NOVI SAD J. MATH.
VOL. 30, NO. 2, 2000, 11-14

Proceedings of the TARA 2000 Conference
Novi Sad, Yugoslavia, September 6-7, 2000

SEGMENTS OF SCORE SEQUENCES

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Abstract. We give characterization of segments and subsequences of tournament score sequences. These characterizations generalize Landau criteria for score sequences and yield bounds for the tournament size.

AMS Mathematics Subject Classification (1991): 05C20

Key words and phrases: graph, tournament, score

A tournament T_n is a graph having vertices $1, 2, \ldots, n$, such that each pair of distinct vertices i and j is joined by one and only one of the oriented edges either ij or ji. The score or outdegree of vertex i is the number s_i of vertices that i dominates. The score vector s of T_n is the ordered n-tuple of scores $(s_1, s_2, \ldots, s_n)$, where the vertices are labeled in such a way to make a nondecreasing sequence $s_1 \leq s_2 \leq \cdots \leq s_n$. A *transitive* tournament has the score sequence

$$0 \leq 1 \leq \cdots \leq n-1.$$

A *regular* tournament R_n has the scores as nearly equal as possible

$$\underbrace{\lfloor e \rfloor = \cdots = \lfloor e \rfloor}_{\lfloor n/2 \rfloor} \leq \underbrace{\lceil e \rceil = \cdots = \lceil e \rceil}_{\lceil n/2 \rceil}, \quad e = (n-1)/2.$$

The Landau theorem [6, 8] gives characterizations for a score sequence: A sequence $s_1 \leq s_2 \leq \cdots \leq s_n$ of non-negative integers is the score sequence of some tournament T_n if and only if

$$(1) \qquad \sum_{i=1}^{k} s_i \geq \binom{k}{2}, \quad 1 \leq k \leq n, \quad \sum_{i=1}^{n} s_i = \binom{n}{2}.$$

The number $s(n)$ of different score sequences of size n can be computed by recursion.

Hrady, Littlewood and Pólya [3] introduced the term *majorization* relation. Let a and b belong to $\mathbf{R}^n$, then a is majorized by b

$$a \prec b \quad \text{if} \quad \sum_{i=1}^{k} a_{[i]} \leq \sum_{i=1}^{k} b_{[i]}, \quad 1 \leq k \leq n, \quad \sum_{i=1}^{n} a_{[i]} = \sum_{i=1}^{n} b_{[i]},$$

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where $c_{[1]} \geq c_{[2]} \geq \ldots \geq c_{[n]}$ denotes the nonincreasing permutation of $c_1, c_2, \ldots, c_n$. The same authors [2] proved that $x \prec y$ if and only if for each convex function ϕ it holds

$$\sum_{i=1}^{n} \phi(x_i) \leq \sum_{i=1}^{n} \phi(y_i).$$

Lorenz [5] firstly used the majorization relation in economy, while [7] concerns with many other applications. By this concept the Landau condition (1) can take a condensed form

$$(s_n, s_{n-1}, \ldots, s_1) \prec (n-1, n-2, \ldots, 0).$$

Theorem 1. *Let $t_1 \leq t_2 \leq \cdots \leq t_m$ be a sequence of nonnegative integers, and $s_1 \leq s_2 \leq \cdots \leq s_n$ be the score sequence of tournament T_n, where $m \leq n$. Then the following properties are equivalent:*

P_1 :	$\sum_{i=1}^{j} t_i \geq \binom{j}{2}$,	$1 \leq j \leq m$;	
P_2 :	$s_j = t_j$,	$1 \leq j \leq m$,	*for some* T_n;
P_3 :	$s_{k+j} = t_j$,	$1 \leq j \leq m$,	*for some* T_n *and* k;
P_4 :	$s_{k_j} = t_j$,	$1 \leq j \leq m$,	*for some* T_n *and* $k_1 < k_2 < \cdots < k_m$.

Proof. One digraph of the proof is the implications cycle: $P_1 \Rightarrow P_2 \Rightarrow P_3 \Rightarrow P_4 \Rightarrow P_1$.
$P_2 \Rightarrow P_3 \Rightarrow P_4$. This is obvious.
$P_i \Rightarrow P_1$, $i = 2, 3, 4$. This follows from the Landau theorem

$$\sum_{i=1}^{j} t_i \geq \sum_{i=1}^{j} s_i \geq \binom{j}{2}, \quad 1 \leq j \leq m.$$

$P_1 \Rightarrow P_2$. Define the supersequence $s_1 \leq s_2 \leq \cdots \leq s_n$ which contains t and which satisfies the Landau condition (1)

$$s_k = \begin{cases} t_k, & 1 \leq k \leq m \\ t_m, & m < k < n \end{cases}.$$

In order to determine n and s_n consider the inequality

$$t_1 + \cdots + t_m + (k-m)t_m \geq k(k-1)/2. \tag{2}$$

This quadratic inequation is strict for $m \leq k < k_0$, it becomes equality for

$$k_0 = \frac{2t_m + 1 + \sqrt{(2t_m+1)^2 - 8(mt_m - t_1 - \cdots - t_m)}}{2},$$

and it is strict converse inequality for $k > k_0$. Hence we can take

$$n = \nu = \lceil k_0 \rceil, \quad s_n = t_m + \binom{n}{2} - t_1 - \cdots - t_m - (n-m)t_m.$$

Obviously, $s_{n-1} \le s_n$ and $s_1 + \ldots + s_n = \binom{n}{2}$. By the Landau theorem there exists a desired tournament T_n with the score sequence s and, therefore, the score subsequence t. □

Theorem 2. *Let $t_1 \le t_2 \le \cdots \le t_m$ be non-negative integers which satisfy P_1. Then there exists a tournament T_n which satisfies P_2, and the minimal size of such a tournament is*

$$\nu = \left\lceil t_m + \frac{1}{2} + \sqrt{\left(m - t_m - \frac{1}{2}\right)^2 + 2\left[\sum_{i=1}^{m} t_i - \binom{m}{2}\right]} \right\rceil.$$

Proof. Suppose that T_q, where $q < n$, is a tournament with score sequence $u_1 \le u_2 \le \cdots \le u_q$ such that $u_i = t_i$, where $1 \le i \le m$. Let s be the score sequence defined in the proof of Theorem 1. Then $u_i \ge u_m = t_m = s_m$, where $m \le i \le q$, and

$$\binom{q}{2} = \sum_{i=1}^{q} u_i \ge \sum_{i=1}^{q} s_i > \binom{q}{2}.$$

This is a contradiction and therefore $q \ge n$. □

For $m = 1$one obtains $\nu = 2t_1 + 1$. If P_1 becomes equality for $j = m$, then $\nu = m$ and the initial score segment becomes the whole score sequence from the Landau theorem.

Letting t be the score subsequence of some tournament T_n, then t is also the score subsequence of some tournament T_q of size q for all $q \ge n$. An example of such a tournament is the extension of T_n with the vertex set $1, \ldots, n, \ldots, q$, and with arcs ij whenever $i > j$ and $i > n$. A tournament is *reducible* if it is possible to partition its vertices into two nonempty sets B and A so that all the vertices in B dominate all the vertices in A [8]. A characterization of reducibility of a tournament is the equality in (1) for some $k < n$. Consider some questions when score subsequences become segments.

Theorem 3. *Let $t_1 \le t_2 \le \ldots \le t_m$ satisfy P_4 and*

$$\sum_{j=1}^{m} t_j = \binom{m}{2}.$$

Then $s_j = t_j$ for all $1 \le j \le m$. Moreover, if $k_l > l$ for some l, then $s_l = \ldots = s_m = \ldots = s_{k_m}$.

Proof. From $t_j = s_{k_j} \ge s_j$, where $1 \le j \le m$, and the Landau theorem

$$\binom{m}{2} = \sum_{j=1}^{m} t_j = \sum_{j=1}^{m} s_{k_j} \ge \sum_{j=1}^{m} s_j \ge \binom{m}{2}$$

follows $\sum_{j=1}^{m} s_j = \binom{m}{2}$ and $s_j = t_j$ for all $1 \le j \le m$. Letting $k_j = j$, where $1 \le j < l$, and $k_l > l$, one obtains $t_l = s_l \le \cdots \le s_{k_l} = t_l$ and $s_l = \ldots = s_{k_l}$. Further we have $k_{l+1} > k_l \ge l+1$ and so $k_{l+1} > l+1$. Hence $s_{l+1} = \cdots = s_{k_{l+1}}$ and the proof concludes by induction. □

Theorem 4. *Let $t_1 \le t_2 \le \cdots \le t_m$ be the score subsequence of the tournament T_n. Then, t is the score sequence of some subtournament T_m of T_n if and only if $\sum_{j=1}^{m} t_j = \binom{m}{2}$.*

Proof. Necessity follows immediately from the Landau theorem.

Sufficiency. Let $s_1 \le s_2 \le \cdots \le s_n$ be the score sequence of T_n. If $m = n$ then $t = s$ is a score sequence itself. If $m < n$ then by Theorem 2.3 one obtains $t_j = s_j$, where $1 \le j \le m$. Hence, T_n is reducible with the partition $A = \{1, \ldots, m\}$ and $B = \{m+1, \ldots, n\}$ and subtournament T_m on the vertex set A is a desired subtournament. □

Theorem 5. *Let integers $t_1 \le t_2 \le \cdots \le t_m$ satisfy Landau condition and let $n \ge m$. Then there exists a tournament T_n with a subtournament T_m such that t is both the score subsequence of T_n and the score sequence of T_m.*

Proof. Follows immediately from the Landau theorem with the reducible T_n. □

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