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A REMARK ON WEAK INTERPOLATION THEOREM FOR INFINITARY LOGICS

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Abstract. Weak Interpolation Theorem for infinitary logics is proved by "standard" methods.

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1. Preliminaries

It is known that Craig Interpolation Theorem fails in all logics $L_{\kappa\lambda}(\mu)$ for $\kappa > \omega_1$. In his Ph. D. thesis Jean-Pierre Keller proved Weak Interpolation Theorem for first-order infinitary logics (semantical $\models$ is replaced by syntactical $\vdash$) applying the forcing method. However, in our opinion the relation he used is not (in cases of real interest) a forcing relation, while on the other hand some of the main properties of forcing relations were employed. We offered an amendment to this proof in [4].

Here we prove the Theorem in a "classical" way, using the proof pattern from finitary logic.

2. Basic axiomatic system for infinitary logics

Let $L_{\kappa\lambda}(\mu)$ be an infinitary first-order logic with equality of the similarity type μ, where κ is an arbitrary regular cardinal and λ (an infinite) cardinal less than or equal to κ. Our metatheory will be $ZFC + GCH$. As the basic logical symbols we will treat connectives $\neg$ and $\bigwedge$ – conjunction of a set of formulae of cardinality less than κ, a quantifier $\forall$, which enables us to "quantify" any set of variables of cardinality less than λ, and the equality relation $=$. Other logical symbols are defined in the usual way; for instance, $\bigvee_{\alpha<\delta}\phi_\alpha$ is standing for $\neg\bigwedge_{\alpha<\delta}\neg\phi_\alpha$. As for non-logical symbols we allow the possibility of having symbols of all sorts (relations, functions, constants). The axiomatic system we will use is the so-called *basic system*. For the reader's convenience we will present it using the notation from [2].

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Propositional axiom-schemes:

(*a*) $\phi \Rightarrow (\psi \Rightarrow \phi)$

(*b*) $(\phi \Rightarrow (\psi \Rightarrow \theta)) \Rightarrow ((\phi \Rightarrow \psi) \Rightarrow (\phi \Rightarrow \theta))$

(*c*) $(\neg\phi \Rightarrow \neg\psi) \Rightarrow (\psi \Rightarrow \phi)$

(*d*) $\bigwedge_{\alpha<\delta}(\psi \Rightarrow \phi_\alpha) \Rightarrow (\psi \Rightarrow \bigwedge_{\alpha<\delta}\phi_\alpha)$ for $0 < \delta < \kappa$

(*e*) $\bigwedge_{\alpha<\delta}\phi_\alpha \Rightarrow \phi_\beta$ for $0 < \delta < \kappa$ i $\beta < \delta$;

Quantificational axiom-schemes

(*f*) $\forall X(\phi \Rightarrow \psi) \Rightarrow (\phi \Rightarrow \forall X\psi)$ for $X \subset Var$ (where Var is the set of variables – $\{v_\alpha \mid \alpha < \kappa\}$), $|X| < \lambda$, provided that no variable from X occurs free in ϕ

(*g*) $\forall X\phi \Rightarrow Sub^{v\in X}_{F(v)}\phi$, where $X \subset Var$, $|X| < \lambda$, $F : X \longrightarrow Term$, where $Term$ is the set of terms of the language $L(\mu)$, and $Sub^{v\in X}_{F(v)}\phi$ is the formula obtained from ϕ by substituting each free occurrence of a variable v ($\in X$) in ϕ by the term $F(v)$, on condition that ϕ contains no free occurrences of the variable v ($\in X$) within the scope of a quantifier binding some variable of the term $F(v)$;

Equality axiom-schemes

(*h*) $t = t$ for any term t

(*i*) $\bigwedge_{\alpha<\delta}(t_\alpha = t'_\alpha) \Rightarrow (Sub^{\langle v_\alpha|\alpha<\delta\rangle}_{\langle t_\alpha|\alpha<\delta\rangle}\phi \Leftrightarrow Sub^{\langle v_\alpha|\alpha<\delta\rangle}_{\langle t'_\alpha|\alpha<\delta\rangle}\phi)$ for $0 < \delta < \kappa$ and $\langle v_\alpha \mid \alpha < \delta\rangle \subseteq Fv(\phi)$ (– the set of free variables of the formula ϕ) provided that no variable in $\bigcup_{\alpha<\delta}(Fv(t_\alpha) \cup Fv(t'_\alpha))$ is bound in ϕ;

Proposition rules of inference

(*j*) Modus ponens $\frac{\phi,\ \phi\Rightarrow\psi}{\psi}$

(*k*) Conjunction $\frac{\phi_\alpha(\alpha<\delta)}{\bigwedge_{\alpha<\delta}\phi_\alpha}$ for $0 < \delta < \kappa$;

Quantificational rule of inference

(*l*) Generalization $\frac{\phi}{\forall X\phi}$ for any $X \subset Var$, $|X| < \lambda$.

We will denote the given axiomatic system by Λ; thus, $\vdash_\Lambda \phi$ will mean that there is a proof (of the length less than κ) of the formula ϕ which uses (only) axiom-schemes and rules of inference from Λ. This system certainly provides that any formula is equivalent to some negation normal formula, i.e. to a formula involving only the symbols $\neg$, $\bigwedge$, $\bigvee$, $\forall$ and $\exists$ in which every negation symbol preceeds an atomic formula; for instance, we have at disposal the theorem $\bigwedge_{\alpha<\delta}\neg\neg\phi_\alpha \Leftrightarrow \bigwedge_{\alpha<\delta}\phi_\alpha$. We will also use the next form of Deduction Theorem: if $\Phi \cup \Psi \vdash_\Lambda \theta$, where Φ is a set of formulae and Ψ a set of sentences of cardinality less than κ, then $\Phi \vdash_\Lambda \bigwedge_{\psi\in\Psi}\psi \Rightarrow \theta$.

Later, we will sometimes write $\bigwedge\Phi$ instead of $\bigwedge_{\phi\in\Phi}\phi$, when the set of formulae Φ is specified. Naturally, $\phi_0 \wedge \phi_1$ will stand for $\bigwedge_{\alpha<2}\phi_\alpha$.

3. Weak Interpolation Theorem

Theorem 3.1. *(Weak Interpolation Theorem). If φ and ψ are sentences of the logic $L_{\kappa\lambda}(\mu)$ (κ regular cardinal, $\lambda \leq \kappa$) such that $\vdash_\Lambda \varphi \Rightarrow \psi$, then there exists a sentence θ of the same logic satsfying the following: $\vdash_\Lambda \varphi \Rightarrow \theta$, $\vdash_\Lambda \theta \Rightarrow \psi$ and all non-logical symbols appearing in θ appear in both φ and ψ.*

Proof. Naturally, in order to avoid trivial cases we assume that neither ϕ is a contradiction nor ψ is a theorem as well as that that in each of the formulae φ, ψ occurs a nonlogical symbols not appearing in the other one. We add firstly to the initial language a set of new constants, C, of cardinality κ and continue to work in the logic $L_{\kappa\lambda}(\mu \cup C)$, shortly denoted by $L_{\kappa\lambda}(\mu')$. The basic terms of this logic will be all constants and all terms of the form $f(c_1, \ldots, c_n)$, where $c_1, \ldots, c_n$ are constants from C. In order to make the proof more eligible we will introduce a few lemmas (whose proofs will be ended by $\Box$).

Let S_φ be the set of all negation normal sentences χ of the logic $L_{\kappa\lambda}(\mu')$ such that each non-logical symbol of the type μ occurring in χ occurs in φ too and such that χ contains less than λ constants from C. In the same way we define the set S_ψ. Furthemore, let P be the set of all sets p of cardinality less than κ of negation normal sentences of the logic $L_{\kappa\lambda}(\mu')$ which can be decomposed into (not necessarily disjoint) union $p_1 \cup p_2$ for which the following holds:

(*i*) $p_1 \subset S_\varphi$, $p_2 \subset S_\psi$

and

(*ii*) there does not exist a sentence $\theta \in S_\varphi \cap S_\psi$ such that $p_1 \vdash_\Lambda \theta$ and $p_2 \vdash_\Lambda \neg\theta$.

Any decomposition of p ($\in P$) satisfying the above two conditions will be called the *correct decomposition* (of p). We are starting with examination of the properties of elements of P.

Lemma 3.2. *If $p_1 \cup p_2$ is a correct decomposition of p ($\in P$) and if $s \subset S_\varphi \cap S_\psi$ is a subset of p_1, then $(p_1 \setminus s) \cup (p_2 \cup s)$ is also a correct decomposition of p.*

The same assertion holds in the othe "direction" too – if s is a subset of p_2, then $(p_1 \cup s) \cup (p_2 \setminus s)$ is a correct decomposition of p.

Proof. Let us suppose that (on the given assumptions) $(p_1 \setminus s) \cup (p_2 \cup s)$ is not a correct decomposition of p. Then for some $\theta \in S_\varphi \cap S_\psi$ $p_1 \setminus s \vdash_\Lambda \theta$ and $p_2 \cup s \vdash_\Lambda \neg\theta$. Hence, $p_2 \vdash_\Lambda \neg(\bigwedge s \wedge \theta)$, while $p_1 \setminus s \vdash_\Lambda \bigwedge s \Rightarrow \theta$; thus also $p_1 \vdash_\Lambda \bigwedge s \wedge \theta$, contradictory to the condition that $p_1 \cup p_2$ is a correct decomposition. $\Box$

Lemma 3.3. *Every element p from P satisfies the following:*

(1) *there exists no atomic sentence ϕ such that both ϕ and $\neg\phi$ belong to p;*

(2) *if $p_1 \cup p_2$ is a correct decompostion of p and $p_1 \vdash_\Lambda \chi$ ($p_2 \vdash_\Lambda \chi$), where $\chi \in S_\varphi$ ($\chi \in S_\psi$), then $p \cup \{\chi\}$ is in P.*

Specially, if $\chi \in S_\varphi \cup S_\psi$ *is a theorem, then* $p \cup \{\chi\} \in P$*;*

(3) *if* $\bigwedge_{\alpha<\delta} \phi_\alpha \in p$ $(\delta < \kappa)$*, then* $p \cup \{\phi_\alpha\} \in P$ *for each* α $(< \delta)$*;*

(4) *if* $\bigvee_{\alpha<\delta} \phi_\alpha \in p$*, then* $p \cup \{\phi_\gamma\} \in P$ *for some* $\gamma < \delta$*;*

(5) *if* $\forall X \phi(X) \in p$*, then* $p \cup \{Sub^{v \in X}_{F(v)}\, \phi\} \in P$ *for all functions* $F : X \longrightarrow C$*;*

(6) *if* $\exists X \phi(X) \in p$ $(X \subset Var)$*, then* $p \cup \{Sub^{v\in X}_{F(v)}\, \phi\} \in P$ *for some function* $F : X \longrightarrow C$*;*

(7) *if for a constant* $c \in C$*, a basic term* t *and an atomic formula* $\phi(v)$ *of the logic* $L_{\kappa\lambda}(\mu')$ *holds* $t = c$*,* $\phi(t) \in p$*, then* $p \cup \{\phi(c)\} \in P$*;*

(8) *for any closed term* t *appearing in some sentence of* p*, there exists a constant* $c \in C$ *such that* $p \cup \{c = t\} \in P$*.*

Proof. (1) Let $p_1 \cup p_2$ be a correct decomposition of p and let us suppose that for some atomic sentence ϕ both ϕ and $\neg\phi$ belong to p. We distinguish the cases: (i) $\phi, \neg\phi \in p_1$, (ii) $\phi \in p_1$, $\neg\phi \in p_2$, (iii) $\neg\phi \in p_1$, $\phi \in p_2$ and (iv) $\phi, \neg\phi \in p_2$. The first case implies $p_1 \vdash_\Lambda \forall v \neg(v = v)$, while clearly $p_2 \vdash_\Lambda \neg\forall\neg v(v = v)$, a contradiction. In the second case we have $\phi \in S_\varphi \cap S_\psi$ and $p_1 \vdash_\Lambda \phi$, $p_2 \vdash_\Lambda \neg\phi$, contradiction again. By the symmetry the cases (iii) and (iv) fail too.

(2) Clear; under the (first) conditions $(p_1 \cup \{\chi\}) \cup p_2$ is a correct decomposition of $p \cup \{\chi\}$.

(3) A direct consequence of the previous item.

(4) Let $p_1 \cup p_2$ be a correct decomposition of p and $\bigvee_{\alpha<\delta} \phi_\alpha \in p_1$. Let us assume that for no α $(< \delta)$ $(p_1 \cup \{\phi_\alpha\}) \cup p_2$ is a correct decomposition of $p \cup \{\phi_\alpha\}$. Hence, for each α there exists a sentence $\theta_\alpha \in S_\varphi \cap S_\psi$ such that $p_1, \phi_\alpha \vdash_\Lambda \theta_\alpha$ and $p_2 \vdash_\Lambda \neg\theta_\alpha$. By Deduction Theorem and the theorem $\vdash_\Lambda \theta_\alpha \Rightarrow \bigvee_{\beta<\delta} \theta_\beta$ for any $\alpha < \delta$ (it is an axiom $\bigwedge_{\beta<\delta} \neg\theta_\beta \Rightarrow \neg\theta_\alpha$) we obtain $p_1 \vdash_\Lambda \phi_\alpha \Rightarrow \bigvee_{\beta<\delta} \theta_\beta$, that is $p_1 \vdash_\Lambda \neg \bigvee_{\beta<\delta} \theta_\beta \Rightarrow \neg\phi_\alpha$. Therefore, $p_1 \vdash_\Lambda \bigwedge_{\alpha<\delta} (\neg \bigvee_{\beta<\delta} \theta_\beta \Rightarrow \neg\phi_\alpha)$, whence by (d) and modus ponens $p_1 \vdash_\Lambda \neg \bigvee_{\beta<\delta} \theta_\beta \Rightarrow \bigwedge_{\alpha<\delta} \neg\phi_\alpha$, i.e. $p_1 \vdash_\Lambda \neg \bigwedge_{\alpha<\delta} \neg\phi_\alpha \Rightarrow \bigvee_{\beta<\delta} \theta_\beta$. Thus $p_1 \vdash_\Lambda \bigvee_{\beta<\delta} \theta_\beta$, but $p_2 \vdash_\Lambda \bigwedge_{\beta<\delta} \neg\theta_\beta$, that is $p_2 \vdash_\Lambda \neg \bigvee_{\beta<\delta} \theta_\beta$, a contradiction.

(5) Let $\forall X \phi(X) \in p_1$, whith $p_1 \cup p_2$ as a correct decomposition of p. Obviously (due to (g) nad (2)), $(p_1 \cup \{Sub^{v\in X}_{F(v)}\, \phi\}) \cup p_2 \in P$ for any mapping $F : X \longrightarrow C$.

(6) Let $\exists X \phi(X) \in p_1$, where (as usual) $p_1 \cup p_2$ is a correct decomposition of p. Since p is of cardinality less than κ and since each sentence from p contains less than λ constants from C there is an injection $F : X \longrightarrow C$ such that no constant from $F(X)$ $(= D)$ appears in the sentences from p. Then $(p_1 \cup \{Sub^{v\in X}_{F(v)}\, \phi\}) \cup p_2$ is in P – in order to simplify notation we will write (temporarily) $\phi(D)$ instead of $Sub^{v\in X}_{F(v)}\, \phi$. For in the opposite case there would be some sentence $\theta \in S_\varphi \cap S_\psi$ such that $p_1, \phi(D) \vdash_\Lambda \theta$ (thus $p_1 \vdash_\Lambda \phi(D) \Rightarrow \theta$) and $p_2 \vdash_\Lambda \neg\theta$; we can and we will assume that no appearance of any constant from D in θ is under the scope of the quantifiers $\exists x$, $\forall x$ for some $x \in X$. By

the choice of D it follows $p_1 \vdash_\Lambda \forall X(\phi(X) \Rightarrow \theta(X))$ and $p_2 \vdash_\Lambda \forall X \neg\theta(X)$. But $\vdash_\Lambda \forall X(\phi(X) \Rightarrow \theta(X)) \Rightarrow (\exists X\phi(X) \Rightarrow \exists X\theta(X))$ gives $p_1 \vdash_\Lambda \exists X\theta(X)$, while still $p_2 \vdash_\Lambda \neg\exists X\theta(X)$.

(7) If $p_1 \cup p_2$ is a correct decomposition of p and $t = c, \phi(t)$ are the elements of p, then by Lemma 3.2 we can immediately suppose that both sentences belong to the same part of the decomposition. The rest follows from (2).

Let us just note that the roles of (a constant) c and (a term) t cannot be reversed, for it could happen, for instance, that both $c = t$ and $\phi(c)$ are the elements of p, while $\phi(t)$ is not in $S_\varphi \cup S_\psi$ at all.

(8) Let t appears in some sentence of p. By (2), $p \cup \{\exists v(t = v)\}$ is in P, and by (6), $p \cup \{\exists v(t = v)\} \cup \{t = c\}$ is in P for some constant c from C; in particular, $p \cup \{t = c\}$ is in P. □

In the proof of the last item we used the following (rather obvious) fact: if p is an element of P and $s \subseteq p$, then $s \in P$; for, if $p_1 \cup p_2$ is a correct decomposition of p, then $(s \cap p_1) \cup (s \cap p_2)$ is a correct decomposition of s.

We are now going to show that any element p of P can be extended to a set of sentences with the properties that provide the existence of its "canonical" model. This set will be the union of a nondecreasing sequence of elements from P (starting with p). Let $\{\phi_\alpha \mid \alpha < \kappa\}$ be an enumeration of the set of all sentences ϕ from $S_\varphi \cup S_\psi$ such that every non-logical symbol of $L(\mu)$ occurring in ϕ has an occurrence in some sentence of p. We define recursively the elements of the sequence p_α, $\alpha < \kappa$, in the following way:

$p_0 = p$.

Let α be a successor. If $p_{\alpha-1} \cup \{\phi_{\alpha-1}\}$ is not in P, then $p_\alpha = p_{\alpha-1}$. If $p_{\alpha-1} \cup \{\phi_{\alpha-1}\} \in P$ we distinguish the cases depending on the form of the sentence $\phi_{\alpha-1}$. If $\phi_{\alpha-1}$ is either a basic sentence or of the form $\bigwedge_{\beta<\delta} \chi_\beta$ or of the form $\forall X\chi(X)$, then $p_\alpha = p_{\alpha-1} \cup \{\phi_{\alpha-1}\}$. If $\phi_{\alpha-1} \equiv \bigvee_{\beta<\delta} \chi_\beta$, then $p_\alpha = p_{\alpha-1} \cup \{\phi_{\alpha-1}, \chi_\gamma\}$, where χ_γ $(\gamma < \delta)$ is a sentence such that $(p_{\alpha-1} \cup \{\phi_{\alpha-1}\}) \cup \{\chi_\gamma\}$ is in P (we can agree that in the case when there are more such sentences we choose the one with the smallest index). Finally, if $\phi_{\alpha-1} \equiv \exists X\chi(X)$, we put $p_\alpha = p_{\alpha-1} \cup \{\phi_{\alpha-1}, \chi(D)\}$, where D is a subset of C such that $(p_{\alpha-1} \cup \{\phi_{\alpha-1}\}) \cup \{\chi(D)\}$ is in P (again we can take the sentence with the smallest index).

If α is a limit, we define $p_\alpha = \bigcup_{\beta<\alpha} p_\beta$.

Surely, the above definition makes sense because of the properties of the elements of P proved in the previous lemma.

If $p_\kappa \stackrel{\text{def}}{=} \bigcup_{\alpha<\kappa} p_\alpha$, we have

Lemma 3.4. p_κ *has the following properties:*

I *there is no atomic sentence* χ *sush that both* χ *and* $\neg\chi$ *belong to* p_κ*;*

II *if* $\bigwedge_{\beta<\delta} \chi_\beta$ *is in* p_κ*, then any "conjuct"* χ_β *belongs to* p_κ*;*

III *if* $\bigvee_{\beta<\delta} \chi_\beta$ *is in* p_κ*, then (at least) one of "disjuncts" belongs to* p_κ*;*

IV *if* $\forall X\chi(X) \in p_\kappa$, *then* $Sub^{v\in X}_{F(v)}\,\chi \in p_\kappa$ *for every mapping* $F : X \longrightarrow C$;

V *if* $\exists X\chi(X) \in p_\kappa$, *then* $Sub^{v\in X}_{F(v)}\,\chi \in p_\kappa$ *for some mapping* $F : X \longrightarrow C$;

VI $c = c \in p_\kappa$ *for any* $c \in C$;

VII *if* $t_1 = t_2 \in p_\kappa$ *(where, of course,* t_1 *and* t_2 *are some closed terms), then also* $t_2 = t_1 \in p_\kappa$;

VIII *if* t_1, t_2 *and* t_3 *are closed terms with non-logical symbols of the language* $L(\mu)$ *appearing either in* φ *or* ψ, *then from* $t_1 = t_2$, $t_2 = t_3 \in p_\kappa$ *follows* $t_1 = t_3 \in p_\kappa$;

IX *if* R *is an n-ary relation symbol of the language* $L(\mu)$ *and* $c_1, \ldots, c_n, d_1, \ldots, d_n$ *are constants from* C *such that* $c_i = d_i \in p_\kappa$ *for* $i = 1, \ldots, n$, *then* $R(c_1, \ldots, c_n) \in p_\kappa$ *iff* $R(d_1, \ldots, d_n) \in p_\kappa$;

X *if* f *is an n-ary function symbol of the language* $L(\mu)$ *appearing in some sentence of* p *and* $c_1, \ldots, c_n, d_1, \ldots, d_n$ *are constants from* C *such that* $c_i = d_i \in p_\kappa$ *for* $i = 1, \ldots, n$, *then* $f(c_1, \ldots, c_n) = f(d_1, \ldots, d_n) \in p_\kappa$.

Proof. More or less everything is obvious.

I is guaranteed by the property (1) of the elements of P.

II Let $\phi_\gamma \equiv \bigwedge_{\alpha<\delta} \chi_\beta \in p_\kappa$, $\phi_{\alpha_\beta} \equiv \chi_\beta$ and $\alpha = sup\{\alpha_\beta \mid \beta < \delta\}$. If $\zeta \geq \alpha, \gamma$ is such that $\bigwedge_{\beta<\delta} \chi_\beta \in p_\zeta$, then by the property (3) (from the previous lemma) for each $\beta < \delta$ holds that $p_\zeta \cup \{\chi_\beta\}$ is in P, whence $p_{\alpha_\beta} \cup \{\chi_\beta\} \in P$ (for each β) and, by the construction of the chain, $\chi_\beta \in p_{\alpha_\beta+1}$.

III Let $\phi_\alpha \equiv \bigvee_{\beta<\delta} \chi_\beta \in p_\kappa$. Let $\gamma \geq \alpha$ be such that $\bigvee_{\beta<\delta} \chi_\beta \in p_\gamma$. Then $p_\alpha \cup \{\bigvee_{\beta<\delta} \chi_\beta\} \in P$ and therefore $p_{\alpha+1} = p_\alpha \cup \{\bigvee_{\beta<\delta} \chi_\beta\} \cup \{\chi_\gamma\}$ for some γ ($< \delta$).

IV Let $\phi_\alpha \equiv \forall X\chi(X) \in p_\kappa$ and let $F : X \longrightarrow C$. If $\phi_\beta \equiv Sub^{v\in X}_{F(v)}\,\chi$ and $\gamma \geq \alpha, \beta$ is such that $\forall X\chi(X) \in p_\gamma$ then $p_\gamma \cup \{Sub^{v\in X}_{F(v)}\,\chi\}$ is in P (property (5)) and so $p_\beta \cup \{Sub^{v\in X}_{F(v)}\,\chi\}$ is in P; hence $Sub^{v\in X}_{F(v)}\,\chi \in p_{\beta+1}$.

V Let $\phi_\alpha \equiv \exists X\chi(X)$ and let $\gamma \geq \alpha$ be such that $\exists X\chi(X) \in p_\gamma$. Again, because of $p_\alpha \cup \{\exists X\chi(X)\} \in P$ we obtain $p_{\alpha+1} = p_\alpha \cup \{\exists X\chi(X)\} \cup \{Sub^{v\in X}_{F(v)}\,\chi\}$ for some $F : X \longrightarrow C$.

VI and VII are (very) trivial. As for VIII let us just note that if $\phi_\alpha \equiv t_1 = t_3$ and $\beta \geq \alpha$ is such that $t_1 = t_2$, $t_2 = t_3 \in p_\beta$, we can immediately assume that $t_1 = t_2$ and $t_2 = t_3$ belong to the same part of one of the correct decompositions of p_β (Lemma 3.2); trivially, $p_\beta \cup \{t_1 = t_3\} \in P$, thus $p_\alpha \cup \{t_1 = t_3\} \in P$ and $t_1 = t_3 \in p_{\alpha+1}$. IX and X are proved similarly.□

Let $\sim_{p_\kappa}$ be a binary relation on the set C defined by: $c \sim_{p_\kappa} d$ iff $c = d \in p_\kappa$. By VI, VII and VIII $\sim_{p_\kappa}$ is an equivalence relation. Let M be the set of the classes of equivalence of the relation $\sim_{p_\kappa}$ – $\{[c] \mid c \in C\}$, where $[c] \stackrel{\text{def}}{=} \{d \in C \mid c \sim_{p_\kappa} d\}$, and let $\mathcal{C}$, $\mathcal{F}$ and $\mathcal{R}$ be the sets of all the constant, function and relation symbols of the language $L(\mu')$ appearing in the sentences of p, respectively (clearly, as for function and relation symbols we have just those which appear in the sentences of p, but the set of constants appearing in sentences of p is supplemented by the set C). For a constant $a \in \mathcal{C}$ let $a^M = [c]$, where $c \in C$ is such that $a = c \in p_\kappa$; for an n-ary function symbol $f \in \mathcal{F}$ we define: $f^M([c_1], \ldots, [c_n]) = [c]$, where $c \in C$ is such that $f(c_1, \ldots, c_n) = c \in p_\kappa$; for an n-ary relation symbol $R \in \mathcal{R}$ we put: $R^M([c_1], \ldots, [c_n])$ iff $R(c_1, \ldots, c_n) \in p_\kappa$. The existence of the constant c ($\in C$) in cases of constants and function symbols is guaranteed by IV (see also (8)). The correctness of the definition (of interpretation of symbols) is obvious as well.

Lemma 3.5. *For any closed term t appearing in some sentence of p_κ and any constant c from C holds: $t = c \in p_k$ iff $\mathbf{M} \models t = c$.*

Proof. Naturally, the interpretation of such terms (for a term t it will be denoted by t^M) is defined recursively by their complexity (where the complexity of a term is determined by the number of function symbols appearing in it). The proof of the lemma is again by induction on the complexity of terms.

If t is a constant, the assertion follows from the very definition. If $t \equiv f(t_1, \ldots, t_n)$ and $t = c$, $t_i = c_i \in p_\kappa$, $i = 1, \ldots, n$, for some $c, c_1, \ldots, c_n$ from C, then by inductive hypothesis holds $\mathbf{M} \models t_i = c_i$ (for each $i = 1, \ldots, n$), whence $\mathbf{M} \models f(t_1, \ldots, t_n) = f(c_1, \ldots, c_n)$. But if $f(c_1, \ldots, c_n) = d \in p_\kappa$ for (some) $d \in C$, then clearly $c = d \in p_\kappa$ and $\mathbf{M} \models f(c_1, \ldots, c_n) = d$, $\mathbf{M} \models d = c$, and so $\mathbf{M} \models t = c$.

Let us suppose now that $\mathbf{M} \models f(t_1, \ldots, t_n) = c$, i.e. $f^M(t_1^M, \ldots, t_n^M) = [c]$. If $t_i^M = [c_i]$, $i = 1, \ldots, n$, then from $f^M([c_1], \ldots, [c_n]) = [c]$ follows $f(c_1, \ldots, c_n) = c \in p_\kappa$, which together with $t_i = c_i \in p_\kappa$ ($i = 1, \ldots, n$), implied by the inductive assumption, gives $f(t_1, \ldots, t_n) = f(c_1, \ldots, c_n) \in p_\kappa$ and $t = c \in p_\kappa$. □

Now we have

Lemma 3.6. $\mathbf{M} = (M, (f^M)_{f \in \mathcal{F}}, (R^M)_{R \in \mathcal{R}}, (a^M)_{a \in \mathcal{C}})$ *is a model of p_κ.*

Proof. In the considerations bellow ϕ will be a sentence from $S_\varphi \cup S_\psi$ such that each non-logical symbol of $L(\mu)$ occurring in ϕ has an occurrence in some sentence of p.

If ϕ is atomic, then it holds: $\phi \in p_\kappa$ iff $\mathbf{M} \models \phi$. Let us prove it. Suppose first that ϕ is in p_κ.

If $\phi \equiv t_1 = t_2$, let c_1, c_2 be constants from C such that $t_1 = c_1, t_2 = c_2 \in p_\kappa$. Then, of course, $c_1 = c_2 \in p_\kappa$ and by the previous lemma $\mathbf{M} \models c_1 = c_2$ as well as $\mathbf{M} \models t_i = c_i$, $i = 1, 2$. Hence $\mathbf{M} \models t_1 = t_2$.

Let now $\phi \equiv R(t_1, \ldots, t_n)$. If $\mathbf{M} \models t_i = c_i$, $i = 1, \ldots, n$, again by the previous lemma holds $t_i = c_i \in p_\kappa$ and, certainly, $R(c_1, \ldots, c_n) \in p_\kappa$. By the definition, $R^M([c_1], \ldots, [c_n])$, i.e. $\mathbf{M} \models R(t_1, \ldots, t_n)$.

In the same way we could prove: $\mathbf{M} \models \phi$ implies $\phi \in p_\kappa$.

If $\phi \equiv \neg\chi$ is in $S_\varphi \cup S_\psi$, then χ has to be an atomic sentence and by the previous result it folllows: $\phi \in p_\kappa$ iff $\chi \notin p_\kappa$ iff $\mathbf{M} \not\models \chi$ iff $\mathbf{M} \models \phi$. Here we have in mind: for each atomic sentence χ (satisfying the above condition) either $\chi \in p_\kappa$ or $\neg\chi \in p_\kappa$; for if $\phi_\alpha \equiv \chi \vee \neg\chi$, then obviously $\phi_\alpha \in p_{\alpha+1}$ and by III either χ or $\neg\chi$ (but not both) belongs to p_κ.

The other cases are trivial as well. For instance, if $\phi \equiv \forall X\chi(X) \in p_\kappa$, then $Sub^{v\in X}_{F(v)}\,\chi \in p_\kappa$ for all functions $F : X \longrightarrow C$. Thus, by inductive assumption, $\chi(X)$ is satisfied in $\mathbf{M}$ for any valuation, whence $\mathbf{M} \models \forall X\chi(X)$. □

In the end, let us suppose that there is no *interpolating sentence* for (*interpolants*) φ and ψ. If $\varphi^\star$ and $(\neg\psi)^\star$ are negative normal sentences logically equivalent to φ and $\neg\psi$, respectively, then $\{\phi^\star, (\neg\psi)^\star\}$ is an element of P (otherwise sentences φ and ψ would have an interpolating sentence). But this means that there is a model satisfying $\varphi^\star$ nad $(\neg\psi)^\star$, contradictory to $\vdash_\Lambda \varphi \Rightarrow \psi$. □

Weak Interpolation Theorem and the already mentioned fact that Craig Interpolation Theorem fails in all infinitary logics but $L_{\omega_1\omega}$ ([2], [7]) imply directly

Corollary 3.7. *The basic system is an incomplete axiomatization for all infinitary logics $L_{\kappa\lambda}$, where κ is a regular cardinal greater than ω_1 (incomplete in the sense that the set of its theorems is a proper subset of the set of valid sentences).*

Proof. Clearly, any counterexample which proves that Craig Interpolation Theorem fails in some infinitary logic $L_{\kappa\lambda}$, κ a regular cardinal greater than ω_1, is at the same time an example of a valid sentence which is not provable in the basic system. □

Note. As it is known Karp's completeness theorem states that the basic system is a complete axiomatization for $L_{\omega_1\omega}$. C. Karp proved as well that the basic system is incomplete for all other infinitary logics and gave a significant contribution to the research of complete axiomatizations of infinitary logics ([6], [7]).

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