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BOUNDED SOLUTIONS OF ABSTRACT EQUATIONS

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Abstract. The existence of bounded solutions for abstract ODE's

$$x'(t) = A(t)x(t) + f(t), \quad t \in \mathbf{R}$$
$$x''(t) + bx'(t) = A(t)x(t) + f(t), \quad t \in \mathbf{R}$$

and for abstract boundary problems for PDE

$$\begin{cases} u'_t(t,x) - u''_{xx}(t,x) \\ \qquad = A(t)u(t,x) + g(t,x), \quad t \in \mathbf{R}, \ x \in [0,\pi] \\ u(t,0) = u(t,\pi) = 0, \quad t \in \mathbf{R}; \end{cases}$$
$$\begin{cases} u''_{tt}(t,x) + bu'_t(t,x) - u''_{xx}(t,x) \\ \qquad = A(t)u(t,x) + g(t,x), \ t \in \mathbf{R}, \ x \in [0,\pi] \\ u(t,0) = u(t,\pi) = 0, \quad t \in \mathbf{R} \end{cases}$$

is considered. Here A is a periodic operator valued function, f is a bounded on $\mathbf{R}$ Banach valued function and $b \in \mathbf{R}$.

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1. Existence of bounded solutions for ODE

Let $(B, ||\cdot||)$ be a complex Banach space, $\overline{0}$ the zero element in B, and $\mathcal{L}(B)$ the Banach space of bounded linear operators on B with the operator norm, denoted also by the symbol $||\cdot||$. For a B – valued function the continuity and differentiability mean correspondingly the continuity and differentiability in the B – norm. For an $\mathcal{L}(B)$ – valued function the continuity means the continuity in the operator norm. For an operator A the sets $\sigma(A)$ and $\rho(A)$ are its spectrum and resolvent set, respectively. Set $S = \{z \in \mathbf{C} \ : \ |z| = 1\}$.

Let $\tau > 0$ and the function

$$A \in C(\mathbf{R}, \mathcal{L}(B)); \quad A(t+\tau) = A(t), \ t \in \mathbf{R}$$

be given. With the function A is associated an operator valued function $U : \mathbf{R} \to \mathcal{L}(B)$ which is the solution of the following problem

$$\begin{cases} U'(t) = A(t)U(t), \ t \in \mathbf{R}; \\ U(0) = I. \end{cases}$$

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Here I is the identity operator.

Define

$$C_b(\mathbf{R}, B) := \{f \in C(\mathbf{R}, B) \mid \|f\|_\infty := \sup_{t\in\mathbf{R}} \|f(t)\| < +\infty\};$$
$$C_b^1(\mathbf{R}, B) := C^1(\mathbf{R}, B) \cap C_b(\mathbf{R}, B);$$
$$C_b^2(\mathbf{R}, B) := \{f \in C^2(\mathbf{R}, B) \mid \|f\|_\infty + \|f'\|_\infty < +\infty\}.$$

The first result is related to the following differential equation in a Banach space.

Theorem 1. *Equation*

$$x'(t) = A(t)x(t) + f(t), \quad t \in \mathbf{R}, \tag{1}$$

has a unique solution $\{x(t) \ : \ t \in \mathbf{R}\}$ *in* $C_b^1(\mathbf{R}, B)$ *for every function* $f \in C_b(\mathbf{R}, B)$ *if and only if*

$$\sigma(U(\tau)) \cap S = \emptyset. \tag{2}$$

Before proving Theorem 1, we prove the following lemma from [1].

Lemma 1. *Let* $D \in \mathcal{L}(B)$ *be a fixed operator. The following two statements are equivalent.*

(a) The equation

$$x(n+1) = Dx(n) + y(n), \quad n \in \mathbf{Z} \tag{3}$$

has a unique bounded solution $\{x(n) \ : \ n \in \mathbf{Z}\}$ *for every bounded sequence* $\{y(n) \ : \ n \in \mathbf{Z}\}$.

(b)

$$\sigma(D) \cap S = \emptyset.$$

Proof. *(a)* implies *(b)*. Let $\lambda_0 \in S$ and $z \in B$. Let $\{x(n) \ : \ n \in \mathbf{Z}\}$ be a unique bounded solution of equation (3) which corresponds to the bounded sequence $\{-\lambda_0^n z \ : \ n \in \mathbf{Z}\}$. Put

$$u(n) := x(n)\lambda_0^{-n}, \quad n \in \mathbf{Z}.$$

From formula (3), it is clear that the equation

$$\lambda_0 u(n+1) = Du(n) - z, \quad n \in \mathbf{Z}$$

has a unique bounded solution $\{u(n) \ : \ n \in \mathbf{Z}\}$ for every $z \in B$. Consequently,

$$\lambda_0(u(n+1) - u(n)) = D(u(n) - u(n-1)), \quad n \in \mathbf{Z}.$$

Thus $u(n) = u(0) =: u$ for every $n \in \mathbf{Z}$. Therefore, given any element $z \in B$, there exists a unique element $u \in B$ such that

$$(A - \lambda_0 I)u = z.$$

Thus by Banach theorem the operator $A - \lambda_0 I$ is invertible. *(b)* implies *(a)*. Let

$$\sigma_- := \sigma(A) \cap \{z \in \mathbf{C} \mid |z| < 1\},\ \sigma_+ := \sigma(A) \setminus \sigma_-$$

and P_-, P_+ be the corresponding Riesz spectral projectors. It is easy to show that

$$L := \sum_{j=0}^{\infty} ||(DP_-)^j|| + \sum_{j=-\infty}^{-1} ||(DP_+)^j|| < +\infty$$

where $(DP_-)^0 := P_-$, see [1], [5]. If $\{y(n) \ :\ n \in \mathbf{Z}\}$ is a bounded sequence in B, the sequence

$$x(n) := \sum_{j \in \mathbf{Z}} G(j) y(n-1-j), \quad n \in \mathbf{Z}$$

where

$$G(j) := \begin{cases} (DP_-)^j, & j \geq 0 \\ -(DP_+)^j, & j \leq -1; \quad j \in \mathbf{Z} \end{cases}$$

is also bounded. Moreover, for every $n \in \mathbf{Z}$ we have

$$\begin{aligned} Dx(n) &= (DP_- + DP_+)x(n) = \\ &= \sum_{j=0}^{\infty} (DP_-)^{j+1} y(n-1-j) - \sum_{j=-\infty}^{-1} (DP_+)^{j+1} y(n-1-j) = \\ &= x(n+1) - P_- y(n) - P_+ y(n) = x(n+1) - y(n). \end{aligned}$$

Now let us prove that the solution $\{x(n) \ :\ n \in \mathbf{Z}\}$ for equation (3) is unique. Let $\{u(n) \ :\ n \in \mathbf{Z}\}$ be a solution of equation (3) which corresponds to $\{y(n) \ :\ n \in \mathbf{Z}\}$. Then the sequence

$$\{v(n) := x(n) - u(n) \ :\ n \in \mathbf{Z}\}$$

satisfies the equation

$$v(n+1) = Dv(n), \quad n \in \mathbf{Z}$$

which is equivalent to the system

$$\begin{cases} v_-(n+1) = (DP_-)v_-(n), \\ v_+(n+1) = (DP_+)v_+(n); \quad n \in \mathbf{Z} \end{cases}$$

with

$$v_-(n) := P_- v(n), \quad v_+(n) := P_+ v(n), \quad n \in \mathbf{Z}.$$

From this we have for $m \geq 1$

$$||v_-(n)|| = ||(DP_-)^m v_-(n-m)|| \leq ||(DP_-)^m|| \sup_{k\in\mathbf{Z}} ||v_-(k)||,$$
$$||v_+(n)|| = ||(DP_+)^{-m} v_+(n+m)|| \leq ||(DP_+)^{-m}|| \sup_{k\in\mathbf{Z}} ||v_+(k)||.$$

Therefore, $v_-(n) = \bar{0} = v_+(n), \quad n \in \mathbf{Z}$. Lemma 1 is proved. □

Proof of Theorem 1. Note that equation (1) is equivalent to the following one

$$x(t) = U(t)U^{-1}(s)x(s) + \int_s^t U(t)U^{-1}(s)f(s)\,ds, \quad s < t. \tag{4}$$

If equation (1) has a unique solution $x \in C_b^1(\mathbf{R}, B)$ for every $f \in C(\mathbf{R}, B)$, then by (4) the following difference equation

$$z(n+1) = U(\tau)z(n) + v(n), \quad n \in \mathbf{Z}$$

has a unique bounded solution $z(n) = x(n\tau)$, $n \in \mathbf{Z}$ for every bounded sequence $\{v(n) \mid n \in \mathbf{Z}\}$. By Lemma 1 we have (2).

Suppose that condition (2) is satisfied and $f \in C_b(\mathbf{R}, B)$. Let $P_-(P_+)$ be the Riesz spectral projectors corresponding to the part of the spectrum $\sigma(U(\tau))$ which is inside (outside) S. For every $t \in \mathbf{R}$ define the element

$$x(t) := \int_{-\infty}^{+\infty} G(t,s)f(s)\,ds,$$

where G is the Green function for equation (1)

$$G(t,s) := \begin{cases} U(t)P_-U^{-1}(s), & s < t; \\ U(t)P_+U^{-1}(s), & s > t. \end{cases}$$

Applying the following well known properties of the operator valued function U ([4])

$$U(t) = U(t - n\tau)U(\tau)^n;$$
$$||U(t)U^{-1}(s)|| \leq \exp\left(\int_s^t ||A(u)||\,du\right), \quad s < t,$$

we have

$$C := \sup_{[0,\tau]} \int_{-\infty}^{+\infty} ||G(t,s)||\,ds < +\infty.$$

Thus x is a bounded solution of (1). □

Remarks.

1. Theorem 1 is a generalization of the following M. G. Krein theorem ([3], p. 54) for $A(t) \equiv A \in \mathcal{L}(B)$.

Theorem 2. *Let $A \in \mathcal{L}(B)$. Equation*

$$x'(t) = Ax(t) + f(t), \quad t \in \mathbf{R}$$

has a unique solution $x \in C_b^1(\mathbf{R}, B)$ for every function $f \in C_b(\mathbf{R}, B)$ if and only if

$$\sigma(A) \cap i\mathbf{R} = \emptyset.$$

2. Theorem 1 can be proved in another way with the use of a dichotomy [2], n. 7.6.

The second result of this section is related to the following second-order equation

$$x''(t) + bx'(t) = A(t)x(t) + f(t), \quad t \in \mathbf{R}, \tag{5}$$

where $b \in \mathbf{R}$. Let us consider a Banach space B^2 of vectors equipped with term-by-term linear operations and with a norm which is equal to the sum of norms of coordinates. Let

$$\mathbf{y}(t) := \begin{pmatrix} x'(t) \\ x(t) \end{pmatrix}, \quad \mathbf{f}(t) := \begin{pmatrix} f(t) \\ 0 \end{pmatrix}, \quad \mathbb{A} := \begin{pmatrix} -Ib & A(t) \\ I & \Theta \end{pmatrix}, \quad t \in \mathbf{R}$$

where Θ is the zero operator. Then the following equation in B^2

$$\mathbf{y}'(t) = \mathbb{A}(t)\mathbf{y}(t) + \mathbf{f}(t), \quad t \in \mathbf{R} \tag{6}$$

is equivalent to the equation (5) in B. Let $\mathbb{U}$ be a unique solution of the following problem in B^2

$$\begin{cases} \mathbb{U}'(t) = \mathbb{A}(t)\mathbb{U}(t), & t \in \mathbf{R} \\ \mathbb{U}(0) = \mathbb{I}, \end{cases}$$

where $\mathbb{I}$ is the identity operator on B^2. Note that

$$\mathbb{U} = \begin{pmatrix} V_1' & V_2' \\ V_1 & V_2 \end{pmatrix}$$

and that the elements $V_1,\ V_2$ are determined by the equations

$$\begin{cases} V_1''(t) + bV_1(t) = A(t)V_1(t), & t \in \mathbf{R}; \\ V_1(0) = \Theta, \quad V_1'(0) = I; \end{cases}$$

$$\begin{cases} V_2''(t) + bV_2(t) = A(t)V_2(t), & t \in \mathbf{R}; \\ V_2(0) = I, \quad V_2(0) = \Theta; \end{cases}$$

Theorem 3. *Suppose that a function A and a number b are such that*

$$\sigma(\mathbb{U}(\tau)) \cap S = \emptyset.$$

Then for every function f bounded on $\mathbf{R}$ the equation (5) has a unique solution $x \in C_b^2(\mathbf{R}, B)$.

Proof. To prove Theorem 3 we need only to apply Theorem 1 to equation (6).□

2. Existence of bounded solutions of boundary value problem for PDE

Let $Q := \mathbf{R} \times [0, \pi]$.

Definition 1. *A B-valued function $u = \{u(t,x) \ : \ (t,x) \in Q\}$ is called bounded if* $\sup_Q ||u|| < +\infty$.

Denote

$$C_0^1 := \{g : \ [0,\pi] \to \mathbf{C} \mid g(0) = g(\pi) = 0\} \cap C^1([0,\pi]);$$
$$C_0^3 := \{g : \ [0,\pi] \to \mathbf{C} \mid g^{(k)}(0) = g^{(k)}(\pi) = 0, \ k = 0,1,3\} \cap C^3([0,\pi]).$$

Given the functions $g \in C_0^1$, $A \in C(\mathbf{R}, \mathcal{L}(B))$ and $f \in C_b(\mathbf{R}, B)$ let us consider the following boundary problem for the heat equation

$$\text{(7)} \qquad \begin{cases} u_t'(t,x) - u_{xx}''(t,x) = A(t)u(t,x) + f(t)g(x), & (t,x) \in Q; \\ u(t,0) = u(t,\pi) = \overline{0}, & t \in \mathbf{R}. \end{cases}$$

Theorem 4. *Let $A \in C(\mathbf{R}, \mathcal{L}(B))$ be a periodic function with the period τ.*

(i) *If for every $g \in C_0^1$ and $f \in C_b(\mathbf{R}, B)$ the boundary problem* (7) *has a unique bounded solution u then the following condition is valid*

$$\text{(8)} \qquad \{e^{k^2\tau + i\alpha} \mid k \in \mathbf{N}, \ \alpha \in [0, 2\pi]\} \subset \rho(U(\tau)).$$

(ii) *If* (8) *is valid, $g \in C_0^3$ and $f \in C_b(\mathbf{R}, B)$ then the boundary problem* (7) *has a unique bounded solution u.*

Proof. (i) Let $k \in \mathbf{N}$ and $\alpha \in \mathbf{R}$ be given. Let u be a unique bounded solution of the boundary problem (7) for the functions

$$g(x) = \sin kx, \quad x \in [0,\pi]; \quad f \in C_b(\mathbf{R}, B).$$

Define

$$v_k(t) := \int_0^\pi u(t,x) \sin kx \, dx, \quad t \in \mathbf{R},$$

the last integral is the Riemann integral for a B–valued continuous function. It can be easily checked that $v_k \in C_b^1(\mathbf{R}, B)$. From (7) we have

$$\text{(9)} \qquad v_k'(t) = (A(t) - k^2 I)v_k(t) + \frac{\pi}{2} f(t), \quad t \in \mathbf{R}.$$

By Theorem 1 equation (9) has a unique solution $\{v_k(t) \ : \ t \in \mathbf{R}\}$ in $C_b^1(\mathbf{R}, B)$ for every function $f \in C_b(\mathbf{R}, B)$ if and only if

$$\sigma(U_k(\tau)) \cap S = \emptyset,$$

where U_k is a solution of the following problem

$$\begin{cases} U_k'(t) = (A(t) - k^2 I)U_k(t), \ t \in \mathbf{R}; \\ U_k(0) = I. \end{cases}$$

Note that

$$U_k(t) = U(t)e^{-k^2 t}, \quad t \in \mathbf{R}.$$

Therefore, we obtain

$$\sigma(U(\tau)) \cap \{e^{k^2\tau + i\alpha} \mid k \in \mathbf{N}, \ \alpha \in [0, 2\pi]\} = \emptyset.$$

(ii) Let be given the functions $g \in C_0^3$ and $f \in C_b(\mathbf{R}, B)$ and the condition (8) be fulfilled. We again use Theorem 1. According to Theorem 1 equation (9) for every $k \geq 1$ has a unique solution $v_k(t)$, $t \in \mathbf{R}$, in $C_b^1(\mathbf{R}, B)$.
The sequence $\{\sin kx, \ \ x \in [0, \pi] \ : \ k \geq 1\}$ is complete in C_0^1; thus we have

$$g(x) = \sum_{k=1}^{\infty} g_k \sin kx, \ x \in [0, \pi]; \ \ \{g_k \ : \ k \geq 1\} \subset \mathbf{C},$$

where the series on the right-hand side uniformly converges on $\mathbf{R}$.

Let us introduce the function

$$u(t, x) := \sum_{k=1}^{\infty} v_k(t) g_k \sin kx, \quad (t, x) \in Q,$$

here

$$g_k := \frac{2}{\pi} \int_0^{\pi} g(x) \sin kx \, dx, \quad k \geq 1.$$

The series for u converges uniformly on Q. Hence $u \in C(Q, B)$. In what follows, we need a more exact estimate for

$$\sup_{t \in \mathbf{R}} ||v_k(t)||$$

for large k. Let k_0 be a minimal natural number such that the set $\sigma(U(\tau)e^{-k_0^2\tau})$ lies inside $\{z \in \mathbf{C} \ : \ |z| \leq 1\}$. For every $k > k_0$, in virtue of the properties of the function U_k and representation of a solution for the equation (9), we have the following estimate

$$\sup_{t \in \mathbf{R}} ||v_k(t)|| \leq \frac{C_1}{k^2 - k_0^2},$$

where

$$C_1 := \frac{a\pi}{2} \sum_{j=0}^{\infty} ||U_{k_0}(\tau)^j|| \cdot ||f||_{\infty}, \ \ a := \exp\left(\int_0^{\tau} ||A(s)|| \, ds\right).$$

Similarly we obtain

$$\begin{aligned} u'_t(t,x) &= \sum_{k=1}^{\infty} v'_k(t) g_k \sin kx \\ &= \sum_{k=1}^{\infty} ((A(t) - k^2 I) v_k(t) + \frac{\pi}{2} f(t)) g_k \sin kx \\ &= A(t) u(t,x) - \sum_{k=1}^{\infty} k^2 v_k(t) g_k \sin kx + f(t) g(x) \end{aligned}$$

and

$$u''_{xx}(t,x) = \sum_{k=1}^{\infty} v_k(t) g_k (-k^2) \sin kx,$$

where the series converge uniformly on Q.

To prove the uniqueness we remark that for every $t \in \mathbf{R}$ elements $\{v_k(t) g_k \mid k \geq 1\}$ are Fourier coefficients of $u(t,\cdot) \in C^2([0,\pi], B)$. The Fourier coefficients and the solutions of equations (9) are uniquely determined. □

Now we shall establish sufficient conditions for the existence of bounded solutions of the following boundary problem for the hyperbolic equation

$$\text{(10)} \qquad \begin{cases} u''_{tt}(t,x) + b u'_t(t,x) - u''_{xx}(t,x) \\ \qquad = A(t) u(t,x) + g(x) f(t), \quad (t,x) \in Q; \\ u(t,0) = u(t,\pi) = \overline{0} \end{cases}$$

where $g : [0,\pi] \to \mathbf{C}$ and $f \in C_b(\mathbf{R}, B)$.

Denote by V_{1k}, V_{2k} the unique solutions of the following problems in B

$$\begin{cases} V''_{1k}(t) = (A(t) - bI - k^2 I) V_{1k}, \quad t \in \mathbf{R}; \\ V_{1k}(0) = \Theta, \quad V'_{1k}(0) = I \end{cases}$$

and

$$\begin{cases} V''_{2k}(t) = (A(t) - bI - k^2 I) V_{2k}, \quad t \in \mathbf{R}; \\ V_{2k}(0) = I, \quad V'_{2k}(0) = \Theta \end{cases}$$

for $k \in \mathbf{N}$. Let

$$\mathbb{U}_k := \begin{pmatrix} V'_{1k} & V'_{2k} \\ V_{1k} & V_{2k} \end{pmatrix}.$$

Definition 2. *A B-valued function $u = \{u(t,x) : (t,x) \in Q\}$ is called a bounded solution of problem* (10) *if*

$$\sup_Q \|u\| < +\infty, \quad \sup_Q \|u'_t\| < +\infty.$$

Theorem 5. *Suppose that*

$$\cup_{k\geq 1}\sigma(\mathbb{U}_k(\tau)) \cap S = \emptyset.$$

Then for every function $g \in C_0^3$ and $f \in C_b(\mathbf{R}, B)$ the boundary problem (10) *has a unique bounded solution.*

Proof. The proof is analogous to the one of Theorem 3. Let $g \in C_0^3$ and $f \in C(\mathbf{R}, B)$ be given. Then

$$g(x) = \sum_{k=1}^{\infty} g_k \sin kx, \quad x \in [0, \pi].$$

Define

$$v_k(t) := \int_0^{\pi} u(t,x) \sin kx \, dx, \quad t \in \mathbf{R}, \quad k \in \mathbf{N}.$$

It follows from Theorem 3 that the following equation

$$v_k''(t) + bv_k'(t) = (A(t) - k^2 I)v_k(t) + \frac{\pi}{2} f(t), \quad t \in \mathbf{R}$$

has a unique solution $v_k \in C_b^2(\mathbf{R}, B)$, $k \geq 1$.

The function

$$u(t,x) := \sum_{k=1}^{\infty} v_k(t) g_k \sin kx, \quad (t,x) \in Q$$

is a unique bounded solution of the problem (10). □

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