NOVI SAD J. MATH.
VOL. 30, NO. 1, 2000, 109-121

SOME TOPOLOGICAL PROPERTIES OF THE ALGEBRA OF GENERALIZED HYPERFUNCTIONS ON THE CIRCLE

Vincent Valmorin[1]

Abstract. An invariant ultrametric distance ω is introduced in the differential algebra $\mathcal{H}(\mathbb{T})$ of generalized hyperfunctions on the unit circle $\mathbb{T}$ due to the author. For the induced topology, addition and mutiplication are continuous maps. It is also shown that the inversion is a continuous endomorphism of the group $\mathcal{H}^*(\mathbb{T})$ of invertible elements of $\mathcal{H}(\mathbb{T})$. Next, the association of an element in $\mathcal{H}(\mathbb{T})$ with a distribution or an analytic function on $\mathbb{T}$ is related to its position with respect to the unit ball.

AMS Mathematics Subject Classification (1991): 30A99, 42A16, 46F15, 54E35, 54H13

Key words and phrases: Fourier coefficients, periodic hyperfunctions, generalized hyperfunctions, ultrametric distance

1. Introduction

Topologies on algebras of generalized functions have been introduced or used by some authors to study the wellposedness of Cauchy problems with generalized data ([2, 3, 7, 8]). Beside this approach the sharp topology introduced by D. Scarpalezos ([7, 8]) is also considered for its own interest related to functorial properties.

In this paper we define in a canonical way, a very natural ultrametric distance on the algebra $\mathcal{H}(\mathbb{T})$ of generalized hyperfunctions on the circle introduced by the author in [16]. This metric will be referred to as "the" metric of $\mathcal{H}(\mathbb{T})$.

In the second section of this paper we give without proofs the relevant materials on $\mathcal{H}(\mathbb{T})$. More details can be found in ([16]). At the beginning of the third section we define the metric ω and give the first properties of the metric algebra $\mathcal{H}(\mathbb{T})$. The last part of this section is devoted to Theorem 3.9, which is our main result.

2. Generalized hyperfunctions on the circle

In the first part of this section we summarize without proofs the relevant material on hyperfunctions on the unit circle $\mathbb{T}$.

[1]Université des Antilles et de la Guyane, Département Math-Info, Campus de Fouillole, 97159 Pointe à Pitre Cedex, France, e-mail: vincent.valmorin@univ-ag.fr

2.1 Basic spaces of functions on $\mathbb{T}$

Let $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$ and $C_r = \{z \in \mathbb{C} : 1/r < |z| < r\}$ where $r > 1$. We denote by $\mathcal{O}_r$ the Banach space of bounded holomorphic functions in C_r normed by $\|f\|_r = \sup_{z \in C_r} |f(z)|$. The space $\mathcal{A}(\mathbb{T})$ of analytic functions on $\mathbb{T}$ is the inductive limit of the spaces $\mathcal{O}_r$ as $r \to 1$. The space $\mathcal{E}'(\mathbb{T})$ of Swartz distributions on $\mathbb{T}$ is the topological dual of the space $\mathcal{E}(\mathbb{T})$ of smooth functions on $\mathbb{T}$.

For $f \in \mathcal{A}(\mathbb{T})$, the coefficient $\widehat{T}(k)$ of z^k in the Laurent expansion of f is its Fourier coefficient of rank k. Complex numbers $c_k, k \in \mathbb{Z}$, are the Fourier coefficients of some analytic function if and only if $\limsup_{|k| \to \infty} |c_k|^{1/|k|} < 1$.

If $k \in \mathbb{Z}$ and $T \in \mathcal{E}'(\mathbb{T})$, the Fourier coefficient of index k of T is given by $\widehat{T}(k) = \overline{T(z \mapsto z^k)}$ and $T = \sum_{k=-\infty}^{\infty} \widehat{T}(k) z^k$ in the topology of $\mathcal{E}'(\mathbb{T})$. Moreover, $(A_k)_k$ is the sequence of Fourier coefficients of a distribution if and only if there are positive constants C and s such that for all $k \in \mathbb{Z}$, $|A_k| \leq C(1 + |k|^s)$. Let $f \in \mathcal{E}(\mathbb{T})$, then $T(f) = \sum_{k=-\infty}^{\infty} \overline{\widehat{T}(k)} \widehat{f}.(k)$

The space $\mathcal{B}(\mathbb{T})$ of hyperfunctions on the circle is the topological dual $\mathcal{A}'(\mathbb{T})$ of $\mathcal{A}(\mathbb{T})$. For $k \in \mathbb{Z}$ and $H \in \mathcal{B}(\mathbb{T})$, the Fourier coefficient of index k of H is defined as $\widehat{H}(k) = \overline{H(z \mapsto z^k)}$. Furthemore, $H = \sum_{k=-\infty}^{\infty} \widehat{H}(k) z^k$ holds for the topology of $\mathcal{B}(\mathbb{T})$ and $(B_k)_k$ is the sequence of Fourier coefficients of some hyperfunction if and only if $\limsup_{|k| \to \infty} |B_k|^{1/|k|} \leq 1$. If $f \in \mathcal{A}(\mathbb{T})$, then $H(f) = \sum_{k=-\infty}^{\infty} \overline{\widehat{H}(k)} \widehat{g}(k)$.

The convolution $S * T$ of the two hyperfunctions S and T is given by:

$$(S * T)(z) = \sum_{-\infty}^{+\infty} \widehat{S}(k) \widehat{T}(k) z^k.$$

It is seen that $S * f \in \mathcal{A}(\mathbb{T})$ if $S \in \mathcal{B}(\mathbb{T})$ and $f \in \mathcal{A}(\mathbb{T})$. In the same way $S * f \in \mathcal{E}(\mathbb{T})$ if $S \in \mathcal{E}'(\mathbb{T})$ and $f \in \mathcal{E}(\mathbb{T})$. If $n \in \mathbb{N}$ we set $\varphi_n(z) = \sum_{|k| \leq n} z^k$. We have $\varphi_n * \varphi_n = \varphi_n$ and $\lim_{n \to \infty} \varphi_n = \delta$ in $\mathcal{E}'(\mathbb{T})$ where δ is the Dirac distribution.

2.2 Definition of $\mathcal{H}(\mathbb{T})$

Let $\mathcal{X}(\mathbb{T})$ denote the set of sequences of functions $(f_n)_n$ where $n \in \mathbb{N}$ and $f_n \in \mathcal{A}(\mathbb{T})$, and let $\mathcal{X}_e(\mathbb{T})$ denote the subset of $\mathcal{X}(\mathbb{T})$ whose elements $(f_n)_n$ are such that:

$$\exists a > 0,\ \exists \eta \in \mathbb{N},\ \exists r > 1 / \forall n > \eta,\ f_n \in \mathcal{O}_r,\ \|f_n\|_r \leq a^n.$$

We denote by $\mathcal{N}_e(\mathbb{T})$ the subset of $\mathcal{X}_e(\mathbb{T})$ whose elements $(f_n)_n$ satisfy the following condition:

$$\forall b \in]0, 1[,\ \exists \eta \in \mathbb{N},\ \exists r > 1 / \forall n > \eta,\ f_n \in \mathcal{O}_r,\ \|f_n\|_r \leq b^n.$$

It is seen that $\mathcal{X}_e(\mathbb{T})$ is an algebra for usual termwise operations and $\mathcal{N}_e(\mathbb{T})$ is an ideal of $\mathcal{X}_e(\mathbb{T})$.

Proposition 2.1. *Let $(f_n)_n$ be a family of functions f_n which are holomorphic in some neighborhood of $\mathbb{T}$. Then we have:*
(i) *$(f_n)_n \in \mathcal{X}_e(\mathbb{T})$ if and only if*

$$\exists a > 0,\ \exists \eta \in \mathbb{N},\ \exists r > 1/\ \forall n > \eta,\ \forall k \in \mathbb{Z},\ |c_{n,k}| \leq a^n r^{-|k|}.$$

(ii) *$(f_n)_n \in \mathcal{N}_e(\mathbb{T})$ if and only if*

$$\forall b \in]0,1[,\ \exists \eta \in \mathbb{N},\ \exists r > 1/\ \forall n > \eta,\ \forall k \in \mathbb{Z},\ |c_{n,k}| \leq b^n r^{-|k|}.$$

Definition 2.1. *The algebra of generalized hyperfunctions on $\mathbb{T}$ is the factor algebra*

$$\mathcal{H}(\mathbb{T}) = \mathcal{X}_e(\mathbb{T})/\mathcal{N}_e(\mathbb{T}).$$

The class of $(f_n)_n$ in $\mathcal{H}(\mathbb{T})$ is denoted by $\mathrm{cl}(f_n)$ or $[f_n]$ and $\mathcal{H}(\mathbb{T})$ is endowed with two differential structures defined by

$$\frac{d}{dz}[f_n] = [\frac{df_n}{dz}] \text{ and } \partial_\theta [f_n] = [\partial_\theta f_n]$$

where $(\partial_\theta f)(z) = iz\frac{df}{dz}(z)$ for $f \in \mathcal{A}(\mathbb{T})$.

Let $\mathcal{Q}(\mathbb{T})$ denote the subset of $\mathcal{A}(\mathbb{T})$ which consists of functions f such that $\widehat{f}(k) = 0$ for $|k|$ large enough. We set $\tilde{\mathcal{X}}_e(\mathbb{T}) = \mathcal{X}_e(\mathbb{T}) \cap \mathcal{Q}(\mathbb{T})^{\mathbb{N}}$ and $\tilde{\mathcal{N}}_e(\mathbb{T}) = \mathcal{N}_e(\mathbb{T}) \cap \mathcal{Q}(\mathbb{T})^{\mathbb{N}}$. Clearly, $\tilde{\mathcal{X}}_e(\mathbb{T})$ is an algebra and $\tilde{\mathcal{N}}_e(\mathbb{T})$ is an ideal of $\tilde{\mathcal{X}}_e(\mathbb{T})$. If $H \in \mathcal{B}(\mathbb{T})$, then $(H * \varphi_n)(z) = \sum_{|k| \leq n} \widehat{H}(k) z^k$ and $\lim_{n\to\infty} H * \varphi_n = H$ in $\mathcal{B}(\mathbb{T})$.

Proposition 2.2. *Let* $\bar{\mathbf{i}} : \begin{array}{ll} \mathcal{B}(\mathbb{T}) & \to \mathcal{H}(\mathbb{T}) \\ H & \mapsto [H * \varphi_n] \end{array}$ and $\bar{\mathbf{i}}_0 : \begin{array}{ll} \mathcal{A}(\mathbb{T}) & \to \mathcal{H}(\mathbb{T}). \\ f & \mapsto [f] \end{array}$
Then, $\bar{\mathbf{i}}$ is a linear embedding and $\bar{\mathbf{i}}_0$ is a one-to-one morphism of algebras such that $\bar{\mathbf{i}}|_{\mathcal{Q}(\mathbb{T})} = \bar{\mathbf{i}}_0$. Moreover, for any $H \in \mathcal{B}(\mathbb{T})$ one has $\bar{\mathbf{i}}(\frac{dH}{dz}) = \frac{d}{dz}\left(\bar{\mathbf{i}}(H)\right)$ and $\bar{\mathbf{i}}(\partial_\theta H) = \partial_\theta\left(\bar{\mathbf{i}}(H)\right)$.

In the sequel, an element of $\bar{\mathbf{i}}(\mathcal{B}(\mathbb{T}))$ will be referred to as a hyperfunction of $\mathcal{H}(\mathbb{T})$.

By construction, the embedding of $\mathcal{B}(\mathbb{T})$ in $\mathcal{H}(\mathbb{T})$ shows that any hyperfunction in $\mathcal{H}(\mathbb{T})$ has a representative in $\tilde{\mathcal{X}}_e(\mathbb{T})$. More generally:

Proposition 2.3. *Let $\sigma : \mathbb{N} \to \mathbb{N}$. Then $\left(f_n - f_n * \varphi_{\sigma(n)}\right)_n \in \mathcal{N}_e(\mathbb{T})$ for all $(f_n)_n \in \mathcal{X}_e(\mathbb{T})$, if and only if the ratio $\lim_{n\to\infty} \sigma(n)/n = \infty$. Consequently, any generalized hyperfunction has infinitely many representatives in $\tilde{\mathcal{X}}_e(\mathbb{T})$.*

This proposition gives

Corollary 2.1. *We have $\mathcal{H}(\mathbb{T}) = \tilde{\mathcal{X}}_e(\mathbb{T})/\tilde{\mathcal{N}}_e(\mathbb{T})$.*

Note that $(\varphi_{\sigma(n)})_n \in \mathcal{X}_e(\mathbb{T})$ if and only if $\sigma(n)/n$ is bounded.

Theorem 2.2. *An element $f \in \mathcal{H}(\mathbb{T})$ is invertible if and only if it admits a representative $(f_n)_n$ such that:*

$$\exists b \in]0,1[,\ \exists r > 1,\ \exists \eta \in \mathbb{N} \,/\, \forall n > \eta,\ \ f_n \in \mathcal{O}_r,\ \inf_{z \in C_r} |f_n(z)| > b^n \quad (**)$$

It is shown that if f is invertible then the above property holds for each representative. In the sequel we denote by $\mathcal{H}^*(\mathbb{T})$ the set of invertible elements of $\mathcal{H}(\mathbb{T})$.

2.3 Generalized numbers of exponential type

Let $\mathcal{S}$ be the set of complex valued sequences $(z_n)_{n\in\mathbb{N}}$. Such an element will be simply denoted as $(z_n)_n$. Let $\mathcal{C}_e$ be set of $(z_n)_n \in \mathcal{S}$ such that:

$$\exists a > 0,\ \exists \eta \in \mathbb{N} \,/\, \forall n > \eta,\ |z_n| \leq a^n.$$

We denote by $\mathcal{I}_e$ the set of elements $(z_n)_n \in \mathcal{C}_e$ such that:

$$\forall b \in]0,1[,\ \exists \eta \in \mathbb{N} \,/\, \forall n > \eta,\ \ |z_n| \leq b^n.$$

It may be seen that $\mathcal{C}_e$ is a subalgebra of $\mathcal{S}$ and that $\mathcal{I}_e$ is an ideal of $\mathcal{C}_e$.

Definition 2.2. *The algebra of complex generalized constants of* exponential type, *is the quotient algebra* $\ \mathcal{C} = \mathcal{C}_e/\mathcal{I}_e$.

An element of $\mathcal{C}$ is called a generalized number or a generalized constant. Every complex number z is identified with a generalized number in a natural way and is referred as a classical number.

Invertible elements of $\mathcal{C}$ are characterized by the following

Theorem 2.3. *Let $x \in \mathcal{C}$. Then $x \in \mathcal{C}^*$ if and only if x admits a representative $(x_n)_n$ such that*

$$\exists b \in]0,1[,\ \exists \eta \in \mathbb{N} \,/\, \forall n > \eta,\ \ |x_n| > b^n \quad (*)$$

The set of invertible elements of $\mathcal{C}$ is denoted by $\mathcal{C}^*$.

3. Topological structure

3.1 The indicator of a generalized hyperfunction

It is seen that $(f_n)_n \in \mathcal{X}_e(\mathbb{T})$ iff there exists $r > 1$ such that $\limsup_{n\to+\infty} \|f_n\|_r^{1/n} < +\infty$. Let $(f_n)_n \in \mathcal{X}_e(\mathbb{T})$ and set

$$\nu((f_n)) = \inf_{r>1} \left(\limsup_{p\to+\infty} \|f_p\|_r^{1/p} \right).$$

Let $(f_n)_n$ and $(g_n)_n$ be two elements of $\mathcal{X}_e(\mathbb{T})$ such that $(f_n - g_n)_n \in \mathcal{N}_e(\mathbb{T})$. Fix $b \in]0,1[$. There are $\eta \in \mathbb{N}, r > 1$ such that $f_n - g_n \in \mathcal{O}_r$ and $\|f_n - g_n\|_r \leq b^n$ for $n > \eta$. It follows that if $n > \eta$, then

$$\|f_n\|_r \leq \|f_n - g_n\|_r + \|g_n\|_r \leq b^n + \|g_n\|_r .$$

Taking $n \geq 1$, we obtain

$$\|f_n\|_r^{1/n} \leq (b^n + \|g_n\|_r)^{1/n} \leq b + \|g_n\|_r^{1/n} .$$

It follows that

$$\limsup_{p \to +\infty} \|f_p\|_r^{1/p} \leq b + \limsup_{p \to +\infty} \|g_p\|_r^{1/p} .$$

Therefore we have

$$\inf_{r>1} \left(\limsup_{p \to +\infty} \|f_p\|_r^{1/p} \right) \leq b + \inf_{r>1} \left(\limsup_{p \to +\infty} \|g_p\|_r^{1/p} \right),$$

and consequently

$$\inf_{r>1} \left(\limsup_{p \to +\infty} \|f_p\|_r^{1/p} \right) = \inf_{r>1} \left(\limsup_{p \to +\infty} \|g_p\|_r^{1/p} \right),$$

from which we get $\nu((f_n)) = \nu((g_n))$.
This enables us to give the following definition

Definition 3.1. *The* indicator *of an element f of $\mathcal{H}(\mathbb{T})$ is*

$$\nu(f) = \inf_{r>1} \left(\limsup_{n \to +\infty} \|f_n\|_r^{1/n} \right),$$

where $(f_n)_n$ is an arbitrary representative of f.

We now give some basic properties of ν.

Proposition 3.1. **(i)** $\forall f \in \mathcal{H}(\mathbb{T}), \nu(f) \geq 0$, $\nu(f) = 0 \iff f = 0$;
(ii) $\forall \lambda \in (\mathbb{C})^*, \nu(\lambda f) = \nu(f)$;
(iii) $\forall f, g \in \mathcal{H}(\mathbb{T}), \nu(fg) \leq \nu(f)\nu(g)$;
(iv) $\forall f, g \in \mathcal{H}(\mathbb{T}), \nu(f+g) \leq \sup(\nu(f), \nu(g))$;
(v) $\forall f \in \mathcal{H}^*(\mathbb{T}), \nu(f^{-1}) \geq (\nu(f))^{-1}$;
(vi) $\forall f, g \in \mathcal{H}(\mathbb{T}), |\nu(f) - \nu(g)| \leq \nu(f-g)$.

Proof. The first part of **(i)** and **(ii)**-**(iv)** are obvious, **(v)** follows from **(iii)** and the fact that $\nu(1) = 1$. Note that more generally, $\nu(\lambda) = 1$ for any $\lambda \in \mathbb{C}^*$. Claim **(vi)** follows from $\nu(f-g) \leq \nu(f) + \nu(g)$ which is a consequence of **(iv)**. We now show the second part of **(i)**. Let $(f_n)_n$ denote a representative of f. Suppose that $\nu(f) = 0$. Let $\varepsilon > 0$. There is $\rho > 1$ such that $\limsup_{n \to +\infty} \|f_n\|_r^{1/n}$

$\leq \varepsilon$ as soon as $1 < r < \rho$. It follows that for all $r \in]1, \rho[$, there is $\eta(r) \in \mathbb{N}$ such that $\|f_n\|_r \leq (2\varepsilon)^n$ for $n > \eta(r)$. Now, let $b \in]0,1[$, and choose $\varepsilon = b/2$. From the previous, we can find $r > 1$ and $\eta \in \mathbb{N}$ large enough such that $f_n \in \mathcal{O}_r$ and $\|f_n\|_r \leq b^n$ for $n > \eta$. That is $(f_n)_n \in \mathcal{N}_e(\mathbb{T})$, and then $f = 0$.

Conversely, suppose that $f = 0$. Let $b \in]0,1[$. There are $r > 1$ and $\eta \in \mathbb{N}$ such that $f_n \in \mathcal{O}_r$ and $\|f_n\|_r \leq b^n$ for $n > \eta$. It follows that $\limsup_{n\to+\infty} \|f_n\|_r^{1/n} \leq b$ and then $\nu(f) \leq b$, from which one gets $\nu(f) = 0$. The proposition is thus proved. □

3.2 The metric ω

The indicator ν enables us to define the metric ω on $\mathcal{H}(\mathbb{T})$.

Proposition 3.2. *The map ω from $\mathcal{H}(\mathbb{T})^2$ to $\mathbb{R}_+$ defined by $\omega(f,g) = \nu(f-g)$ satisfies the following properties for all $f, g, h \in \mathcal{H}(\mathbb{T})$.*
(i) $\omega(f,g) = \omega(g,f)$,
(ii) $\omega(f,g) = 0 \iff f = g$,
(iii) $\omega(f,g) \leq \sup(\omega(f,h), \omega(h,g))$,
(iii) $\omega(f+h, g+h) = \omega(f,g)$.
ω is then an invariant ultrametric distance on $\mathcal{H}(\mathbb{T})$.

Proof. This is a straigthforward consequence of Proposition 3.1. □
Consider on $\mathcal{H}(\mathbb{T})^2$ the metric D defined by $D[(f,g),(u,v)] = \sup(\omega(f,u), \omega(g,v))$, which defines the nutural topology of $\mathcal{H}(\mathbb{T})^2$. Then we have

Proposition 3.3. *ν is a continuous map from $\mathcal{H}(\mathbb{T})^2$ to $\mathbb{R}_+$. Moreover if (f,g) and (u,v) are elements of $\mathcal{H}(\mathbb{T})^2$, then:*
(ii) $\omega(f+g, u+v) \leq D[(f,g),(u,v)]$;
(iii) $\omega(fg, uv) \leq \sup(\nu(f), \nu(g))\, D[(f,g),(u,v)]$.

Proof. The first part follows from **(vi)** of Proposition 3.1 and **(i)** is obvious, let us show **(ii)**. Writing $fg - uv = f(g-v) + v(f-u)$ yields

$$\omega(fg, uv) \leq \sup[\nu(f(g-v)), \nu(v(f-u))] \leq \sup[\nu(f)\nu(g-v), \nu(v)\nu(f-u)],$$

from which **(ii)** follows □

A main result from **(ii)** and **(iii)** above is

Corollary 3.1. *Addition and multiplication are continuous maps from $\mathcal{H}(\mathbb{T})^2$ to $\mathcal{H}(\mathbb{T})$.*

Proposition 3.4. *Let $f, g \in \mathcal{H}^*(\mathbb{T})$. If $\nu(g-f)\nu\left(\frac{1}{f}\right) < 1$ then*

$$\nu\left(\frac{1}{f} - \frac{1}{g}\right) \leq \frac{\left[\nu\left(\frac{1}{f}\right)\right]^2}{1 - \nu\left(\frac{1}{f}\right)\nu(g-f)}\nu(g-f).$$

Proof. Let $f, g \in \mathcal{H}^*(\mathbb{T})$ and write

$$\frac{1}{fg} = \frac{1}{fg} - \frac{1}{f^2} + \frac{1}{f^2} = \frac{1}{f}\left(\frac{1}{g} - \frac{1}{f}\right) + \frac{1}{f^2}.$$

Then we have

$$\nu\left(\frac{1}{fg}\right) \leq \nu\left(\frac{1}{f}\right)\nu\left(\frac{1}{f} - \frac{1}{g}\right) + \left[\nu\left(\frac{1}{f}\right)\right]^2.$$

Since

$$\nu\left(\frac{1}{f} - \frac{1}{g}\right) = \nu\left(\frac{f-g}{gf}\right) \leq \nu\left(\frac{1}{fg}\right)\nu(g-f),$$

the inequality found above gives

$$\left[1 - \nu\left(\frac{1}{f}\right)\nu(g-f)\right]\nu\left(\frac{1}{f} - \frac{1}{g}\right) \leq \left[\nu\left(\frac{1}{f}\right)\right]^2 \nu(g-f).$$

Now, if $1 - \nu\left(\frac{1}{f}\right)\nu(g-f) > 0$ the expected inequality follows. □

In particular we have

Corollary 3.2. *The map $f \mapsto \frac{1}{f}$ is continuous from $\mathcal{H}^*(\mathbb{T})$ to $\mathcal{H}^*(\mathbb{T})$ with respect to the metric induced by ω.*

Proof. Taking $\nu(g-f) \leq \frac{1}{2}\left[\nu\left(\frac{1}{f}\right)\right]^{-1}$ in the previous proposition yields $\omega\left(\frac{1}{f}, \frac{1}{g}\right)$ $\leq 2\left[\nu\left(\frac{1}{f}\right)\right]^2 \omega(f,g)$ from which the continuity follows straigthforwardly. □

3.3 Association with respect to a subspace

Definition 3.2. *Let E denote some topological space of functions on $\mathbb{T}$ such that $\mathcal{A}(\mathbb{T}) \subset E \subset \mathcal{H}(\mathbb{T})$ and $\lim_{n\to 0} f_n = 0$ in E whenever $(f_n)_n \in \mathcal{N}_e(\mathbb{T})$. Two sequences $(f_n)_n$ and $(g_n)_n$ in $\mathcal{X}_e(\mathbb{T})$ are said to be* associated with respect to E, *if $\lim_{n\to 0}(f_n - g_n) = 0$ in E.*

Let $f, g \in \mathcal{H}(\mathbb{T})$ such that $f = [f_n]$ and $g = [g_n]$. It is seen that if $(f_n)_n$ and $(g_n)_n$ are associated with respect to E, then the same holds for all representatives of f and g. Hence we have the following definition

Definition 3.3. *Two generalized hyperfunctions f and g are said to be* associated with respect to E *if they admit two representatives $(f_n)_n$ and $(g_n)_n$ which are associated with respect to E. We note $f \approx g(E)$.*

Association is defined in the same way in $\mathcal{C}$. Note that in general association is not compatible with multiplication in $\mathcal{C}$ or in $\mathcal{H}(\mathbb{T})$.

3.4 A dense subalgebra of $\mathcal{H}(\mathbb{T})$

Let Λ denote the set of maps σ from $\mathbb{N}$ to $\mathbb{R}_+$ such that $\sigma(n) = O(n)$ as $n \to \infty$. We denote by $\mathcal{H}_\Lambda(\mathbb{T})$ the subset of $\mathcal{H}(\mathbb{T})$ whose elements f have a representative $(f_n)_n$ such that $f = [g_n]$ where $g_n = \sum_{|k|\leq\sigma(n)} \widehat{f_n}(k)z^k$ for some σ in Λ. It is seen that $(g_n)_n \in \mathcal{X}_e(\mathbb{T})$. By Proposition 2.3 there is no such σ satisfying this condition for all elements f of $\mathcal{H}(\mathbb{T})$ because from the hypothesis $\limsup_{n\to\infty} \frac{\sigma(n)}{n} < \infty$. However, we have the following proposition.

Proposition 3.5. *$\mathcal{H}_\Lambda(\mathbb{T})$ is a dense subalgebra of $\mathcal{H}(\mathbb{T})$. In particular, for any $f \in \mathcal{H}(\mathbb{T})$, there exists $\gamma > 0$ such that*

$$cl\left(\sum_{|k|\leq\gamma n} \widehat{f_n}(k)z^k\right) \approx f\left(\mathcal{A}(\mathbb{T})\right).$$

Proof. Let $f \in \mathcal{H}(\mathbb{T})$. Choose $a > 0$, $r > 1$, $\eta \in \mathbb{N}$ such that for all $n > \eta$ and $k \in \mathbb{Z}$, $\left|\widehat{f_n}(k)\right| \leq a^n r^{-|k|}$. It follows that there exists $C > 0$ such that

$$\forall n \in \mathbb{N},\ \forall k \in \mathbb{Z} \left|\widehat{f_n}(k)\right| \leq Ca^n r^{-|k|}.$$

Let $\varepsilon > 0$. Set $\rho = \sqrt{r}$ and choose $\gamma > 0$ such that $a\rho^{-\gamma} < \varepsilon$. For $|k| > \gamma n$, we find

$$\left|\widehat{f_n}(k)\right| \leq Ca^n\rho^{-|k|}\rho^{-|k|} < C\left(a\rho^{-\gamma}\right)^n \rho^{-|k|}.$$

It follows that

$$\|f_n - g_n\|_{\sqrt{\rho}} \leq C\left(a\rho^{-\gamma}\right)^n \sum_{|k|>\gamma n} \sqrt{\rho}^{-|k|} \leq C_1\left(a\rho^{-\gamma}\right)^n.$$

Hence, we have $\nu(f-g) \leq a\rho^{-\gamma} < \varepsilon$, showing the density of $\mathcal{H}_\Lambda(\mathbb{T})$ in $\mathcal{H}(\mathbb{T})$. That $\mathcal{H}_\Lambda(\mathbb{T})$ is a subalgebra of $\mathcal{H}(\mathbb{T})$ is obvious.
Taking $\varepsilon \leq 1$, we obtain from $\|f_n - g_n\|_{\sqrt{\rho}} \leq C_1\left(a\rho^{-\gamma}\right)^n$, that $f_n - g_n \to 0$ in $\mathcal{O}_{\sqrt{\rho}}$ when $n \to +\infty$. The proposition is thus proved. □

In order to deal with Fourier coefficients, we give another formula for $\nu(f)$.

Proposition 3.6. *The following formula holds for $f \in \mathcal{H}(\mathbb{T})$.*

$$\nu(f) = \inf_{r>1}\left\{\limsup_{n\to+\infty}\left[\sup_k\left(r^{|k|}\left|\widehat{f_n}(k)\right|\right)\right]^{1/n}\right\}.$$

Proof. For all positive ε, there are $r > 1, \eta \in \mathbb{N}$ such that $f_n \in \mathcal{O}_r$ and $\|f_n\|_r < (\nu(f)+\varepsilon)^n$ for $n > \eta$. Cauchy's inequalities lead to $r^{|k|}\left|\widehat{f_n}(k)\right| \leq \|f_n\|_r < (\nu(f)+\varepsilon)^n$ for all $k \in \mathbb{Z}$ if r, η and n chosen as above. Therefore

$\sup_k \left(r^{|k|}\left|\widehat{f_n}(k)\right|\right) \leq (\nu(f)+\varepsilon)^n$ and $\limsup_{n\to+\infty}\left[\sup_k\left(r^{|k|}\left|\widehat{f_n}(k)\right|\right)\right]^{1/n} \leq \nu(f)+\varepsilon$. It follows that $\inf_{r>1}\left\{\limsup_{n\to+\infty}\left[\sup_k\left(r^{|k|}\left|\widehat{f_n}(k)\right|\right)\right]^{1/n}\right\} \leq \nu(f)$. Conversely, let μ denote the right side of the equality to be shown. For every positive ε, there is $r>1$ such that $\sup_k\left(r^{|k|}\left|\widehat{f_n}(k)\right|\right) < (\mu+\varepsilon)^n$ for n large enough. It follows that for all $k \in \mathbb{Z}$ $\left|\widehat{f_n}(k)\right| < (\nu(f)+\varepsilon)^n r^{-|k|}$ and

$$\|f_n\|_{\sqrt{r}} < \sum_{k=-\infty}^{+\infty} (\mu+\varepsilon)^n r^{-|k|}(\sqrt{r})^k = (\mu+\varepsilon)^n \sum_{k=-\infty}^{+\infty} r^{-|k|}(\sqrt{r})^k.$$

Consequently $\limsup_{n\to+\infty}\left(\|f_n\|_{\sqrt{r}}^{1/n}\right) \leq \mu+\varepsilon$ and then $\nu(f)\leq\mu$, proving the proposition □

We now state one of our main results.

Theorem 3.3. *Let $f \in \mathcal{H}(\mathbb{T})$. Then the following holds:*
(i) *If $\nu(f)<1$ then $f \approx 0\,(\mathcal{A}(\mathbb{T}))$;*
(ii) *There exists $g \in \mathcal{H}(\mathbb{T})$ such that $\nu(g)=1$ and $g \approx 0\,(\mathcal{A}(\mathbb{T}))$;*
(iii) *If $f \approx 0\,(\mathcal{B}(\mathbb{T}))$ then $\nu(f) \leq 1$.*

Proof. **(i)** Suppose $\nu(f)<1$ and let $\varepsilon>0$ be such that $\nu(f)+\varepsilon<1$. From $\nu(f) = \inf_{r>1}\left(\limsup_{n\to+\infty}\|f_n\|_r^{1/n}\right)$, it follows that there exists $r>1$ such that $\limsup_{n\to+\infty}\|f_n\|_r^{1/n} < \nu(f)+\varepsilon$. Thus $\|f_n\|_r \leq (\nu(f)+\varepsilon)^n$ for n large enough showing that $\lim_{n\to+\infty} f_n = 0$ in $\mathcal{O}_r$. Hence $f \approx 0\,(\mathcal{A}(\mathbb{T}))$.
(ii) Let $g=[g_n]$ where g_n is such that $\widehat{g_n}(k) = \frac{1}{n}\left(1-\frac{1}{n}\right)^n \rho^{-|k|}$ with $\rho>1$. Let $1<r<\rho$. We have $r^{|k|}\left|\widehat{g_n}(k)\right| = \frac{1}{n}\left(1-\frac{1}{n}\right)^n\left(\frac{r}{\rho}\right)^{|k|}$ and $\sup_{k\in\mathbb{Z}}\left(r^{|k|}\left|\widehat{g_n}(k)\right|\right) = \frac{1}{n}\left(1-\frac{1}{n}\right)^n$. It follows that $\limsup_{n\to+\infty}\left[\sup_{k\in\mathbb{Z}}\left(r^{|k|}\left|\widehat{g_n}(k)\right|\right)\right]^{1/n} = 1$; thus $\nu(g)=1$. On the other hand, $\left|\widehat{g_n}(k)\right| \leq \frac{1}{n}\rho^{-|k|}$ gives $g \approx 0\,(\mathcal{A}(\mathbb{T}))$.
(iii) Let $f=[f_n]\neq 0$. Take $\rho>1, \eta_0\in\mathbb{N}$ such that $\left|\widehat{f_n}(k)\right| \leq (\nu(f)+\varepsilon)^n \rho^{-|k|}$ for $n>\eta_0$ and all $k\in\mathbb{Z}$, where ε satisfies $0<\varepsilon<\nu(f)$. Let $r\in]1,\rho[$.
From $\limsup_{n\to+\infty}\left[\sup_{k\in\mathbb{Z}}\left(r^{|k|}\left|\widehat{f_n}(k)\right|\right)\right]^{1/n} \geq \nu(f)$, there exists a subsequence $(f_{n_p})_p$ of $(f_n)_n$ such that $\lim_{p\to+\infty}\left[\sup_{k\in\mathbb{Z}}\left(r^{|k|}\left|\widehat{f_{n_p}}(k)\right|\right)\right]^{1/n_p} \geq \nu(f)$. Therefore,

$$\exists \eta_1\in\mathbb{N},\ \forall p>\eta_1,\ \sup_{k\in\mathbb{Z}}\left(r^{|k|}\left|\widehat{f_{n_p}}(k)\right|\right) \geq (\nu(f)-\varepsilon)^{n_p} > 0.$$

As a consequence of that, we have

$$\exists \eta_1\in\mathbb{N},\ \forall p>\eta_1,\ \forall\alpha\in]0,1[,\ \exists k_p\in\mathbb{Z},\ r^{|k_p|}\left|\widehat{f_{n_p}}(k_p)\right| \geq (\nu(f)-\varepsilon)^{n_p}\alpha.$$

It follows that: $\exists \eta \in \mathbb{N},\ \forall p > \eta,\ \forall \alpha \in]0,1[,\ \exists k_p \in \mathbb{Z}$,

$$\left(\frac{r}{\rho}\right)^{|k_p|} (\nu(f)+\varepsilon)^{n_p} \geq r^{|k_p|} \left|\widehat{f_{n_p}}(k_p)\right| \geq (\nu(f)-\varepsilon)^{n_p} \alpha.$$

Thus we have for $p > \eta$ and α fixed in $]0,1[$:

$$\left(\frac{\rho}{r}\right)^{|k_p|} \leq \frac{1}{\alpha} \left(\frac{\nu(f)+\varepsilon}{\nu(f)-\varepsilon}\right)^{n_p}$$

Set $\mu = \frac{\ln\left[\frac{1}{\alpha}\left(\frac{\nu(f)+\varepsilon}{\nu(f)-\varepsilon}\right)\right]}{\ln\left(\frac{\rho}{r}\right)}$. For $p > \eta$, $|k_p| \leq \mu n_p$ and then

$$r^{-|k_p|} \left|\widehat{f_{n_p}}(k_p)\right| \geq r^{-2|k_p|} (\nu(f)-\varepsilon)^{n_p} \alpha \geq r^{-2\mu n_p} (\nu(f)-\varepsilon)^{n_p} \alpha.$$

We thus have

$$\forall p > \eta,\ r^{-|k_p|} \left|\widehat{f_{n_p}}(k_p)\right| \geq \left(\frac{\nu(f)-\varepsilon}{r^{2\mu}}\right)^{n_p} \alpha.$$

We now assume that $\nu(f) > 1$ and first prove the following lemma.

Lemma 3.4. *Let $H,\ H_n \in \mathcal{B}(\mathbb{T})$, $n \in \mathbb{N}$. Then $(H_n)_n$ converges to H for the topology of $\mathcal{B}(\mathbb{T})$ iff*
(i) $\forall k \in \mathbb{Z},\ \lim_{n\to+\infty} \widehat{H_n}(k) = \widehat{H}(k)$;
(ii) $\forall r > 1,\ \exists A_r \geq 0,\ \forall n \in \mathbb{N},\ \forall k \in \mathbb{Z},\ \left|\widehat{H_n}(k)\right| \leq A_r r^{|k|}$.

Proof of Lemma 3.4. This proof is adapted from ([[4], Lemma p. 54]) which is related to distributions.
First suppose that both the conditions **(i)** and **(ii)** are satified. Let $\varphi \in \mathcal{A}(\mathbb{T})$. We have $H(\varphi) = \sum_{k=-\infty}^{\infty} \widehat{H}(k)\widehat{\varphi}(k)$ with $\left|\widehat{H}(k)\widehat{\varphi}(k)\right| \leq \left|\widehat{H}(k)\right| B_r r^{-|k|}$ for some $r > 1$ and an associated positive constant B_r. Let $\rho = \sqrt{r}$. From **(ii)**, there exists $A_\rho > 0$ such that for all $(n,k) \in \mathbb{N}\times\mathbb{Z}$, $\left|\widehat{H}(k)\right| \leq A_\rho \rho^{-|k|}$. It follows that $\left|\widehat{H}(k)\widehat{\varphi}(k)\right| \leq A_\rho B_r \rho^{-|k|}$ showing that $\sum_{k=-\infty}^{\infty} \left|\widehat{H}(k)\widehat{\varphi}(k)\right| < +\infty$. Therefore by **(i)** and Lebesgue's bounded convergence theorem $\lim_{n\to+\infty} H_n(\varphi) = H(\varphi)$, that is $\lim_{n\to+\infty} H_n = H$ in $\mathcal{B}(\mathbb{T})$.

Conversely, suppose that H_n converges to H in $\mathcal{B}(\mathbb{T})$. Since $\widehat{H_n}(k) = H_n(z \mapsto z^k)$, **(i)** is verified. Now, contrary to **(ii)**, suppose that:

$$\exists r > 1,\ \forall C \geq 0,\ \exists (n,k) \in \mathbb{N}\times\mathbb{Z},\ \left|\widehat{H_n}(k)\right| > C r^{2|k|} \quad (*)$$

We construct by induction two sequences of integers $(n_p)_p$ and $(k_p)_p$ with $n_p \in \mathbb{N}$ and $k_p \in \mathbb{Z}$, in the following way:

$n_1, ..., n_{q-1}; k_1, ..., k_{q-1}$ being determined for an integer $q \geq 2$, we set

$$a_p = \left|\widehat{H_{n_p}}(k_p)\right|^{-1/2}, \ 1 \leq p \leq q-1; \ \ A_p = \sup_{n\in\mathbb{N}} \left| H_n \left(\sum_{p=1}^{q-1} a_p z^{k_p} \right) \right|.$$

Set $\rho = \sqrt{r}$. From the continuity of the H_n's, it follows that

$$\exists B_q \geq 0, \forall \varphi \in \mathcal{O}_\rho, \ |H_n(\varphi)| \leq B_q \|\varphi\|_\rho; \ n \leq n_{q-1}.$$

From $(*)$, there exist n_q and k_q such that

$$|k_q| > |k_{q-1}|, \ \left|\widehat{H_{n_q}}(k_q)\right| > (A_q + B_q + q + 1)^2 \, r^{2|k_q|}.$$

That k_q can be chosen so that $|k_q| > |k_{q-1}|$ it follows from the boundedness of $\left(\widehat{H_n}(k)\right)_n$ for each fixed k.

Since $a_q \leq \frac{r^{-|k|}}{A_q+B_q+q+1}$, the series $\sum_{p=1}^{+\infty} a_p z^{k_p}$ converges in $\mathcal{O}_\rho$ to a function φ and

$$H_{n_q}(\varphi) = H_{n_q} \left(\sum_{p=1}^{q-1} a_p z^{k_p} + a_q z^{k_q} + \sum_{p=q+1}^{+\infty} a_p z^{k_p} \right),$$

from which we obtain

$$\left|H_{n_q}(\varphi)\right| \geq |a_q| \left|\widehat{H_{n_q}}(k_q)\right| - \left| H_{n_q} \left(\sum_{p=1}^{q-1} a_p z^{k_p} \right) \right| - \left| H_{n_q} \left(\sum_{p=q+1}^{+\infty} a_p z^{k_p} \right) \right|;$$

and finally

$$\left|H_{n_q}(\varphi)\right| \geq \left|\widehat{H_{n_q}}(k_q)\right|^{-1/2} - A_q - B_q \left\| \sum_{p=q+1}^{+\infty} a_p z^{k_p} \right\|_\rho .$$

The series $\sum_{p=1}^{+\infty} a_p z^{k_p}$ being convergent in $\mathcal{O}_\rho$, it follows that there exists $q_0 \in \mathbb{N}$ such that $\left\| \sum_{p=q+1}^{+\infty} a_p z^{k_p} \right\|_\rho \leq 1$ for $q \geq q_0$. Therefore

$$\forall q \geq q_0, \ \left|H_{n_q}(\varphi)\right| \geq \left|\widehat{H_{n_q}}(k_q)\right|^{-1/2} - A_q - B_q > q + 1.$$

This contradicts the fact that $\lim_{n\to+\infty} H_n(\varphi) = H(\varphi)$, which proves the lemma. □

We now come back to the proof of Theorem 3.3. Take $1 < \rho < [\nu(f)]^{1/9}$, $1 < r < \rho^{1/2}$, $\varepsilon = \frac{\rho-1}{\rho+1}\nu(f)$ and $\alpha = 1/\rho$. Then we find:

$$\mu = \frac{2 \ln \left[\rho \left(\frac{\nu(f)+\varepsilon}{\nu(f)-\varepsilon} \right) \right]}{\ln \rho}; \ \nu(f) - \varepsilon = \frac{2}{\rho+1}\nu(f) > \frac{\nu(f)}{\rho} > \rho^8 > r^{2\mu}.$$

It follows that $\frac{\nu(f)-\varepsilon}{r^{2\mu}} > 1$ and consequently $\lim_{p\to+\infty} r^{-|k_p|}\left|\widehat{f_{n_p}}(k_p)\right| = +\infty$ which contradicts Lemma 3.4, **(ii)**. It follows that $\nu(f) \leq 1$ whenever $f \approx 0\,(\mathcal{B}(\mathbb{T}))$, proving the theorem □

Next, we obtain straightforwardly

Corollary 3.5. *The following holds:*
(i) *If f is a nonzero hyperfunction of $\mathcal{H}(\mathbb{T})$, then $\nu(f) = 1$.*
(ii) *ν is a surjective map from $\mathcal{H}(\mathbb{T})$ to $\mathbb{R}_+$.*

Proof. Let $f \in \bar{\mathrm{i}}(\mathcal{B}(\mathbb{T}))$. Then we have $f = [f_n]$ with $f_n = H * \varphi_n$ where $H \in \mathcal{B}(\mathbb{T})$. First we show that $\nu(f) \leq 1$. For, let $\rho > 1$. From the Fourier characterization of $\mathcal{B}(\mathbb{T})$, there is $C > 0$ such that $|\widehat{H}(k)| \leq C\rho^{|k|}$ for all $k \in \mathbb{Z}$. Since $\widehat{f_n}(k) = \widehat{H}(k)$ for $|k| \leq n$ and $\widehat{f_n}(k) = 0$ for $|k| > n$, then for all $r > 1$ and all $k \in \mathbb{Z}$, we have $r^{|k|}\left|\widehat{f_n}(k)\right| \leq C(r\rho)^n$. It follows that $\left[\sup_k \left(r^{|k|}\left|\widehat{f_n}(k)\right|\right)\right]^{1/n} \leq C^{1/n} r\rho$. From Proposition 3.6, we find $\nu(f) \leq \rho$. Since this is valid for all $\rho > 1$, it follows that $\nu(f) \leq 1$. Now assume that $\nu(f) \neq 1$. From the above we have $\nu(f) < 1$. Therefore by **(i)** of Theorem 3.3, $f \approx 0\,(\mathcal{A}(\mathbb{T}))$ that is $H * \varphi_n \to 0$ in $\mathcal{A}(\mathbb{T})$ as $n \to +\infty$. Since $H * \varphi_n \to H$ in $\mathcal{B}(\mathbb{T})$ as $n \to +\infty$, it follows that $H = 0$ and then $f = 0$ proving **(i)**.
Let $\lambda \in \mathbb{R}_+$ and set $g_n = \lambda^n \varphi_n$. It is seen that $(g_n)_n \in \mathcal{X}_e(\mathbb{T})$. Let $g = [g_n]$. By **(i)** we have $\nu([\varphi_n]) = 1$, from which it is shown that $\nu(g) = \lambda$ proving the surjectivity of ν. □

References

[1] Berenstein, C.A., Gay R., Complex Analysis and Special Topics in Harmonic Analysis, Springer, 1995.

[2] Biagioni, H.A., Colombeau, J-F., New generalized functions and C^∞ functions with values in generalized complex numbers, J. London Math Soc(2) 33 (1986), 169-179.

[3] Biagioni, H.A., A Nonlinear Theory of Generalized Functions, Lect. Notes Math. 1421, Springer Verlag, Berlin - Heidelberg - New York 1990.

[4] Chou, C-C., Séries de Fourier et théorie des distributions, Science Press, Beijing, 1983.

[5] Colombeau, J-F., New Generalized Functions and Multiplication of Distributions, North-Holland Math. Stud. 84, 1984.

[6] Colombeau, J-F., Elementary Introduction to New Generalized Functions, North-Holland Math. Stud. 113, 1985.

[7] Delcroix, A., Scarpalezos, D., Topology and asymptotic algebras of generalized functions and applications, Mh. Math. 129 (2000), 1-14.

[8] Delcroix, A., Scarpalezos, D., Sharp topology on $(\mathcal{C}, \mathcal{E}, \mathcal{P})$-algebras, in Michael Grosser and al., Chapman & Hall/Crc Research Notes in Mathematics, Nonlinear Theory of Generalized Functions, 165-173, 1999.

[9] Kaneko, A., Introduction to Hyperfunctions, Kluwer Academic Publishers, 1988.

[10] Morimoto, M., Analytic Functionals on the Sphere, Translations of Mathematical Monographs. Volume 178, 1998.

[11] Oberguggenberger, M., Multiplication of distributions and application to partial differential equations, Pitman Res. Notes Math. Ser. 259, Longman, Harlow, 1992.

[12] Valmorin, V., Fonctions Généralisées périodiques et problème de goursat, C. R. Acad. Sci. Paris Série I 320 (1995), 537-540.

[13] Valmorin, V., A new algebra of periodic generalized functions, Journal for Analysis and its Applications, Volume 15 (1996), N^0 1, 57-74.

[14] Valmorin, V., Fonctions généralisées périodiques et applications, Dissertationes Mathematicae, N^0 CCCLXI (1997).

[15] Valmorin, V., On the multiplication of periodic hyperfunctions of one variable, in Michael Grosser and al., Chapman & Hall/Crc Research Notes in Mathematics, Nonlinear Theory of Generalized Functions, 219-228, 1999.

[16] Valmorin, V., Generalized hyperfunctions on the circle, to appear in Journal of Mathematical Analysis and Applications.

Received by the editors May 5, 2000.