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A REPRESENTATION OF THE MINIMAL P-NORM SOLUTION

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Abstract. Using the determinantal representation and conditions for the existence of the Drazin inverse from the paper of Stanimirović and Djordjević (Linear Algebra Appl. **311** (2000), 131-151), we introduce a determinantal representation of the minimal P-norm solution of a given linear system. More precisely, we represent elements of the minimal P-norm solution $A^D b$ as fractions of two expressions involving minors of the order $\operatorname{rank}(A^k)$, $k = \operatorname{ind}(A)$, taken from the matrix A and its rank invariant powers A^l, $l \geq k$.

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1. Introduction

The set of all $m \times n$ matrices of rank r whose elements are taken from an integral domain $\mathbb{I}$ is denoted by $\mathbb{I}_r^{m\times n}$. By $\operatorname{Tr}(A)$, $\operatorname{ind}(A)$ and $|A|$ we denote the trace, index and the determinant of a square matrix A, respectively.

For any matrix $A \in \mathbb{C}^{m\times n}$ consider the following equations in G:

$$(1)\quad AGA=A,\quad (2)\quad GAG=G,\quad (3)\quad (AG)^*=AG,\quad (4)\quad (GA)^*=GA$$

where the superscript $*$ denotes the conjugate and transpose matrix. Also, in the case $m = n$, consider the following equations:

$$(5)\quad AG = GA \qquad (1^k)\quad A^{k+1}G = A^k$$

for a positive integer $k = \operatorname{ind}(A) = \min\{p : \operatorname{rank}(A^{p+1}) = \operatorname{rank}(A^p)\}$.

For a sequence $\mathcal{S}$ of the elements from the set $\{1, 2, 3, 4, 5\}$, the set of matrices obeying the equations represented in $\mathcal{S}$ is denoted by $A\{\mathcal{S}\}$. A matrix from $A\{\mathcal{S}\}$ is called an $\mathcal{S}$-inverse of A and denoted by $A^{(\mathcal{S})}$. If G satisfies the system of equations (1)-(2), it is said to be a reflexive g-inverse of A, whereas the Moore-Penrose inverse $G = A^\dagger$ of A satisfies the set of the equations (1)–(4). A matrix $G = A^D$ is said to be the Drazin inverse of A if (1^k) (2) and (5) are satisfied.

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The group inverse $A^{\#}$ is the unique $\{1,2,5\}$ inverse of A, and it exists if and only if $\operatorname{ind}(A)=1$.

We use the following notation from [3]. Let A be an $m \times n$ matrix of rank r; let $\alpha = \{\alpha_1, \ldots, \alpha_p\}$ and $\beta = \{\beta_1, \ldots, \beta_p\}$ be subsets of $\{1, \ldots, m\}$ and $\{1, \ldots, n\}$, respectively, of the order $1 \le p \le \min\{m,n\}$. Then $\left|A^\alpha_\beta\right|$ denotes the minor of A determined by the rows indexed by α and the columns indexed by β. Also, A^α_β denotes the submatrix of A determined by the rows indexed by α and the columns indexed by β.

For $1 \le k \le n$, denote the collection of strictly increasing sequences of k integers chosen from $\{1, \ldots, n\}$, by

$$\mathcal{Q}_{k,n} = \{\alpha : \ \ \alpha = (\alpha_1, \ldots, \alpha_k), \ 1 \le \alpha_1 < \cdots < \alpha_k \le n\}.$$

Let $\mathcal{N} = \mathcal{N}_r = \mathcal{Q}_{r,m} \times \mathcal{Q}_{r,n}$. For fixed $\alpha \in \mathcal{Q}_{k,m}$, $\beta \in \mathcal{Q}_{k,n}$, $1 \le k \le r$, let

$$\begin{aligned} \mathcal{I}(\alpha) \quad &= \mathcal{I}_r(\alpha) = \{I : \ I \in \mathcal{Q}_{r,m}, \ I \supseteq \alpha\}, \\ \mathcal{J}(\beta) \quad &= \mathcal{J}_r(\beta) = \{J : \ J \in \mathcal{Q}_{r,n}, \ J \supseteq \beta\}, \\ \mathcal{N}(\alpha,\beta) \quad &= \mathcal{N}_r(\alpha,\beta) = \mathcal{I}(\alpha) \times \mathcal{J}(\beta). \end{aligned}$$

If A is a square matrix, then the coefficient of $\left|A^\alpha_\beta\right|$ in the Laplace expansion of $|A|$ is denoted by $\frac{\partial}{\partial\left|A^\alpha_\beta\right|}|A|$. For the special case $\alpha = \{i\}$, $\beta = \{j\}$, we get the cofactor of a_{ij}: $\frac{\partial}{\partial a_{ij}}\,|A|$.

We use $C_p(A)$ to denote the pth compound matrix of A with rows indexed by p-element subsets of $\{1, \ldots, m\}$, columns indexed by p-element subsets of $\{1, \ldots, n\}$, and the (α, β) entry defined by $\left|A^\alpha_\beta\right|$, for $\alpha \in \mathcal{Q}_{p,m}$, $\beta \in \mathcal{Q}_{p,n}$.

By $A\,(i \to z)$, $i \in \{1, \ldots, n\}$ is denoted the matrix obtained from A replacing its column i by the vector z.

Also, we use the following extension of the presented notation from [6]:

$$\mathcal{N}_k = \mathcal{Q}_{k,m} \times \mathcal{Q}_{k,n}, \ \text{where } k \le \operatorname{rank}(A);$$

for fixed α, $\beta \in \mathcal{Q}_{k,n}$, $1 \le k \le p \le \operatorname{rank}(A)$, let

$$\begin{aligned} \mathcal{I}_p(\alpha) \quad &= \{I : \ I \in \mathcal{Q}_{p,m}, \ I \supseteq \alpha\}, \quad \mathcal{J}_p(\beta) = \{J : \ J \in \mathcal{Q}_{p,n}, \ J \supseteq \beta\}, \\ \mathcal{N}_p(\alpha,\beta) \quad &= \mathcal{I}_p(\alpha) \times \mathcal{J}_p(\beta). \end{aligned}$$

Finally, for an arbitrary n-dimensional vector b with the components $b_1, \ldots, b_n$ we denote by ${}_\alpha b$ the vector whose components are $b_{\alpha_1}, \ldots b_{\alpha_p}$.

It is known that the vector $x = A^D b$ is the unique solution of the following problem: for a given A and a given vector $b \in \mathcal{R}(A^k)$, find a vector $x \in \mathcal{R}(A^k)$ satisfying $Ax = b$, $\operatorname{ind}(A) = k$ (see for example [7], [8]).

The motivation for this paper is as follows. In [2] Berg derived an explicit determinantal representation of the best-approximate solution of the system of linear equations $Ax = b$ over the field of complex numbers. Using this representation, Berg also proved that the best-approximate solution is a convex combination of all uniquely solvable $r \times r$ partial subsystems, where $r = \text{rank}(A)$. An equivalent determinantal representation for the least-squares solution of an overdetermined linear system is derived in [1]. Using this formula, Ben-Tal proved that the least-squares solution lies in the convex hull of the solutions of the square subsystems of the original system. Also, a determinantal representation of the weighted best-approximate solution of a linear system over an integral domain is derived in [4]. Recently, in [5] it is proved that an arbitrary solution $A^{(1,2)}b$ of a linear system $Ax = b$ can be represented as a linear combination of the solutions $x^{(\alpha,\beta)}$ of all uniquely solvable $r \times r$ subsystems $A^{\alpha}_{\beta}x = {}_{\alpha}b$ of the starting system, where $r = \text{rank}(A)$. We now answer the following question: Are analogous representation and characterization valid for the minimal P-norm solution?

We introduce a determinantal formula for the minimal P-norm solution of a given singular linear system $Ax = b$. This representation is obtained by representing elements of the vector $A^D b$ in terms of minors of the order $r_k = \text{rank}(A^k)$, $k = \text{ind}(A)$, selected from matrices A and A^l, $l \geq k$. Such an approach, as far as we know, has not been employed before. The determinantal representation of various solutions of the linear system, using only $r \times r$ minors with $r = \text{rank}(A)$, are investigated in [1], [2], [4] and [5].

2. A representation of the minimal P-norm solution

Theorem 2.1. *Consider an arbitrary matrix* $A \in \mathbb{I}^{n\times n}$ *satisfying* $\text{ind}(A) = k$ *and a linear system* $Ax = b$, $x \in \mathcal{R}(A^k)$. *Assume that* $l \geq k$ *is an arbitrary integer. If full-rank factorizations* $A = PQ$ *and* $A^l = P_{A^l}Q_{A^l}$ *are allowed, then the following conditions are equivalent:*

(i) $A^D b \neq 0$.

(ii) $Q_{A^l}AP_{A^l}$ *is invertible matrix in* $\mathbb{I}$ *and* $Q_{A^l}AP_{A^l}b \neq 0$.

(iii) $Q_{A^l}P_{A^l}$ *is invertible matrix in* $\mathbb{I}$ *and* $Q_{A^l}P_{A^l}b \neq 0$.

(iv) $u = \sum_{(\gamma,\delta)\in\mathcal{N}_{r_k}} \left|(A^l)^{\delta}_{\gamma}\right| |A^{\gamma}_{\delta}|\, b = \text{Tr}(C_{r_k}(A^{l+1}))b \neq 0$.

Moreover, in each of these cases, the non-zero minimal P*-norm solution of this system possesses the following determinantal representation:*

$$
\begin{aligned}
(A^D b)_i &= \left(\sum_{(\gamma,\delta)\in\mathcal{N}_{r_k}} \left|(A^l)^{\delta}_{\gamma}\right| |A^{\gamma}_{\delta}| \right)^{-1} \sum_{(\alpha,\beta)\in\mathcal{N}_{r_k},\, i\in\beta} \left|(A^l)^{\beta}_{\alpha}\right| \left|A^{\alpha}_{\beta}(i \to {}_{\alpha}b)\right| \\
&= \left(\text{Tr}(C_{r_k}(A^{l+1}))\right)^{-1} \sum_{(\alpha,\beta)\in\mathcal{N}_{r_k},\, i\in\beta} \left|(A^l)^{\beta}_{\alpha}\right| \left|A^{\alpha}_{\beta}(i \to {}_{\alpha}b)\right|,
\end{aligned} \tag{2.1}
$$

for each $i=1,\ldots,n$*, where and* $r_k = \operatorname{rank}(A^l)$.

Also, each component $x_i^D = (A^D b)_i$ *of the non-zero minimal P-norm solution can be represented as the convex linear combination of the solutions* $x_i^{(\alpha,\beta)}$ *of all uniquely solvable* $r_k \times r_k$ *subsystems* $A_\beta^\alpha x = {}_\alpha b$ *of the starting system* $Ax = b$:

$$x_i^D = (A^D b)_i = \sum_{(\alpha,\beta)\in\mathcal{N}_{r_k}, i\in\beta} p_\alpha q_\beta\, x_i^{(\alpha,\beta)}, \quad p_\alpha, q_\beta \in \mathbb{I}, \tag{2.2}$$

where the coefficients p_α *and* q_β *satisfy*

$$\sum_{\alpha\in\mathcal{Q}_{r_k,n}} p_\alpha = 1, \qquad \sum_{\beta\in\mathcal{Q}_{r_k,n}} q_\beta = 1. \tag{2.3}$$

Proof. The equivalence of the conditions (i)–(iv) follows from the known conditions for the existence of the Drazin inverse from [6] and the following: the non-zero minimal P-norm solution of the system $Ax = b$ exists if and only if there exists the nonzero Drazin inverse A^D of A and $A^D b \neq 0$.

Using the determinantal representation of the Drazin inverse from [6], one can verify the following:

$$\begin{aligned}
(A^D b)_i &= \sum_{k=1}^{n} (A^D)_{ij} b_j \\
&= \left(\sum_{(\gamma,\delta)\in\mathcal{N}_{r_k}} \left|(A^l)_\gamma^\delta\right| \left|A_\delta^\gamma\right| \right)^{-1} \sum_{(\alpha,\beta)\in\mathcal{N}_{r_k}(j,i)} \left|(A^l)_\alpha^\beta\right| \frac{\partial}{\partial a_{ji}} \left|A_\beta^\alpha\right| \cdot b_j \\
&= \left(\sum_{(\gamma,\delta)\in\mathcal{N}_{r_k}} \left|(A^l)_\gamma^\delta\right| \left|A_\delta^\gamma\right| \right)^{-1} \sum_{(\alpha,\beta)\in\mathcal{N}_{r_k}, i\in\beta} \left|(A^l)_\alpha^\beta\right| \left|A_\beta^\alpha(i \to {}_\alpha b)\right|
\end{aligned}$$

Consequently, we verify the representation (2.1).

If $\left|A_\beta^\alpha\right|$ is an invertible element in $\mathbb{I}$, then the canonical embedding of the solution of the system $A_\beta^\alpha x = {}_\alpha b$ into the r_k-dimensional space, denoted by $x_i^{(\alpha,\beta)}$, is equal to

$$x_i^{(\alpha,\beta)} = \left(\left|A_\beta^\alpha\right|\right)^{-1} \left|A_\beta^\alpha(i \to {}_\alpha b)\right|.$$

In the singular case we define $x^{(\alpha,\beta)}$ to be the zero vector.

Using the full-rank factorizations $A^l = P_{A^l} Q_{A^l}$ and $A = PQ$ we obtain

$$\sum_{(\gamma,\delta)\in\mathcal{N}_{r_k}} \left|(A^l)_\gamma^\delta\right| \left|A_\delta^\gamma\right| = \sum_{(\gamma,\delta)\in\mathcal{N}_{r_k}} \left|(P_{A^l})^\delta\right| \left|(Q_{A^l})_\gamma\right| \left|P^\gamma\right| \left|Q_\delta\right|. \tag{2.4}$$

On the other hand,

$$\begin{aligned}
&\left|(A^l)^{\beta}_{\alpha}\right| \ \left|A^{\alpha}_{\beta}(i \to_{\alpha} b)\right| \\
(2.5)\qquad &= \left|(P_{A^l})^{\beta}\right| \left|(Q_{A^l})_{\alpha}\right| \left|P^{\alpha}\right| \left|Q_{\beta}\right| \left(\left|A^{\alpha}_{\beta}\right|\right)^{-1} \left|A^{\alpha}_{\beta}(i \to_{\alpha} b)\right| \\
&= \left|(P_{A^l})^{\beta}\right| \left|(Q_{A^l})_{\alpha}\right| \left|P^{\alpha}\right| \left|Q_{\beta}\right| \cdot x_i^{(\alpha,\beta)}.
\end{aligned}$$

Hence, from (2.1), (2.4) and (2.5), we get (2.2) for the case

$$p_{\alpha} = \left(\sum_{\gamma \in Q_{r_k,n}} \left|(Q_{A^l})_{\gamma}\right| \left|P^{\gamma}\right| \right)^{-1} \left|(Q_{A^l})_{\alpha}\right| \left|P^{\alpha}\right|, .$$

$$q_{\beta} = \left(\sum_{\delta \in Q_{r_k,n}} \left|(P_{A^l})^{\delta}\right| \left|Q_{\delta}\right| \right)^{-1} \left|(P_{A^l})^{\beta}\right| \left|Q_{\beta}\right|.$$

Equation (2.3) can simply be verified. □

Corollary 2.1. *The determinantal representation of an arbitrary element of the minimal P-norm solution of a given singular linear system $Ax = b$, $A \in \mathbb{C}_r^{n\times n}$, possesses the form*

$$(2.6)\qquad (A^D b)_i = \sum_{(\alpha,\beta)\in \mathcal{N}_{r_k}(j,i)} \lambda_{\alpha,\beta} \left|A^{\alpha}_{\beta}(i \to_{\alpha} b)\right|, \quad i = 1,\dots,n,$$

where the matrix $\Lambda = (\lambda_{\alpha,\beta})$ satisfies the following conditions

$$(2.7)\qquad \operatorname{rank}(\Lambda) = 1, \qquad \sum_{(\alpha,\beta)\in \mathcal{N}_{r_k}} \lambda_{\alpha,\beta} \left|A^{\alpha}_{\beta}\right| = 1.$$

Proof. Let $l \geq k = \operatorname{ind}(A)$ is an arbitrary integer and let $A^l = P_{A^l} Q_{A^l}$ be a rank factorization of A^l. Consider the matrix Λ defined by

$$\begin{aligned}
\Lambda &= \left(\left|Q_{A^l} A P_{A^l}\right|\right)^{-1} C_{r_k}\left((A^l)^T\right) \\
&= \left(\sum_{(\gamma,\delta)\in \mathcal{N}_{r_k}} \left|(A^l)^{\delta}_{\gamma}\right| \left|A^{\gamma}_{\delta}\right| \right)^{-1} C_{r_k}\left((A^l)^T\right) \\
&= \left(\operatorname{Tr}(C_{r_k}(A^{l+1}))\right)^{-1} C_{r_k}\left((A^l)^T\right).
\end{aligned}$$

In this case $\lambda_{\alpha,\beta} = \left(\operatorname{Tr}(C_{r_k}(A^{l+1}))\right)^{-1} \left|A^{\alpha}_{\beta}\right|$, and the representation (2.6) follows immediately from Theorem 2.1. The conditions (2.7) can be easily verified. □

Corollary 2.2. *Let A be an $n \times n$ matrix of rank r over $\mathbb{I}$, and let $A = PQ$ be a full-rank factorization of A. If* $\mathrm{ind}(A) = 1$, *then the minimal P-norm solution $A^{\#}b$ satisfies*

$$\begin{aligned}(A^{\#}b)_i &= u^{-2} \sum_{(\alpha,\beta)\in\mathcal{N}_r} \left|A_\alpha^\beta\right| \left|A_\beta^\alpha(i \to_\alpha b)\right| \\ &= \sum_{(\alpha,\beta)\in\mathcal{N}_r} p_\alpha q_\beta x_i^{(\alpha,\beta)},\end{aligned}$$

where $u = \mathrm{Tr}(C_r(A))$, *the coefficients $p_\alpha, q_\beta \in \mathbb{I}$ satisfy*

$$\sum_{\alpha\in Q_{r,n}} p_\alpha = 1, \qquad \sum_{\beta\in Q_{r,n}} q_\beta = 1$$

and $x_i^{(\alpha,\beta)}$ is defined by

$$x_i^{(\alpha,\beta)} = \begin{cases} \left(\left|A_\alpha^\beta\right|\right)^{-1} \left|A_\beta^\alpha(i \to_\alpha b)\right|, & \left|A_\alpha^\beta\right| \text{ is invertible,} \\ 0, & \text{otherwise.} \end{cases}$$

Remark 2.1. The determinantal representation of the vector $A^{\#}b$ can also be derived as a partial case from the determinantal representation of the vector $A^{(1,2)}b$, introduced in [5].

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