha > 0$

as $\varepsilon \to 0$.

Proof. Obviously, a solution to $y' = -Cy\,|y|^p$, $p > 1$ decreases for positive y and increases for negative y.

First, note that a solution to

$$y_\varepsilon'(t) = f(t, y_\varepsilon(t)) + \alpha\phi_\varepsilon'(t)$$

can be written in the following form

$$y_\varepsilon(t) = y_{1\varepsilon}(t) + \alpha\phi_\varepsilon(t),$$

where $y_{1\varepsilon}$ is a solution to

$$y_{1\varepsilon}'(t) = f(t, y_{1\varepsilon}(t) + \alpha\phi_\varepsilon(t))$$

with the same initial data. Before the point $t = -a\varepsilon$, the solution to (4) is a classical function $\overline{y}_1(t)$, independent on ε. Let us denote $y_{0\varepsilon} = \overline{y}_1(-a\varepsilon)$.

Suppose that $y_{0\varepsilon} > 0$.

By using the comparison theorem one can see that the solution to (4) with initial data $y_{1\varepsilon}(-a\varepsilon) = y_{0\varepsilon}$ is less or equal to a solution to

$$v'(t) = f(t, v(t) + g_\varepsilon(t)), \ v(-a\varepsilon) = y_{0\varepsilon}, \tag{5}$$

where $g_\varepsilon(t) \le \alpha\phi_\varepsilon(t)$ and

$$g_\varepsilon(t) = \begin{cases} 0, & t < -\overline{a}_\varepsilon \\ \overline{\xi}_\varepsilon, & t \in [-\overline{a}_\varepsilon, \overline{b}_\varepsilon] \\ 0, & t > \overline{b}_\varepsilon \end{cases}$$

for some $\overline{a}_\varepsilon \le a\varepsilon$, $\overline{b}_\varepsilon \le b\varepsilon$ and for $\overline{\xi}_\varepsilon \to \infty$, as $\varepsilon \to 0$.

This means that $y_{1\varepsilon}(t) \le v(t)$, where

$$v'(t) = \begin{cases} -Cv\,|v|^p, & v(-a\varepsilon) = y_{0\varepsilon}, \quad t \in [-a\varepsilon, -\overline{a}_\varepsilon] \\ -C\left(v + \overline{\xi}_\varepsilon\right)\left|v + \overline{\xi}_\varepsilon\right|^p, & v(-\overline{a}_\varepsilon) = \overline{y}_{0\varepsilon}, \quad t \in [-\overline{a}_\varepsilon, \overline{b}_\varepsilon] \\ -Cv\,|v|^p, & v(\overline{b}_\varepsilon) = \widetilde{y}_{0\varepsilon}, \quad t \in [\overline{b}_\varepsilon, b\varepsilon] \end{cases}. \tag{6}$$

Solution of the equation

$$v'(t) - C\left(v + \overline{\xi}_\varepsilon\right)\left|v + \overline{\xi}_\varepsilon\right|^p, \ v(-\overline{a}_\varepsilon) = \overline{y}_{0\varepsilon}, \ t \in [-\overline{a}_\varepsilon, \overline{b}_\varepsilon]$$

is given by

$$v(t) = \frac{\overline{y}_{0\varepsilon} + \overline{\xi}_\varepsilon}{\sqrt[p]{(-1)^p + Cp\left(\overline{y}_{0\varepsilon} + \overline{\xi}_\varepsilon\right)^p (t + \overline{a}_\varepsilon)}} - \overline{\xi}_\varepsilon.$$

Note that

$$\overline{\xi}_\varepsilon \cdot \overline{a}_\varepsilon \le C_\phi < 1,$$

where C_ϕ is a constant which depends on ϕ. Then

$$\overline{\xi}_\varepsilon^p \cdot \overline{a}_\varepsilon = \overline{\xi}_\varepsilon \cdot \overline{a}_\varepsilon \cdot \overline{\xi}_\varepsilon^{p-1} \le C_\phi \cdot \overline{\xi}_\varepsilon^{p-1},$$

and

$$v(t) \le \frac{\overline{\xi}_\varepsilon}{\text{const} \cdot \overline{\xi}_\varepsilon^{\frac{p-1}{p}}} - \overline{\xi}_\varepsilon,$$

for ε small enough. The term on the right-hand side tends to $-\infty$ as ε tends to zero (since $p > 1$). One can easily see that $v(\overline{b}_\varepsilon)$ is not bounded as $\varepsilon \to 0$.

Obviously, the first and the third equation in (6) do not change point $v(b\varepsilon)$ significantly (i.e. only finite shifts can occur).

We know that $\phi_\varepsilon(t) = 0$ for $t > b\varepsilon$ and

$$y'_{1\varepsilon}(t) = -Cy_{1\varepsilon}(t)\,|y_{1\varepsilon}(t)|^p, \ y_{1\varepsilon}(b\varepsilon) = \widetilde{\xi}_\varepsilon,$$

where $\widetilde{\xi}_\varepsilon$ is some point which tends to $-\infty$ as $\varepsilon \to 0$. In other words

$$y'_{1\varepsilon}(t) = \left(\widetilde{\xi}_\varepsilon^{-p} + p(t - b\varepsilon)\right)^{-\frac{1}{p}},$$

and $y'_{1\varepsilon}(t) \to \overline{y}_1(t) = -(pt)^{-\frac{1}{p}}$ for $t > 0$, as ε tends to zero.

So, the solution to

$$y'_{1\varepsilon}(t) = -Cy_{1\varepsilon}(t)\,|y_{1\varepsilon}(t)|^p + \alpha\phi_\varepsilon(t)$$

converges to $\overline{y}_1(t) + \alpha\delta(t)$, where

$$\overline{y}_1(t) = \begin{cases} -(pt)^{-\frac{1}{p}}, & t > 0 \\ y_0\left((-1)^p + Cp(t+1)\,y_0^p\right)^{-\frac{1}{p}}, & t < 0 \end{cases}.$$

One can similarly analyze the case $y_{0\varepsilon} < 0$. □

Lemma 2. *Let f be a monotone function such that $f(t,0) = 0$ for every t and let the following holds:*

If $y > 0$

(7) $$f(t,y) \le -C_1 y|y|^{p_1},\ t \in [-t_0, t_0],\ \textit{for some } C_1 > 0,\ \textit{and } p_1 > 1,$$

and if $y < 0$

(8) $$f(t,y) \ge -C_2 y|y|^{p_2},\ t > 0,\ \textit{for some } C_2 > 0,\ \textit{and } p_2 > 1.$$

Then, the net of solutions to

$$y'_\varepsilon(t) = f(t, y_\varepsilon) + \alpha\phi'_\varepsilon(t),\ \varepsilon \in (0,1)$$

is bounded by $y = 0$ and by solution to $y' = -C_2 y|y|^{p_2}$ (see Lemma 1) for $t > 0$ and for every non-negative $\phi \in C_0^\infty(\mathbb{R})$, $\alpha \ge 0$. Additionally, $y_{1\varepsilon}(0) \to -\infty$ as $\varepsilon \to 0$.

Proof. Let us consider

$$y'_{1\varepsilon}(t) = f\left(t, y_{1\varepsilon}(t) + \alpha\phi_\varepsilon(t)\right),\ y_{0\varepsilon} = \overline{y}_1(-a\varepsilon) > 0.$$

By comparison theorem (f is monotone) its solution is less or equal to a solution to

$$v'(t) = f\left(t, v(t) + g_\varepsilon(t)\right),\ v(-a\varepsilon) = y_{o\varepsilon},$$

where

$$g_\varepsilon(t) = \begin{cases} 0, & t < -\overline{a}_\varepsilon \\ \overline{\xi}_\varepsilon, & t \in \left[-\overline{a}_\varepsilon, \overline{b}_\varepsilon\right] \\ 0, & t > \overline{b}_\varepsilon \end{cases}$$

for some $\overline{a}_\varepsilon \le a\varepsilon$, $\overline{b}_\varepsilon \le b\varepsilon$ and for $\overline{\xi}_\varepsilon \to \infty$, as $\varepsilon \to 0$.

This implies that $y_{1\varepsilon}(t) \le v(t)$, where

(9) $$v''t) = \begin{cases} f(t,v), & v(-a\varepsilon) = y_{0\varepsilon}, & t \in [-a\varepsilon, -\overline{a}_\varepsilon] \\ f\left(t, v + \overline{\xi}_\varepsilon\right), & v(-\overline{a}_\varepsilon) = \overline{y}_{0\varepsilon}, & t \in \left[-\overline{a}_\varepsilon, \overline{b}_\varepsilon\right] \\ f(t,v), & v\left(\overline{b}_\varepsilon\right) = \widetilde{y}_{0\varepsilon}, & t \in \left[\overline{b}_\varepsilon, b\varepsilon\right] \end{cases}.$$

Note $v_0 + \overline{\xi}_\varepsilon$ is positive, for ε small enough. This implies that v starts to decrease. It will be a decreasing function until $v = -\overline{\xi}_\varepsilon$.

Suppose that t_0 is the first point when $v(t_0) = -\overline{\xi}_\varepsilon$. Then the unique solution to

$$v'(t) = f\left(t, v + \overline{\xi}_\varepsilon\right) = f(t, 0) = 0, \; v(t_0) = -\overline{\xi}_\varepsilon$$

is $v(t) = \text{const} = -\overline{\xi}_\varepsilon$ in some interval around t_0. This is a contradiction with the fact that v decreases for $t < t_0$. Specially, $v + \overline{\xi}_\varepsilon > 0$. Now we obtain that

$$f\left(t, v + \overline{\xi}_\varepsilon\right) \le -C_1\left(v + \overline{\xi}_\varepsilon\right)\left|v + \overline{\xi}_\varepsilon\right|^{p_1}, \; p_1 > 1.$$

This implies that a solution to

$$v'(t) = f\left(t, v + \overline{\xi}_\varepsilon\right)$$

is less or equal to a solution to

$$v'(t) = -C_1\left(v + \overline{\xi}_\varepsilon\right)\left|v + \overline{\xi}_\varepsilon\right|^{p_1}, \; p_1 > 1.$$

By using Lemma 1, we have

$$y(\overline{b}_\varepsilon) \in \left(-\overline{\xi}_\varepsilon, \frac{\overline{\xi}_\varepsilon}{\text{const} \cdot \overline{\xi}_\varepsilon^{\frac{p-1}{p}}} - \overline{\xi}_\varepsilon\right).$$

As in the proof of Lemma 1 one can see that the first and the third equations in (9) do not change this point significantly (only for a finite value).

By comparison theorem, one can see that (8) implies the assertion. □

Theorem 2. *Let f be a function depending only on y, $f(y)$, and let it satisfy (7), (8), $f(0) = 0$, and the general assumption from the beginning of the paper. Then, a net of solutions to*

$$y_\varepsilon'(t) = f(y_\varepsilon(t)) + \alpha\phi_\varepsilon'(t), \; y(-1) = y_0, \; \varepsilon < 1,$$

converges to $\overline{y}(t) + \alpha\delta(t)$, where $\overline{y}(t)$ is a solution to

$$y'(t) = f(y(t)), \; y(-1) = y_0, \; t \in [-1, 0)$$

given implicitly by

$$\int_{-\infty}^{\overline{y}} dy/f(y) = t, \; t > 0,$$

i.e. $\overline{y}$ is the unique solution to

$$y'(t) = f(y(t)), \; y(0) = -\infty, \; t \in (0, T).$$

Proof. We have seen in Lemma 2 that both $\overline{y}$ and y_ε are less than 0. Let us notice that $1/f(y) > 0$ for negative y and that $\overline{y}$ and y_ε are given by

$$\int_{-\infty}^{\overline{y}} dy/f(y) = t \text{ and } \int_{-\xi_\varepsilon}^{y_\varepsilon} dy/f(y) = t - b\varepsilon,\ \varepsilon < 1,$$

respectively (plain separation of the variables). Then

$$\int_{-\infty}^{\overline{y}} dy/f(y) = \int_{-\xi_\varepsilon}^{y_\varepsilon} dy/f(y) + b\varepsilon$$

for every $t > 0$. This implies

$$\left|\int_{-\xi_\varepsilon}^{\overline{y}} dy/f(y) - \int_{-\xi_\varepsilon}^{y_\varepsilon} dy/f(y)\right| = \left|\int_{\overline{y}}^{y_\varepsilon} dy/f(y)\right| = \left|b\varepsilon + \int_{-\infty}^{-\xi_\varepsilon} dy/f(y)\right|. \tag{10}$$

Since $\overline{y}$ is finite for every $t > 0$, $f(y) > 0$ and the right-hand side of (10) tends to zero as $\varepsilon \to 0$, it follows that $y_\varepsilon(t) \to \overline{y}(t)$ as $\varepsilon \to 0$ for every t. □

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