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PARAMETER ESTIMATION FOR UNIFORM AUTOREGRESSIVE PROCESSES

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Abstract. Chernick [2] and Chernick and Davis [3] have described the stationary uniform first-order autoregressive processes with positive and negative lag one autocorrelation function, respectively. In this paper, we discuss some properties of these processes. We also apply some estimation methods to estimate the parameters of these processes. It is shown that the conditional least squares estimators are strongly consistent and asymptotically normal.

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1. Introduction

The uniform first-order autoregressive process with positive lag one autocorrelation function (UAR(+)) was introduced by Chernick [2] and is defined as follows.

Let k be an integer such that $k \geq 2$. The stationary sequence $\{X_n\}$ is defined recursively by the equation

$$X_n = \frac{1}{k} X_{n-1} + \varepsilon_n, \; n \in Z = \{0, \pm 1, \pm 2, \ldots\} \tag{1.1}$$

where $\{\varepsilon_n\}$ is sequence of independent and identically distributed (i.i.d.) random variables with the distribution

$$\begin{pmatrix} 0 & 1/k & \ldots & (k-1)/k \\ 1/k & 1/k & \ldots & 1/k \end{pmatrix}$$

and the sequences $\{X_n\}$ and $\{\varepsilon_n\}$ are semi-independent, i.e. the random variables X_m and ε_n are independent iff is $m < n$.

Chernick [2] showed that if $x \geq 0$ and $u_n = 1 - x/n$, then

$$P\left[M_n \leq u_n\right] \to \exp\{-(k-1)x/k\} \text{ as } n \to \infty$$

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where $M_n = \max\{X_0, X_1, \dots, X_n\}$.

The second process, the uniform first-order autoregressive process with negative lag one autocorrelation function (UAR(−)) was introduced by Chernick and Davis [3] and is defined as follows.

Let k be an integer such that $k \geq 2$. The stationary sequence $\{X_n\}$ is defined recursively by the equation

$$X_n = -\frac{1}{k} X_{n-1} + \eta_n, \ \ n = 0, \pm 1, \pm 2, \dots \tag{1.2}$$

where $\{\eta_n\}$ is a sequence of i.i.d. random variables with the distribution

$$\begin{pmatrix} 1/k & 2/k & \dots & (k-1)/k & 1 \\ 1/k & 1/k & \dots & 1/k & 1/k \end{pmatrix}$$

and the sequences $\{X_n\}$ and $\{\eta_n\}$ are semi-independent.

Chernick and Davis [3] showed that if $x \geq 0$ and $u_n = 1 - x/n$, then

$$P[M_n \leq u_n] \to \exp\{-(k^2-1)x/k^2\} \text{ as } n \to \infty.$$

2. Some properties of the uniform autoregressive processes

In this section we discuss some properties of the uniform first-order autoregressive processes with positive and negative lag one autocorrelation function as the autocovariance and autocorrelation functions. We also discuss the regression and the joint distribution of two successive elements of the sequence.

Theorem 2.1. *The $UAR(+)$ process has:*

(i) the real valued absolutely summable autocovariance function

$$\gamma_X(i) = \frac{1}{12k^{|i|}}, \ \ i \in Z.$$

(ii) the real valued absolutely summable autocorrelation function

$$\rho_X(i) = \frac{1}{k^{|i|}}, \ \ i \in Z.$$

(iii) the real valued spectral density

$$f(\lambda) = \frac{k^2-1}{24\pi(k^2 - 2k\cos\lambda + 1)}, \ \ \lambda \in [-\pi, \pi].$$

Theorem 2.2. *The $UAR(-)$ process has:*

(i) the real valued absolutely summable autocovariance function

$$\gamma_X(i) = \frac{(-1)^{|i|}}{12k^{|i|}}, \quad i \in Z.$$

(ii) the real valued absolutely summable autocorrelation function

$$\rho_X(i) = \frac{(-1)^{|i|}}{k^{|i|}}, \quad i \in Z.$$

(iii) the real valued spectral density

$$f(\lambda) = \frac{k^2-1}{24\pi(k^2+2k\cos\lambda+1)}, \quad \lambda \in [-\pi, \pi].$$

By using (1.1) and the Markovian properties of the process $\{X_n\}$, the joint Laplace-Stieltjes transform of X_n and X_{n-1} for UAR(+) process can be obtained as

$$\begin{aligned} \Phi_{X_n,X_{n-1}}(s,t) &\equiv E(\exp\{-sX_n - tX_{n-1}\}) \\ &= \Phi_X(s/k+t)\Phi_\varepsilon(s) \\ &= \frac{1-e^{-\frac{s+kt}{k}}}{s+kt}\sum_{i=0}^{k-1} e^{-\frac{is}{k}}. \end{aligned} \tag{2.1}$$

which is not symmetrical in s and t. As a consequence, the process $\{X_n\}$ is not a time-reversible one.

We now obtain the joint p.d.f. of X_n and X_{n-1}. By inverting (2.1), the joint p.d.f. of X_n and X_{n-1} is given by

$$f(x_n, x_{n-1}) = \frac{1}{k}\, I_{(0,1)}(x_{n-1}) \sum_{i=0}^{k-1} \delta\left(x_n - \frac{1}{k}x_{n-1} - \frac{i}{k}\right).$$

where $\delta(\cdot)$ is Dirac's delta function and $I_{(a,b)}(x)$ is given by

$$I_{(a,b)}(x) = \begin{cases} 1, & x \in (a,b), \\ 0, & \text{otherwise.} \end{cases}$$

The regression of X_n on $X_{n-1} = x$ and of X_{n-1} on $X_n = x$ follows from the theorem:

Theorem 2.3. *Let $\{X_n\}$ be the UAR(+) process defined by* (1.1).

(i) The regression of X_n on $X_{n-1} = x$ is

$$E(X_n|\, X_{n-1} = x) = \frac{1}{k}x + \frac{k-1}{2k}, \quad x \in (0,1).$$

(ii) The regression of X_{n-1} *on* $X_n = x$ *is*

$$E(X_{n-1}|\ X_n = x) = \begin{cases} kx + i(k-2) & , kx \in (i, i+1),\ i = \overline{0, k-1} \\ k(k-1) & , x > 1. \end{cases}$$

(iii) From (i) and (ii) follows that the process $\{X_n\}$ *is not time-reversible.*

Proof. (i) The best linear regression of X_n on X_{n-1} by means of the conditional expectation follows

$$E(X_n|\ X_{n-1} = x) = \frac{1}{k}\ x + E(\varepsilon_n) = \frac{1}{k}\ x + \frac{1}{2}\left(1 - \frac{1}{k}\right) = \frac{1}{k}\ x + \frac{k-1}{2k}.$$

(ii) To obtain the regression of X_{n-1} on $X_n = x$ we differentiate (2.1) with respect to t, set $t \to 0+$, invert with respect to s and then divide by $-I_{(0,1)}(x)$. □

The joint Laplace-Stieltjes transform of X_n and X_{n-1} for the UAR(−) process is

$$\Phi_{X_n, X_{n-1}}(s,t) = \frac{1 - e^{\frac{s-kt}{k}}}{kt - s} \sum_{i=1}^{k} e^{-\frac{is}{k}}.$$

Since this transform is not symmetric in s and t, the UAR(−) is not time-reversible.

The regression of X_n on $X_{n-1} = x$ and of X_{n-1} on $X_n = x$ for the UAR(−) process follows from the theorem:

Theorem 2.4. *Let* $\{X_n\}$ *be the* $UAR(-)$ *process defined by* (1.2).

(i) The regression of X_n *on* $X_{n-1} = x$ *is*

$$E(X_n|\ X_{n-1} = x) = -\frac{1}{k}x + \frac{k+1}{2k},\ x \in (0,1).$$

(ii) The regression of X_{n-1} *on* $X_n = x$ *is*

$$E(X_{n-1}|\ X_n = x) = \begin{cases} -kx + 1 + (1-i)(k-2) & , kx \in (i-1, i),\ i = \overline{1, k} \\ k(1-k) & , x > 1. \end{cases}$$

(iii) From (i) and (ii) follows that the process $\{X_n\}$ *is not time-reversible.*

3. Random coefficient representation and conditional least squares estimation

Random coefficient representation gives linear form to the models (1.1) and (1.2). Consider the process $\{X_n\}$ generated by

$$X_n = \alpha X_{n-1} + \theta_n, \ n \in Z, \tag{3.1}$$

where $\alpha = 1/k$ and $\theta_n = \varepsilon_n$ or $\alpha = -1/k$ and $\theta_n = \eta_n$.

Let $\sigma_n = \{\theta_s, \ s \leq n\}$ denotes the σ-field.

The following Lemma will be needed to prove Theorem 3.1.

Lemma 3.1. *The random difference equation* (3.1) *has a unique, strictly stationary, σ_n-measurable and ergodic solution of the form*

$$X_n = \sum_{i=0}^{\infty} \alpha^i \theta_{n-i}.$$

We can now estimate the parameter α using conditional least squares method.

The equation (3.1) can be rewritten as

$$Y_n = \alpha Y_{n-1} + \beta_n, \tag{3.2}$$

where $Y_n = X_n - 1/2$ and $\beta_n = \theta_n - (1-\alpha)/2$. This translation does not disturb the existence of the solution of difference equation (3.2), by meaning that the solution exists and it is also unique, stationary, strictly stationary and ergodic.

Let $(X_0, X_1, \ldots, X_N)$ be a sample of size $N+1$. If we translate each observation of this sample in the following way $Y_n = X_n - 1/2$, we obtain the sample $(Y_0, Y_1, \ldots, Y_N)$.

The conditional least squares estimator $\hat{\alpha}$ of the parameter α is obtained by minimizing the function

$$S(\alpha) = \sum_{n=1}^{N} \{Y_n - \alpha Y_{n-1}\}^2$$

with respect to α. So, it is of the form

$$\hat{\alpha} = \frac{\sum_{n=1}^{N} Y_n Y_{n-1}}{\sum_{n=1}^{N} Y_{n-1}^2}.$$

The following Theorem gives the limit distribution of the conditional least squares estimator $\hat{\alpha}$.

Theorem 3.1. *The estimator $\hat{\alpha}$ is a strongly consistent estimator for α and $\sqrt{N}(\hat{\alpha} - \alpha)$ has an asymptotic $\mathcal{N}(0, 1-\alpha^2)$ distribution.*

Proof. Note that

$$\hat{\alpha} - \alpha = \frac{N^{-1} \sum\limits_{n=1}^{N} Y_{n-1}\beta_n}{N^{-1} \sum\limits_{n=1}^{N} Y_{n-1}^2}.$$

Applying the ergodic theorem on the strictly stationary and ergodic processes $\{Y_n^2\}$ and $\{Y_{n-1}\beta_n\}$, we obtain that these sequences converge almost surely to $1/12$ and 0, respectively. This implies that $\hat{\alpha} - \alpha$ converges almost surely to 0.

According to the central limit theorem for martingales (Billingsley [1]), each element of the random sequence $\{N^{-1/2} \sum_{i=1}^{N} qY_{n-1}\beta_n\}$ has asymptotically normal distribution with mean value zero and variance

$$E(q^2 Y_{n-1}^2 \beta_n^2) = q^2(1-\alpha^2)/144,$$

for any real q.

Furthermore, by the ergodic theorem, the sequence $N^{-1} \sum_{i=1}^{N} Y_{n-1}^2$ converges in probability to $1/12$.

So, $\sqrt{N}(\hat{\alpha} - \alpha)$ has an asymptotically normal distribution with mean value zero and variance $1 - \alpha^2$. □

The results of estimating α by means of conditional least squares for three modeled samples are given in Tables 1 and 2.

N	$k = 2$	$k = 10$	$k = 100$
500	1.8539350	12.2339067	———
1000	1.9126812	9.0984889	———
5000	1.9131263	9.1551268	429.0522275
10000	1.9536114	9.4156990	399.5720612
15000	1.9673772	9.3654976	381.5429510

Table 1: Estimators of the parameter k for three different modeled samples for the UAR(+) process

N	$k = 2$	$k = 10$	$k = 100$
500	1.7167570	9.5541922	19.8852167
1000	1.8894550	7.4311021	16.4995337
5000	1.9824494	9.9830257	23.8153963
10000	1.9810025	10.5540275	43.2904491
15000	1.9823614	10.2341603	63.2459873

Table 2: Estimators of the parameter k for three different modeled samples for the UAR(−) process

Consider the UAR(+) process. Since

$$X_n = \frac{1}{k}X_{n-1} + \varepsilon_n \geq \frac{1}{k}X_{n-1},$$

it implies that we can use

$$\tilde{k} = \max_{1\leq i\leq N}\left\{\frac{X_{i-1}}{X_i}\right\}$$

as the estimator for k.

Now, consider the UAR(−) process. Since

$$1 - X_n = \frac{1}{k}X_{n-1} + 1 - \eta_n \geq \frac{1}{k}X_{n-1},$$

it implies that we can use

$$\tilde{k} = \max_{1\leq i\leq N}\left\{\frac{X_{i-1}}{1 - X_i}\right\}$$

as the estimator for k.

The simulations show that $\tilde{k}$ is considerably better than the conditional least squares estimator $\hat{k}$.

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