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EUCLIDIAN PERIMETER OF CLASSES OF OPTIMAL CONVEX LATTICE $2k$–GONS

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Abstract. Convex lattice polygon is said to be optimal in sense of l_p metric if its l_p–perimeter is minimal with respect to the number of its vertices.

In this paper, the asymptotic expression for the Euclidian perimeter of optimal convex lattice $2k$–gons is derived as a function of the number of its vertices. The optimality is taken in sense of l_p metric for every integer p, and also for $p = \infty$.

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1. Introduction

Convex lattice polygon is a polygon whose vertices are points on the integer lattice and whose interior angles are strictly smaller than π radians (no three vertices are collinear). A convex lattice polygon with n vertices is called an n–gon.

Some problems concerning the convex polygons determined by lattice points on strictly convex curves cutting the maximal number of lattice points with respect to the length of the curve, were studied in [2], [4], [5] and [6]. In [3], [7] and [8], some asymptotic properties of convex lattice polygons were studied.

In this paper we consider a class of convex lattice n–gons which have a minimal l_p–perimeter with respect to the number of their vertices. So, if a convex lattice n–gon has that property, its l_p–perimeter is equal to

$$\min\left\{ \sum_{e \text{ is edge of } Q} l_p \text{ length of } e \;\middle|\; Q \text{ is a convex lattice } n\text{–gon} \right\},$$

and such a polygon is denoted by $Q_p(n)$. This polygon is not necessarily unique for a given integer n.

The main purpose of the paper is to describe the asymptotic behavior of the Euclidean (l_2) perimeter of the optimal convex lattice polygon $Q_p(2k)$ (where $2k$

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is an even integer) as a function of the number of its vertices. As a consequence, an asymptotic upper bound for the area of $Q_p(2k)$ is obtained.

These results generalize the results from [8] where this problem was studied in case $p = 1$, and also from [4] where this problem was studied in case $p = 2$.

2. Preliminaries

Let $e = ((x_1, y_1), (x_2, y_2))$ be an edge of a convex lattice polygon. The l_p–distance length of e is defined as

$$l_p(e) = \sqrt[p]{|x_2 - x_1|^p + |y_2 - y_1|^p}.$$

We shall denote the differences $|x_2 - x_1|$ and $|y_2 - y_1|$ by $x(e)$ and $y(e)$, respectively. The quotient of these differences $y(e)/x(e)$ is defined to be the slope of e.

The perimeter in sense of l_p metric of a convex lattice polygon Q is defined by

$$\mathrm{per}_p(Q) = \sum_{e \text{ is edge of } Q} l_p(e).$$

If a and b are integers, $a \perp b$ means that the greatest common divisor for a and b is 1. Also, we shall say that $1 \perp 0$.

By $\mu(n)$ we shall denote the Möbius function, defined as:

$$\mu(1) = 1;$$

if $n > 1$ and $n = p_1^{a_1} \cdots p_k^{a_k}$ is the prime decomposition of n, then

$$\mu(n) = \begin{cases} (-1)^k, & \text{if } a_1 = \ldots = a_k = 1 \\ 0, & \text{otherwise} \end{cases} .$$

For $n > 1$, $U_p(n)$ represents the partition function which counts the number of the positive solutions of the equation $n = x^p + y^p$, where x, y are relatively prime integers. If $n = 1$, we define $U_p(1) = 1$ for $p = 1, 2, \ldots, \infty$ (we take $x = 1$, $y = 0$ as a solution).

In [7], the following sequence of integers is introduced

$$n_p(t) = 4 \sum_{i=1}^{t} U_p(i), \qquad t = 1, 2, 3, \ldots \quad .$$

First, we shall consider optimal lattice polygons with $n_p(t)$ $(t = 1, 2, \ldots)$ vertices. It is shown in [7] by explicit construction that the optimal convex lattice polygon $Q_p(n_p(t))$ is determined uniquely. For each integer t, $Q_p(n_p(t))$ is constructed as follows, using "greedy algorithm".

The polygon consists of four isometric arcs, whose edge slopes coincide with the set

$$S_p(t) = \left\{ \frac{k}{l} \;\middle|\; k,\, l \text{ are integers, } k^p + l^p \le t,\ k \perp l \right\}.$$

We shall denote the vertices of $Q_p(n_p(t))$ by

$$A_0 = (x_0, y_0), \quad A_1 = (x_1, y_1), \quad \ldots \quad , A_n = (x_{n_p(t)}, y_{n_p(t)}) = A_0$$

in counterclockwise order.

Let e_1, e_2, $\ldots e_{n_p(t)}$ be the edges determined by consecutive points from the previous sequence, i.e.

$$e_1 = A_0A_1, \quad e_2 = A_1A_2, \quad \ldots \quad , e_{n_p(t)} = A_{n_p(t)-1}A_{n_p(t)}.$$

Then, the edges $e_1, e_2, \ldots, e_{n_p(t)}$ can be arranged into four arcs. If the angle between the positively oriented x–axis and the edge $A_{i-1}A_i$ is observed, then

- the south–east arc contains only the edges whose angles belong to $[0, \frac{\pi}{2})$;
- the north–east arc contains only the edges whose angles belong to $[\frac{\pi}{2}, \pi)$;
- the north–west arc contains only the edges whose angles belong to $[\pi, \frac{3\pi}{2})$;
- the south–west arc contains only the edges whose angles belong to $[\frac{3\pi}{2}, 2\pi)$.

The vertex A_0 is chosen to be one of the vertices having the minimal y–coordinate, which has the minimal x–coordinate (the "left lowest" point), and then the vertex $A_{\frac{1}{4}n_p(t)}$ will be the one of the vertices having the maximal x–coordinate, which has the minimal y–coordinate (the "lowest outermost right" point). For convenience and without loss of generality, let us assume $A_0 = (0,0)$. Since the slope of the edge e_i is equal to $y(e_i)/x(e_i)$ it follows that the vertices of the south–east arc of the polygon $Q_p(n_p(t))$ are:

$$A_0 = (0,0),$$

$$A_1 = (x(e_1),\ y(e_1)),$$

$$A_2 = (x(e_1) + x(e_2),\ y(e_1) + y(e_2)),$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$A_{\frac{1}{4}n_p(t)} = (x(e_1) + x(e_2) + \ldots + x(e_{\frac{1}{4}n_p(t)}),$$
$$y(e_1) + y(e_2) + \ldots + y(e_{\frac{1}{4}n_p(t)})).$$

The slopes belonging to the south–east arc have to be arranged in the increasing order

$$\frac{0}{1} = \frac{y(e_1)}{x(e_1)} < \frac{y(e_2)}{x(e_2)} < \ldots < \frac{y(e_{\frac{1}{4}n_p(t)})}{x(e_{\frac{1}{4}n_p(t)})},$$

and

$$S_p(t) = \left\{ \frac{y(e_1)}{x(e_1)}, \frac{y(e_2)}{x(e_2)}, \ldots, \frac{y(e_{\frac{1}{4}\cdot n_p(t)})}{x(e_{\frac{1}{4}\cdot n_p(t)})} \right\}.$$

The remaining three arcs are obtained by the rotations for $\frac{\pi}{2}, \pi$ and $\frac{3\pi}{2}$ radians about the point $(0, y(e_1) + y(e_2) + \ldots + y(e_{\frac{1}{4}n_p(t)}))$.

It is proved in [7] that a polygon constructed this way is a unique convex lattice polygon with $n_p(t)$ vertices whose l_p–perimeter is minimal.

Thus, we have a sequence of integers representing numbers of vertices of optimal convex lattice polygons (in sense of l_p metric) explicitly constructed.

The following theorem gives the asymptotic expression for $n_p(t)$.

Theorem 1. [7] *The function $n_p(t)$ can be estimated by*

$$n_p(t) = \frac{6A_p}{\pi^2} t^{2/p} + \mathcal{O}(t^{1/p}),$$

where A_p equals the area of the planar shape $|x|^p + |y|^p \leq 1$.

Similar method is used to construct $Q_p(2k)$, where $2k$ is an even integer.

For every even integer $2k$, there exists an integer t such that $n(t-1) \leq 2k < n(t)$. The polygon $Q_p(2k)$ is constructed by adding edges to $Q_p(n(t-1))$. More precisely, $(2k - n(t-1))/2$ edges having the length $\sqrt[p]{t}$ are added to the south–east arc of $Q_p(n(t-1))$, and $(2k - n(t-1))/2$ edges with the same slopes are added to the north–west arc of $Q_p(n(t-1)$, i.e. for each edge e added to the south–east arc, there is an edge e' added to the north–west arc such that $y(e')/x(e')=y(e)/x(e)$ ($x(e) \perp y(e)$ and $x(e') \perp y(e')$ are satisfied). Now it is easy to check that the $2k$–gon obtained by this construction is optimal in sense of l_p metric.

3. Euclidian perimeter of $Q_p(2k)$

Let $P(v)$ be the number of lattice points (a, b, c) different from the origin satisfying $a \perp b$, which belong to the 3–dimensional body

$$D(v) = \left\{ (x, y, z) \,\middle|\, |x|^p + |y|^p \leq v,\, 0 < z \leq \sqrt{x^2 + y^2} \cdot \left(\frac{t}{v}\right)^{1/p} \right\},$$

and let $T(v)$ be the number of all lattice points (a, b, c) different from the origin, which belong to $D(v)$ ($a \perp b$ not required), where v is any positive number.

Now we shall consider the case $v = t$. The condition $|x|^p + |y|^p \le t$ implies that $|x| \le t^{1/p}$ and $|y| \le t^{1/p}$. Therefore, we have that the Euclidian perimeter of optimal convex lattice polygon $Q_p(n_p(t))$ equals $P(t) + \mathcal{O}(t^{2/p})$.

The following lemma gives the asymptotic expression for $T(v)$, for an arbitrary integer p.

Lemma 1. *The following asymptotic equality holds*

$$T(v) = C_p\, t^{1/p} v^{2/p} + \mathcal{O}((vt)^{1/p}),$$

where

$$C_p = \iint_{|x|^p+|y|^p\le 1} \sqrt{x^2+y^2}\, dx\, dy.$$

Proof. From the definition of $T(v)$, we have

$$T(v) = \text{volume}(D(v)) + \mathcal{O}(\text{area}(D(v))).$$

Thus,

$$\begin{aligned} T(v) &= \iint_{|x|^p+|y|^p\le v} \left(\frac{t}{v}\right)^{1/p} \sqrt{x^2+y^2}\, dx\, dy + \mathcal{O}((vt)^{1/p}) \\ &= \iint_{|x|^p+|y|^p\le 1} t^{1/p} v^{2/p} \sqrt{x^2+y^2}\, dx\, dy + \mathcal{O}((vt)^{1/p}) \\ &= t^{1/p} v^{2/p} \iint_{|x|^p+|y|^p\le 1} \sqrt{x^2+y^2}\, dx\, dy + \mathcal{O}((vt)^{1/p}) \\ &= C_p\, t^{1/p} v^{2/p} + \mathcal{O}((vt)^{1/p}). \end{aligned}$$

□

Now we can derive the asymptotic expression for the Euclidian perimeter of optimal lattice polygons $Q_p(n_p(t))$, for an arbitrary choice of p.

Theorem 2. *The Euclidian perimeter of optimal convex lattice polygons $Q_p(n_p(t))$ can be expressed as*

$$EP(n_p(t)) = \frac{6C_p}{\pi^2} t^{3/p} + \mathcal{O}(t^{2/p} \cdot \log t)\,.$$

Proof. Obviously, the following equalities hold

$$T(t) = P(t) + P\left(\frac{t}{2^p}\right) + P\left(\frac{t}{3^p}\right) + \ldots,$$

$$T\left(\frac{t}{a^p}\right) = P\left(\frac{t}{a^p}\right) + P\left(\frac{t}{(2a)^p}\right) + P\left(\frac{t}{(3a)^p}\right) + \ldots.$$

In the proof we shall use Lemma 1, as well as two following well–known equalities ([1]):

$$\sum_{a|l} \mu(a) = \begin{cases} 1, & l = 1 \\ 0, & l > 1 \end{cases} ;$$

$$\sum_{n=1}^{\infty} \frac{\mu(n)}{n^2} = \frac{1}{\zeta(2)} = \frac{\pi^2}{6} \ \Big(\zeta \text{ denotes the Riemann zeta function}\Big).$$

Also, we shall use the following inequality

$$\sum_{n=[\sqrt[p]{t}]+1}^{\infty} \frac{\mu(n)}{n^2} \le \sum_{n=[\sqrt[p]{t}]+1}^{\infty} \frac{1}{n^2} = \mathcal{O}\Big(\frac{1}{\sqrt[p]{t}}\Big).$$

Thus, we have

$$\begin{aligned}
P(t) &= \sum_{l=1}^{\infty} P\Big(\frac{t}{l^p}\Big)\Big(\sum_{a|l}\mu(a)\Big) \\
&= \sum_{n=1}^{[\sqrt[p]{t}]} \mu(n)\Big(\sum_{m=1}^{\infty} P\Big(\frac{t}{n^p m^p}\Big)\Big) \\
&= \sum_{n=1}^{[\sqrt[p]{t}]} \mu(n) T\Big(\frac{t}{n^p}\Big) \\
&= \sum_{n=1}^{[\sqrt[p]{t}]} \mu(n)\Big(t^{1/p}\Big(\frac{t}{n^p}\Big)^{2/p} C_p + \mathcal{O}\Big(\Big(\frac{t^2}{n^p}\Big)^{1/p}\Big)\Big) \\
&= \sum_{n=1}^{[\sqrt[p]{t}]} \mu(n)\frac{t^{3/p}}{n^2} C_p + \mathcal{O}\Big(\sum_{n=1}^{[\sqrt[p]{t}]} \Big|\mu(n)\Big| \frac{t^{2/p}}{n}\Big) \\
&= C_p t^{3/p}\Big(\sum_{n=1}^{\infty}\frac{\mu(n)}{n^2} - \sum_{n=[\sqrt[p]{t}]+1}^{\infty}\frac{\mu(n)}{n^2}\Big) + \mathcal{O}(t^{2/p}\cdot \log t) \\
&= C_p t^{3/p}\frac{6}{\pi^2} + \mathcal{O}(t^{2/p}\cdot \log t).
\end{aligned}$$

Since the Euclidian perimeter of the optimal convex lattice polygon $Q_p(n_p(t))$ is equal to $P(t) + \mathcal{O}(t^{2/p})$, the theorem is proved. □

Note. In case $p = \infty$ we need a slight modification of this proof. If we take

$$D(v) = \Big\{(x, y, z) \Big|\ |x| \le v,\ |y| \le v,\ 0 < z \le \sqrt{x^2 + y^2} \cdot \frac{t}{v}\Big\},$$

the expression given in the following theorem is obtained.

Theorem 3. *The Euclidian perimeter of the optimal convex lattice polygons* $Q_\infty(n_\infty(t))$ *can be expressed as*

$$EP(n_\infty(t)) = \frac{6C_\infty}{\pi^2}t^3 + \mathcal{O}(t^2 \cdot \log t)\,,$$

where

$$C_\infty = \iint_{|x|\le 1,\, |y|\le 1} \sqrt{x^2+y^2}\, dx\, dy.$$

It is easy to check that

$$C_\infty = \lim_{p\to\infty} C_p = \lim_{p\to\infty} \iint_{|x|^p+|y|^p\le 1} \sqrt{x^2+y^2}\, dx\, dy.$$

Now we can give the asymptotic expression for the Euclidian perimeter of all optimal $2k$–gons.

Theorem 4. *If* $2k$ *is an even integer, the Euclidian perimeter of the optimal convèx lattice polygon* $Q_p(2k)$ *expressed as a function of the number of its vertices* $2k$*, is*

$$EP(2k) = \frac{C_p\pi}{\sqrt{6A_p^3}}(2k)^{3/2} + \mathcal{O}((2k)\cdot\log(2k)).$$

Proof. For every even integer $2k$, there is an integer t such that

$$n_p(t-1) \le 2k < n_p(t).$$

From the last inequality and Theorem 1 we have that the asymptotic expression for $2k$ is

$$2k = \frac{6A_p}{\pi^2}t^{2/p} + \mathcal{O}\left(t^{1/p}\right).$$

Since $t = \mathcal{O}((2k)^{p/2})$, we can obtain the asymptotic expression for t as a function of $2k$:

$$t = \frac{\pi^p}{\sqrt{(6A_p)^p}}(2k)^{p/2} + \mathcal{O}\left((2k)^{(p-1)/2}\right).$$

The next inequality follows from the construction of $Q_p(2k)$.

$$\mathrm{EP}(n_p(t-1)) \le \mathrm{EP}(2k) \le \mathrm{EP}(n_p(t))$$

From Theorem 2 we have that

$$\mathrm{EP}(2k) = \frac{6C_p}{\pi^2}t^{3/p} + \mathcal{O}(t^{2/p}\cdot\log t).$$

Thus, by eliminating the variable t in the expression representing $\mathrm{EP}(2k)$ we have

$$\begin{aligned}
\mathrm{EP}(2k) &= \frac{6C_p}{\pi^2}\Big(\frac{\pi^p}{\sqrt{(6A_p)^p}}(2k)^{p/2} \\
&\quad + \mathcal{O}\Big((2k)^{(p-1)/2}\Big)\Big)^{3/p} + \mathcal{O}((2k)\cdot\log(2k)) \\
&= \frac{6C_p\pi}{(6A_p)^{3/2}}(2k)^{3/2}\Big(1+\mathcal{O}((2k)^{-1/2})\Big)^{3/p} \\
&\quad + \mathcal{O}((2k)\cdot\log(2k)) \\
&= \frac{C_p\pi}{\sqrt{6A_p^3}}(2k)^{3/2} + \mathcal{O}((2k)\cdot\log(2k)). \qquad \square
\end{aligned}$$

The next theorem is an application of the main result of this paper, giving the asymptotic upper bound for the area of the optimal convex lattice polygons $Q_p(2k)$.

Theorem 5. *The following asymptotic inequality holds*

$$P(Q_p(2k)) \leq \frac{C_p^2\pi}{24A_p^3}(2k)^3 + \mathcal{O}((2k)^{5/2}\cdot\log(2k)),$$

where $2k$ is an even integer and $P(Q)$ denotes the area of planar shape Q.

Proof. It is known that of all planar shapes with fixed perimeter circle has the greatest area. Since we have the asymptotic expression for the Euclidian perimeter of the optimal convex lattice polygons $Q_p(2k)$ as a function of $2k$ (Theorem 4), the statement of the theorem follows directly. □

In case $p = 2$, we have that

$$P(Q_2(2k)) \leq \frac{1}{54}(2k)^3 + \mathcal{O}((2k)^{5/2}\cdot\log(2k)).$$

This means that the upper bound for the minimal area of a convex lattice $2k$–gon is $1/54\ (2k)^3$, which improves the result from [3] (leading coefficient $15/784$), and confirms the result from [8].

Finally, here are some numerical values for the leading coefficient from Theorem 4:

p	A_p	C_p	$\frac{C_p \pi}{\sqrt{6A_p^3}}$
1	2.0	1.08215	0.490700
2	3.14159	2.09439	0.482401
3	3.53328	2.50422	0.483593
4	3.70815	2.70090	0.485117
5	3.80060	2.80938	0.486303
10	3.94293	2.98507	0.488991
∞	4.0	3.06078	0.490700

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