NOVI SAD J. MATH.
VOL. 30, NO. 2, 2000, 155-158

THE CARDINALITY OF CLONES CONTAINING MINIMAL CLONES GENERATED BY SEMIPROJECTIONS ON A THREE ELEMENT SET

Jovanka Pantović[1], **Dušan Vojvodić**[2]

Abstract. All minimal clones on the three-element set are determined in [2]. In this paper we solved the problem of cardinality of the set of clones which contain some of these clones.

AMS Mathematics Subject Classification (1991): 0840

Key words and phrases: clone, cardinality

1. Notation and Preliminaries

Denote by $\mathbf{N}$ the set $\{1,2,\ldots\}$ of positive integers and for $k,n \in \mathbf{N}$, set $E_k = \{0,1,\ldots,k-1\}$. We say that f is an *i-th projection of arity n* $(1 \leq i \leq n)$ if $f \in P_k^{(n)}$ and f satisfies the identity $f(x_1,\ldots,x_n) \approx x_i$.

For $n,m \geq 1, f \in P_k^{(n)}$ and $g_1,\ldots,g_n \in P_k^{(m)}$, *the superposition of f* and $g_1,\ldots,g_n$, denoted by $f(g_1,\ldots,g_n)$, is defined by $f(g_1,\ldots,g_n)(a_1,\ldots,a_m) = f(g_1(a_1,\ldots,a_m),\ldots,g_n(a_1,\ldots,a_m))$ for all $(a_1,\ldots,a_m) \in E_k^m$.

A set C of operations on E_k is called a *clone* if it contains all projections and is closed under composition.

For an arbitrary set F of operations on E_k there exists the least clone containing F. This clone is called *the clone generated by F*, and will be denoted by $\langle F\rangle_{\mathrm{CL}}$. Instead of $\langle\{f\}\rangle_{\mathrm{CL}}$ we will write simply $\langle f\rangle_{\mathrm{CL}}$. For a clone C and $n \geq 1$ we denote by $C^{(n)}$ the set of $n-$ary operations from C.

The clones on E_k form an algebraic lattice $Lat(E_k)$ whose least element is the clone of all projections and whose greatest element is the clone of all operations on E_k. The atoms (dual atoms) of $Lat(E_k)$ are called *minimal(maximal) clones.*

A full description of all clones, hence of all minimal and maximal clones for $k = 2$ was given by Post; for $k = 3$ a complete list of all maximal clones was found by Yablonskii and all minimal clones were determined by Csákány.

Let h be a positive integer. A subset ρ of E_k (i.e. a set of $h-$tuples over E_k) is an h-ary *relation on E_k*. An n-ary operation f on E_k *preserves ρ* if for every $(h \times n)$ matrix $X = [x_{ij}]$ over E_k whose columns are all $h-$tuples from

[1]**Faculty of Engineering, University of Novi Sad, Trg Dositeja Obradovića 3, 21000 Novi Sad, Yugoslavia, e-mail: pantovic@uns.ns.ac.yu

[2]University of Novi Sad, Trg Dositeja Obradovića 4, 21000 Novi Sad, Yugoslavia e-mail: vojvodic@unsim.ns.ac.yu

ρ we have $(f(x_{00},\ldots,x_{0(n-1)}),\ldots,f(x_{(h-1)0},\ldots,x_{(h-1)(n-1)}))\in\rho$. The set of all operations on E_k preserving a given relation ρ is denoted Polρ.

Let $k=3$ and let ϕ be a permutation of E_3. To each $n-$ary function f we assign f^ϕ, called *conjugate* of f, defined by $f^\phi(x_0,\ldots,x_{n-1})=\phi(f(\phi^{-1}(x_0),\ldots,\phi^{-1}(n-1))$. The map $f\to f^\phi$ carries each clone C onto the clone C^ϕ; in particular $\langle f\rangle^\phi_{\mathrm{CL}}=\langle f^\phi\rangle_{\mathrm{CL}}$, and $g\in\langle f\rangle_{\mathrm{CL}}$ implies $g^\phi\in\langle f^\phi\rangle_{\mathrm{CL}}$. We can permute the variables of f as well: for a permutation ψ of E_n put $f_\psi(x_0,\ldots,x_{n-1})=f(x_{\psi(0)},\ldots,x_{\psi(n-1)}))$. Remark that always $(f^\phi)_\psi=(f_\psi)^\phi$. Note also that $\langle f_\psi\rangle_{\mathrm{CL}}=\langle f\rangle_{\mathrm{CL}}$ for any ψ. The conjugations and permutations of variables generate a permutation group T_n of order $3n!$ on the set of all n-ary functions on E_3.

It is proved in [2] that every minimal clone on E_3 containing an essential ternary operation is a conjugate of exactly one of the following eight clones: $\langle m_i\rangle_{\mathrm{CL}}$ with $i\in\{0,109,624\}$ and $\langle s_i\rangle_{\mathrm{CL}}$ with $i\in\{0,8,26,76,424\}$. The following tables display minimal clones on E_3 generated by semiprojections and majority functions, respectively.

	(01)	(02)	(12)	(012)	(021)
$\langle s_0\rangle_{\mathrm{CL}}$	$\langle s_{364}\rangle_{\mathrm{CL}}$	$\langle s_{728}\rangle_{\mathrm{CL}}$			
$\langle s_8\rangle_{\mathrm{CL}}$	$\langle s_{368}\rangle_{\mathrm{CL}}$	$\langle s_{80}\rangle_{\mathrm{CL}}$	$\langle s_{36}\rangle_{\mathrm{CL}}$	$\langle s_{40}\rangle_{\mathrm{CL}}$	$\langle s_{692}\rangle_{\mathrm{CL}}$
$\langle s_{26}\rangle_{\mathrm{CL}}$	$\langle s_{449}\rangle_{\mathrm{CL}}$		$\langle s_{37}\rangle_{\mathrm{CL}}$		
$\langle s_{76}\rangle_{\mathrm{CL}}$	$\langle s_{684}\rangle_{\mathrm{CL}}$	$\langle s_{332}\rangle_{\mathrm{CL}}$			
$\langle s_{424}\rangle_{\mathrm{CL}}$					

	(01)	(02)	(12)
$\langle m_0\rangle_{\mathrm{CL}}$	$\langle m_{324}\rangle_{\mathrm{CL}}$	$\langle m_{728}\rangle_{\mathrm{CL}}$	
$\langle m_{109}\rangle_{\mathrm{CL}}$		$\langle m_{473}\rangle_{\mathrm{CL}}$	$\langle m_{510}\rangle_{\mathrm{CL}}$
$\langle m_{624}\rangle_{\mathrm{CL}}$			

2 Results

Theorem 2.1 *The cardinality of the set of clones on E_3 containing a minimal clone $\langle s_j\rangle_{\mathrm{CL}}$ is a continuum for each $j\in\{0,8,36,40,80,364,368,692,728\}$.*

Proof. The proof is based on the operations of Yanov-Muchnik.

We shall define a countable set of operations F and an operation g so that for all $f\in F$, $f\notin\langle(F\setminus\{f\})\cup\{g\}\rangle_{\mathrm{CL}}$. This implies that for each $G,H\subseteq F$, from $G\neq H$ it follows $\langle G\cup\{g\}\rangle_{\mathrm{CL}}\neq\langle H\cup\{g\}\rangle_{\mathrm{CL}}$. In this way we get a set of distinct clones of a continuum cardinality.

For $i=1,\ldots,m$ denote by $\mathbf{e}_i$ the $m-$tuples $(1,\ldots,1,2,1,\ldots,1)$ with 2, at the $i-$th place. Let $A_m=\{\mathbf{e}_1,\ldots,\mathbf{e}_m\}$.

For $m>2$, consider the $m-$ary operation f_m (Yanov-Muchnik, [4]) which takes the value 1 on A_m and 0 otherwise: $f_m(\mathbf{x})=\begin{cases}1 & if\ x\in A_m\\ 0 & otherwise\end{cases}$.

Define the following relations $\rho_m \subseteq E_3^m$ on E_3 for $m > 2$: $\rho_m = \{0,1,2\}^m \setminus B_m$, where $B_m = \{1,2\}^m \setminus A_m$.

In what follows we prove that f_i and s_j preserve ρ_m for each $i \neq m$ and $j \in \{0,8\}$, while f_m does not.

Let $X = [x_{ij}]$ be the $m \times m$ matrix with $x_{11} = \ldots x_{mm} = 2$ and $x_{ij} = 1$ otherwise. The i-th column of X is $\mathbf{e}_i \in \rho_m (i = 1, \ldots, m)$ whereas the values of f_m on the rows of X form $(f_m(\mathbf{e}_1), \ldots, f_m(\mathbf{e}_m)) = (1, \ldots, 1) \notin \rho$. Hence, $f_m \notin Pol \rho_m$.

Suppose to the contrary that f_i does not preserve ρ_m for some $i \neq m$. Then there is an $m \times i$ matrix X with all columns in ρ_m and with rows $\mathbf{a}_1, \ldots, \mathbf{a}_m$ such that $\mathbf{b} := (f_i(\mathbf{a}_1), \ldots, f_i(\mathbf{a}_m)) \notin \rho$. Since $imf_i = \{0,1\}$ and clearly $\mathbf{b} = (1, \ldots, 1)$. By the definition of f_i there exist $1 \leq j_1, \ldots, j_m \leq i$ such that $\mathbf{a}_k = \mathbf{e}_{j_k}$ for all $k = 1, \ldots, m$. If $j_k = j_l$ for some $1 \leq k < l \leq m$ then the j_k-th column of X contains at least two 2s and so does not belong to ρ_m. As $i \neq m$ we can choose $k \in \{1, \ldots, i\} \setminus \{j_1, \ldots, j_m\}$. Clearly, the k-th column of X is $(1, \ldots, 1) \notin \rho_m$.

If $s_j, j \in \{0,8\}$ does not preserve ρ then there is an $m \times 3$ matrix X with all columns in ρ such that $(s_j(x_{11}, x_{12}, x_{13}), \ldots, s_j(x_{m1}, x_{m2}, x_{m3})) \notin \rho$. By the definition of $s_j, j \in \{0,8\}$ it follows that $(s_j(x_{11}, x_{12}, x_{13}), \ldots, s_j(x_{m1}, x_{m2}, x_{m3})) = \mathbf{x}_1$ since $s_j(x_{l1}, x_{l2}, x_{l3}) = 1$ implies $x_{l1} = 1$ and $s_j(x_{l1}, x_{l2}, x_{l3}) = 2$ implies $x_{l1} = 2$. So, we get a contradiction.

The set of clones of the form $\langle G \cup \{s_0, s_8\}\rangle_{\mathrm{CL}}, G \subseteq \{f_2, f_3, \ldots\}$ has a continuum cardinality. (Similarly for the conjugates.) □

Theorem 2.2 *The cardinality of the set of clones on E_3 containing a minimal clone $\langle s_j \rangle_{\mathrm{CL}}$ is at least $\aleph_0$ for each $j \in \{26, 37, 76, 332, 449, 684\}$.*

Proof. Let $\{0,1,2\} = \{p,q,r\}$, and for $i = 1, \ldots, m$ denote by $\mathbf{e}_i$ the $m-$tuples $(p, \ldots, p, r, p, \ldots, p)$ with r at the $i-$th place. Let $A_m = \{\mathbf{e}_1, \ldots, \mathbf{e}_m\}$.

For $m > 2$, consider the $m-$ary operation f_m (Yanov-Muchnik,[4]) which takes the value p on A_m and q otherwise: $f_m(\mathbf{x}) = \begin{cases} p & if \;\; x \in A_m \\ q & otherwise \end{cases}$.

Define the following relations $\rho_m \subseteq E_3^m$ on E_3 for $m > 2$: $\rho_m = E_3^m \setminus \{(p, \ldots, p)\}$.

In what follows we prove that f_i preserve ρ_m for each $i > m$ and f_i does not preserve ρ_m for each $i \leq m$.

Suppose to the contrary that f_i does not preserve ρ_m for some $i > m$. Then there is an $m \times i$ matrix X with all columns in ρ_m and with the rows $\mathbf{a}_1, \ldots, \mathbf{a}_m$ such that $\mathbf{b} := (f_i(\mathbf{a}_1), \ldots, f_i(\mathbf{a}_m)) \notin \rho$, i.e. $\mathbf{b} = (p, \ldots, p)$. By the definition of f_i there exist $1 \leq j_1, \ldots, j_m \leq i$ such that $\mathbf{a}_k = \mathbf{e}_{j_k}$ for all $k = 1, \ldots, m$. So, we can choose $k \in \{1, \ldots, i\} \setminus \{j_1, \ldots, j_m\}$. Clearly, the k-th column of X is $(p, \ldots, p) \notin \rho_m$.

Let $i \leq m$ and $X = [x_{ln}]$ be the $m \times i$ matrix with $x_{11} = \ldots = x_{(i-1)(i-1)} = 2, x_{li} = 2$ for $i \leq l \leq m$ and $x_{ln} = 1$ otherwise. The values of f_i on the rows of X form $(p, \ldots, p) \notin \rho$.

We shall prove that s_{26} preserves ρ_m with $r = 1, p = 2$ and $q = 0$; and s_{76} preserves ρ_m with $p = 1, q = 2$, and $r = 0$.

Suppose to the contrary that $s_j, j \in \{26, 76\}$ does not preserve ρ_m. Then, there is an $m \times 3$ matrix with columns in ρ_m such that $(s_j(x_{11}, x_{12}, x_{13}), \dots, s_j(x_{m1}, x_{m2}, x_{m3})) = (p, \dots, p)$. Therefore by the definition of b_j clearly $x_{l1} = p$ for each $j \in \{26, 76\}$ and $1 \le l \le m$. Thus, the first column of X is $(p, \dots, p) \notin \rho$, a contradiction.

So, we proved that the set $\{\bigcup_{m>2} f_m\}$ satisfies $\langle \bigcup_{i>m} \{f_m\} \cup \{s_j\}\rangle_{\mathrm{CL}} \supset \langle \bigcup_{i>m+1} \{f_m\} \cup \{s_j\}\rangle_{\mathrm{CL}} \supset \dots \supset \langle \bigcup_{i>m+1} \{f_m\} \cup \{s_j\}\rangle_{\mathrm{CL}} \dots$, for each $j \in \{26, 76\}$. (The proof is similar for conjugates.) □

Clones generated by majority functions are discussed by Baker and Pixley ([8], pp. 34-35) and it remains an open problem to determine the cardinality of clones which contain s_{424}.

References

[1] Burris S., Sankappanavar H.P., A Course in Universal Algebra, Akademiai Kiadó, Budapest, 1989 (Springer-Verlag,GTM78,1981) (in Hungarian)

[2] Csákány B., All minimal clones on the three-element set, Acta Cybernetica (Szeged), 6 (1983), 227-238.

[3] Csákány B., Three-element grupoids with minimal clones, Acta Sci. Math. (Szeged), v. 45, 1983, pp. 111-117.

[4] Jablonskiǐ S.V., Introduction to Discrete Mathematics, Nauka, Moscow, 1979 (in Russian).

[5] Pantović J., Tošić R., Vojvodić G., The cardinality of functionally complete algebras on a three element set, Algebra Universalis, 38 (1997), 136-140.

[6] Pantović J., Vojvodić D., Note on the cardinality of some sets of clones, Acta Cybernetica, 14 (1999), 491-495.

[7] Rosenberg I.G.., Uber die funktionale Vollständigkeit in den mehrwertigen Logiken (Struktur der Funktionen von mehreren Veränderlichen auf endlichen Mengen), Rozpravy Československe Akad. Věd. Řada Mat. Přirod. Věd., V. 80, 1970, 3-93.

[8] Szendrei Á., Clones in Universal Algebra, Séminaire de Mathématiques Superieures, vol. 99, Les Presses de l'Université de Montréal, Montréal, 1986.

Received by the editors July 9,1999.