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THE NUMBER OF 1–FACTORS IN SOME POLYHEXES

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Abstract. We define a new class of benzoids (hexagonal systems), in continuation of the previous works [1,3]. Some combinatorial K formulas are derived, where K designates the Kekulé structure counts, which generalizes some previously known K formulas. We point out the connections with some combinatorial problems.

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1. General formula

All definitions used in this paper are taken from [3].

Let H be any hexagonal system. If we replace each vertical edge of H (together with both incedent vertices) with a vertex, we obtain a square system S_H. We say that S_H is associated square system for H (Fig. 1).

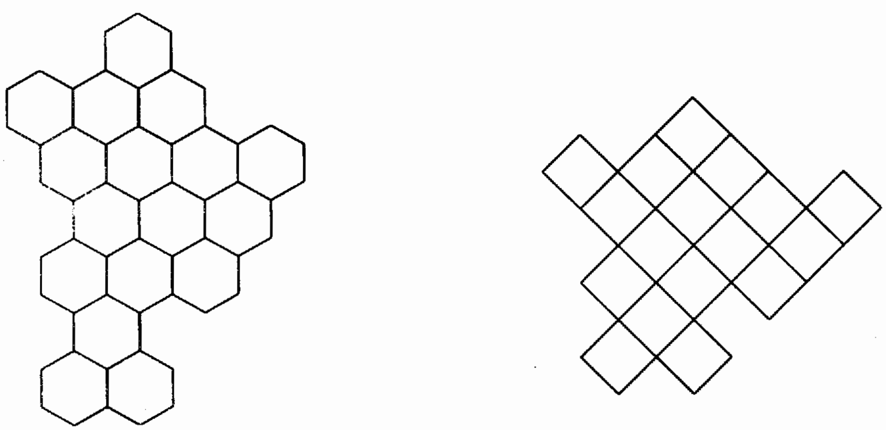

Fig. 1

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We can choose any class of parallel edges to be vertical. So, in fact, to each hexagonal system we can associate three square systems, which are non-isomorphic in general case.

This operation preserves peaks and vallies. Moreover, it is easy to see that the number of monotonic path systems is the same for a hexagonal system and its associated square system (Fig. 2).

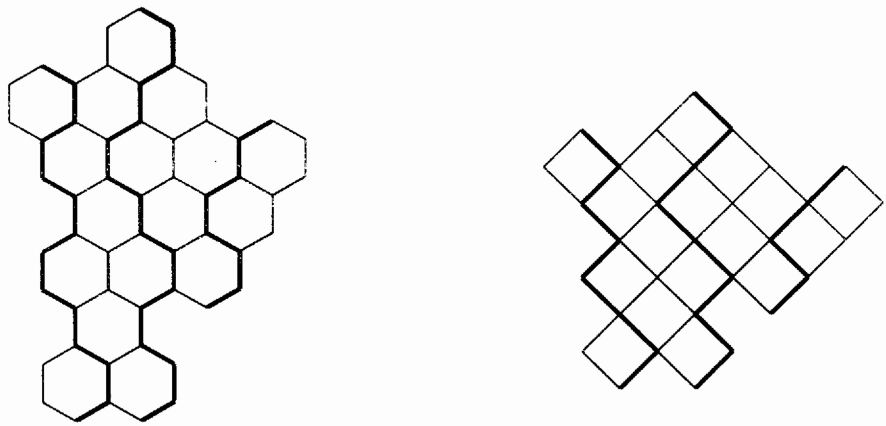

Fig. 2

So, we can determine the number K of a hexagonal system H by determining the number of monotonic path systems of its associated square system S_H.

The advantage of this approach is that we can use some known combinatorial results concerning square systems, i.e. concerning path systems in the square grid.

Denote by $S(n,s,\mu)$ the hexagonal system consisting of n layers with the corresponding number of hexagons s, $s+\mu$, $s+2\mu$,..., $s+(n-1)\mu$. In fact, such a system can be obtained from a parallelogram $L(n,\mu(n-1)+s)$ by removing $(n-i)\mu$ rightmost hexagons from the i-th layer, $i=1,2,\ldots,n-1$. An $S(5,5,3)$ is given in Fig. 3.

The associated square system of $S(5,5,3)$ is given in Fig. 4.

In [2] is given a result which can be formulated in the following way:

The number of monotonic paths from point A to the point B in the square system given in Fig. 4 is

$$\frac{s+1}{(\mu+1)n+s+1}\binom{(\mu+1)n+s+1}{n}.$$

So, we have the folowing theorem.

Theorem 1.

$$K\{S(n,s,\mu)\} = \frac{s+1}{(\mu+1)n+s+1}\binom{(\mu+1)n+s+1}{n}.$$

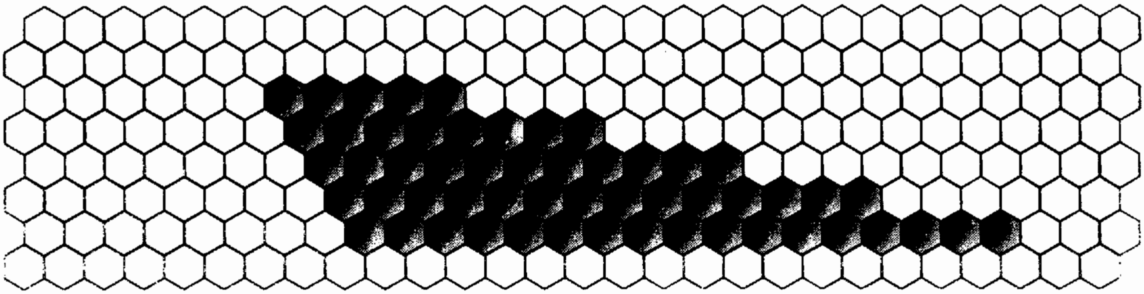

Fig. 3

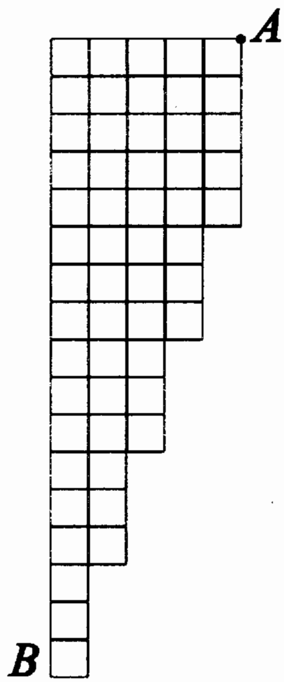

Fig. 4

2. Some special cases

If $s=\mu$, we denote $S(n,\mu,\mu)$ by $R(n,\mu)$. $R(n,\mu)$ may also be considered as $S(n+1,\emptyset,\mu)$. In Fig. 5 an $R(4,3)$ is given.

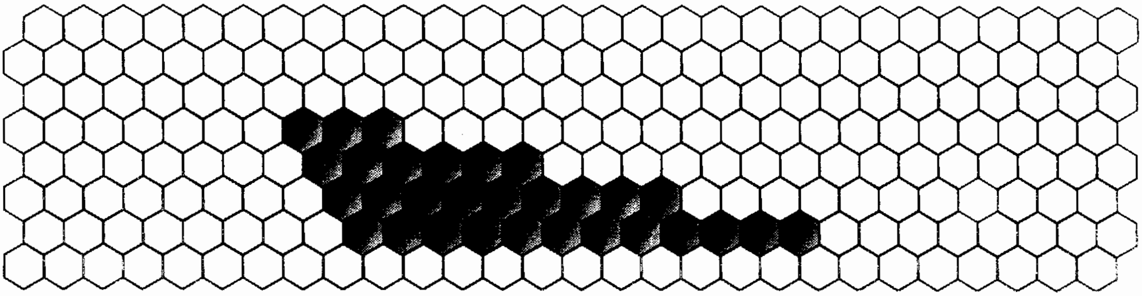

Fig. 5

So from Theorem 1 we obtain:

Corollary 1.

$$K\{R(n,\mu)\} = \frac{1}{(\mu+1)(n+1)+1}\binom{(\mu+1)(n+1)+1}{n+1}.$$

Another specialization of Theorem 1 is obtained by taking $\mu = 1$. The hexagonal system $S(n,s,1)$ is a trapeze $T(n, n+s-1, n-1)$, considered also in [3]. A $T(5,7,4)$ is given in Fig. 6.

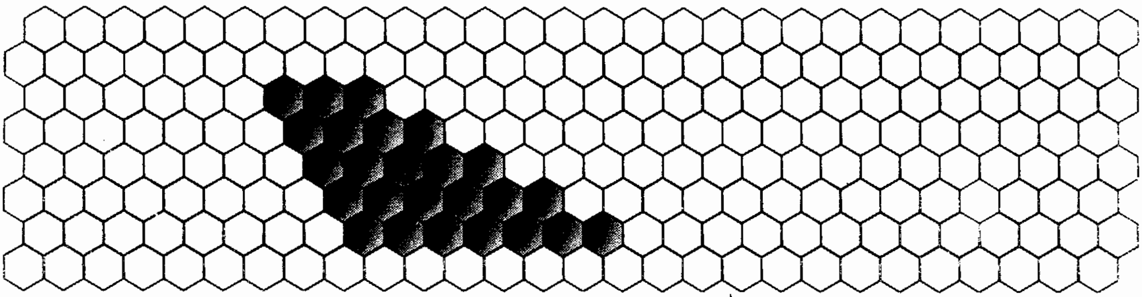

Fig. 6

From Theorem 1, it follows:

Corollary 2.

$$K\{T(n, n+s-1, n-1)\} = \frac{s+1}{2n+s+1}\binom{2n+s+1}{n}.$$

Puting $m = n + s - 1$, we have

$$K\{T(n, m, n-1)\} = \frac{m-n+2}{m+n+2}\binom{m+n+2}{n}.$$

In [3], it is proved that

$$K\{T(n, m, n-1)\} = \binom{m+n}{n} - \binom{m+n}{n-2}.$$

Hence follows the identity

$$\binom{m+n}{n} - \binom{m+n}{n-2} = \frac{m-n+2}{m+n+2}\binom{m+n+2}{n},$$

which can also be proved in a different way.

In order to specialize further Theorem 1, we take at the same time $s = \mu = 1$. We obtain $R(n, 1)$ which may be also considered as $S(n+1, 0, 1)$. In [1] and [3], this hexagonal system is denoted by $T^i(n)$. A $T^i(5)$ is given in Fig. 7.

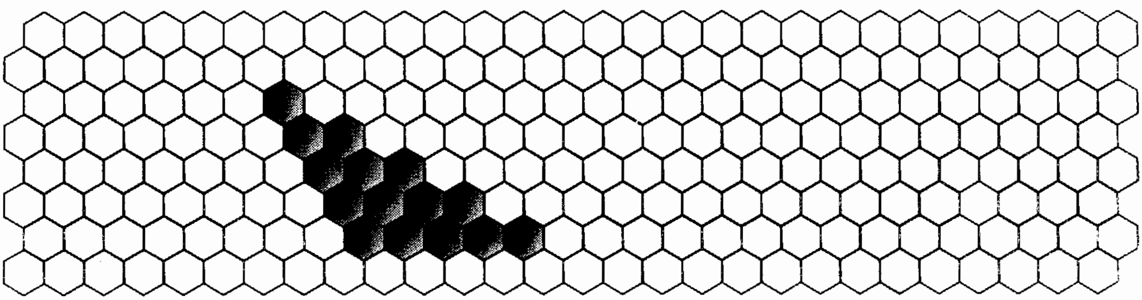

Fig. 7

From Corrolary 2, we obtain:

Corollary 3.

$$K\{T^i(n)\} = \frac{1}{2n+3}\binom{2n+3}{n+1},$$

i.e.

$$K\{T^i(n)\} = C_n + 1,$$

where C_i is the i–th Catalan number. The last formula is also derived in [3].

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