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SPECTRAL APPROXIMATION AND NONLOCAL BOUNDARY VALUE PROBLEMS

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Abstract

We shall consider the boundary layer problems described by second order differential equation with small perturbation parameter multiplying the highest derivative and the appropriate boundary conditions of nonlocal type. This kind of problems represent mathematical models of a large number of phenomena in catalytic processes in chemistry and biology, as well as in the theory of semiconductors in electronics.

The solution inside the boundary layer will be constructed using truncated orthogonal series, and the solution out of the layer will be approximated by the solution of the reduced problem. The layer will be determined in terms of the perturbation parameter and the degree of the chosen truncated orthogonal series.

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1. Introduction

We shall consider the singularly perturbed problem described by the differential equation

(1) $$Ly \equiv -\varepsilon^2 y''(x) + g(x)y(x) = h(x) \quad 0 \le x \le 1\,,$$

where

(2) $$g(x) \geq K^2 > 0, \quad K \in \mathbf{R},$$

and nonlocal boundary conditions

$$y(0) = 0$$

and one of the following cases:

Case 1. Samarski–Bicadze simple condition $y(1) = cy(s) + d, \quad 0 < s < 1,$

Case 2. Samarski–Bicadze general condition $y(1) = \sum_{i=1}^{m} c_i y(s_i) + d, \quad s_i \in (0,1),$

Case 3. Integral condition $\int_0^1 y(x)dx = d.$

In his paper [3] Chegis has proved the following theorem

Theorem 1. *If we denote by $u(x)$ the solution of the corresponding local boundary value problem*

(3) $$Lu \equiv -\varepsilon^2 u''(x) + g(x)u(x) = 0, \quad u(0) = 0, \quad u(1) = 1$$

then the problem (1) with nonlocal boundary conditions has the unique solution if and only if

Case 1. $cu(s) \neq 1$

Case 2. $\sum_{i=1}^{m} c_i u(s_i) \neq 1$

Case 3. $\int_0^1 u(x)dx \neq 0.$

This result gives a sufficient condition for the existence of a unique solution of the considered problem. If we denote by

(4) $$u_0(x) = \frac{sh\frac{Kx}{\varepsilon}}{sh\frac{K}{\varepsilon}}.$$

and if we assume (2), then in

Case 1. $-\infty < c < \frac{1}{u_0(s)}$,

Case 2. $-\infty < \sum_{i=1}^{m} c_i u_0(s_i) < 1$,

Case 3. the unique solution always exists.

The problem (1)-(2) with nonlocal conditions of Samarski–Bicadze type has already been treated by a number of authors (see i.e. [1], [2], and [3]). In this paper we shall state some results obtained earlier by the author, and give some new results concernig *Case 3.*

In the first part we shall perform the transformation of the given problem, adapting it to the idea of approximating only the layer solutions by the truncated orthogonal series.

In the second part we shall perform the domain decomposition by determining the appropriate division points through the special procedure, based on the introduction of the *resemblance function.*

In the third part we shall construct the spectral approximation for the layer solutions, using an arbitrary orthogonal polynomial basis. We shall give the system that determines the coefficients of the truncated orthogonal series for each case of nonlocal boundary conditions separately.

In the fourth part we shall ilustrate theoretical results by a numerical example.

2. Transformation of the problem

The solution of the reduced problem (for $\varepsilon = 0$) is

(5)
$$y_R(x) = \frac{h(x)}{g(x)}.$$

It is well known that if $y_R(0) \neq 0$ the exact solution has boundary layers at both endpoints $x = 0$ and $x = 1$. The size of the layers is $O(\varepsilon)$.

We shall represent the exact solution in the form

(6)
$$y(x) = y_R(x) + y_L(x),$$

and we shall approximate $y_L(x)$ by

$$u_L(x) = \begin{cases} u_l(x) & 0 \le x \le x_0 \\ 0 & x_0 \le x \le 1 - x_0 \\ u_r(x) & 1 - x_0 \le x \le 1 \end{cases}, \tag{7}$$

where $u_l(x)$ is the left layer solution and it is determined by

$$Lu_l \equiv -\varepsilon^2 u_l''(x) + g(x)u_l(x) = \varepsilon^2 y_R''(x), \quad 0 \le x \le x_0 \tag{8}$$

$$u_l(0) = -y_R(0),\ u_l(x_0) = 0 \tag{9}$$

and $u_r(x)$ is the right layer solution and it is determined by the differential equation

$$Lu_r \equiv -\varepsilon^2 u_r''(x) + g(x)u_r(x) = \varepsilon^2 y_R''(x), \quad 1 - x_0 \le x \le 1\,, \tag{10}$$

left boundary condition

$$u_r(1 - x_0) = 0, \tag{11}$$

and nonlocal boundary condition of one of the following types:

Case 1.

a) If $s \in (0, x_0)$

$$u_r(1) = cy_R(s) + cu_l(s) + d - y_R(1)\,. \tag{12}$$

b) If $s \in [x_0, 1 - x_0]$

$$u_r(1) = cy_R(s) + d - y_R(1)\,. \tag{13}$$

c) If $s \in (1 - x_0, 1)$

$$u_r(1) - cu_r(s) = cy_R(s) + d - y_R(1)\,. \tag{14}$$

Case 2.

$$u_r(1) - \sum_{i=l+1}^{m} c_i u_r(s_i) = \sum_{i=1}^{j} c_i u_l(s_i) + \sum_{i=1}^{m} y_R(s_i) + d - y_R(1) \tag{15}$$

where $s_i \in (0, x_0)$ for $i \le j$, $s_i \in (x_0, 1 - x_0)$ for $j < i \le l$ and $s_i \in (1 - x_0, 1)$ for $i > l$.

Case 3.

$$\int_{1-x_0}^{1} u_r(x)dx = d - \int_0^1 y_R(x)dx - \int_0^{x_0} u_l(x)dx \,. \tag{16}$$

3. The division point

As the size of the boundary layer is $O(\varepsilon)$, the idea is to perform the domain decomposition, using the division point $x_0 = c\varepsilon$, in such a way that c depends on the degree n of the spectral approximation for the layer functions.

The spectral solution $v_n(x)$, which approximates $u_l(x)$, is represented in the form of the truncated orthogonal series

$$v_n(x) = \sum_{k=0}^{n} a_k T_k^*(x). \tag{17}$$

$T_k^*(x)$ denote arbitrary orthogonal polynomials upon $[0, x_0]$.

We shall determine the value c in $x_0 = c\varepsilon$ by a special procedure, based on introduction of the *resemblance function* for the layer solution $u_l(x)$.

Definition 1. *The resemblance function is the polynomial $p_n(x)$ of degree $n \geq 2$, such that*

- *a) $p_n(0) = -y_R(0)$ and $p_n(c\varepsilon) = 0$, i.e. $p_n(x)$ satisfies the boundary conditions in (9), and*
- *b) x_0 is the only stationary point for $p_n(x)$.*

Lemma 1. *The resemblance function is given by*

$$p_n(x) = -\frac{h(0)}{g(0)}\left(1 - \frac{x}{c\varepsilon}\right)^n, \; n \geq 2. \tag{18}$$

Proof. We verify the conditions in Definition 1.

a)

$$p_n(0) = -\frac{h(0)}{g(0)}\left(1 - \frac{0}{c\varepsilon}\right)^n = -y_R(0)$$

and

$$p_n(c\varepsilon) = -\frac{h(0)}{g(0)}\left(1 - \frac{c\varepsilon}{c\varepsilon}\right)^n = 0$$

b) From

$$p_n'(x) = \frac{nh(0)}{c\varepsilon g(0)}\left(1 - \frac{x}{c\varepsilon}\right)^{n-1} = 0$$

we conclude that x_0 is the only stationary point for $p_n(x)$.

In order to determine the division point x_0 we shall ask that the resemblance function satisfies the differential equation at the layer point $x = 0$. This will give us

$$\frac{n(n-1)h(0)}{c^2 g(0)} - h(0) = \varepsilon^2 y_R''(0).$$

If we solve this equation for c, $c > 0$, assuming that ε is very small, we shall obtain

(19)
$$c = \sqrt{\frac{n(n-1)}{g(0)}}.$$

Once the division point x_0 is determined we find the spectral approximation $v_n(x)$ for the problem (8),(9) using the standard procedure (see [1]).

4. Approximation to the right layer solution

We shall approximate the solution $u_r(x)$ of the problem (10)-(16) by the solution of the problem described by the differential equation

(20)
$$Lw(x) \equiv -\varepsilon^2 w(x) + g(x)w(x) = \varepsilon^2 y_R''(x), \quad x \in [1 - x_0, 1],$$

left boundary condition

(21)
$$w(1 - x_0) = 0,$$

and nonlocal boundary condition of one of the following types:

Case 1.

a) If $s \in (0, x_0)$

(22)
$$w(1) = cy_R(s) + cv_n(s) + D\ , \quad D = d - y_R(1)\ .$$

b) If $s \in [x_0, 1-x_0]$

$$w(1) = cy_R(s) + D\,, \quad D = d - y_R(1)\,. \tag{23}$$

c) If $s \in (1-x_0, 1)$

$$w(1) - cw(s) = cy_R(s) + D\,, \quad D = d - y_R(1)\,. \tag{24}$$

Case 2.

$$w(1) - \sum_{i=l+1}^{m} c_i w(s_i) = D\,, \quad D = \sum_{i=1}^{j} c_i v_n(s_i) + \sum_{i=1}^{m} y_R(s_i) + d - y_R(1), \tag{25}$$

where $s_i \in (0, x_0)$ for $i \le j$, $s_i \in (x_0, 1-x_0)$ for $j < i \le l$ and $s_i \in (1-x_0, 1)$ for $i > l$.

Case 3.

$$\int_{1-x_0}^{1} w(x)dx = D\,, \quad D = d - \int_0^1 y_R(x)dx - \int_0^{x_0} v_n(x)dx\,. \tag{26}$$

The spectral solution $w_n(x)$, which approximates $w(x)$, is represented in the form of truncated orthogonal series

$$w_n(x) = \sum_{k=0}^{n} b_k T_k(x), \tag{27}$$

where $T_k(x)$ denote orthogonal polinomials upon $[1-x_0, 1]$.

The coefficients b_k are determined by the collocation method using the Gauss–Lobatto nodes t_j, $j = 1, ..., n-1$,

Theorem 2. *The coefficients b_k are obtained as the solution of the system*

$$\sum_{k=0}^{n} \left(-\varepsilon^2 T_k''(t_j) + g(t_j)T_k(t_j)\right) b_k = \varepsilon^2 y_R''(t_j),\ j = 1, \ldots, n-1$$

$$\sum_{k=0}^{n} T_k(1-x_0) b_k = 0$$

and one of the following equations: In

Case 1.

a) If $s \in (0, x_0)$ $\sum_{k=0}^{n} b_k T_k(1) = c y_R(s) + c v_n(s) + D\,, \quad D = d - y_R(1)\,.$

b) If $s \in [x_0, 1 - x_0]$ $\sum_{k=0}^{n} b_k T_k(1) = c y_R(s) + D\,, \quad D = d - y_R(1)\,.$

c) If $s \in (1 - x_0, 1)$ $\sum_{k=0}^{n} b_k (T_k(1) - c T_k(s)) = c y_R(s) + D\,, \quad D = d - y_R(1)\,.$

Case 2.

$$\sum_{k=0}^{n} b_k \Big(T_k(1) - \sum_{i=l+1}^{m} c_i T_k(s_i)\Big) = D\,, \quad D = \sum_{i=1}^{j} c_i v_n(s_i) + \sum_{i=1}^{m} y_R(s_i) + d - y_R(1),$$

where $s_i \in (0, x_0)$ *for* $i \le j$, $s_i \in (x_0, 1 - x_0)$ *for* $j < i \le l$ *and* $s_i \in (1 - x_0, 1)$ *for* $i > l$.

Case 3.

$$\sum_{k=0}^{n} b_k \int_{1-x_0}^{1} T_k(x)dx = D\,, \quad D = d - \int_0^1 y_R(x)dx - \int_0^{x_0} v_n(x)dx\,.$$

Proof. The theorem is proved by introducing (27) into (20)-(26) and asking that the first equation is satisfied at Gauss–Lobatto nodes.

5. Numerical examples

As the numerical example we shall consider the problem

$$-\varepsilon^2 y(x) + y(x) = 1, \quad 0 \le x \le 1,$$

$$y(0) = 0, \quad y(1) = 0.2y(0.1) + 0.5y(0.2) + 0.3y(0.999).$$

The reduced solution is $y_R(x) = 1$, so that we have two boundary layers. We shall use Chebyshev polynomials as the orthogonal basis.

Table 1 gives the values of Chebyshev coefficients b_k, $k = 0, \ldots, n$ for $\varepsilon^2 = 10^{-5}$ and $\varepsilon^2 = 10^{-7}$ when truncated orthogonal series of the tenth degree is used.

b_k	$n = 10$
b_0	0.37753456
b_1	0.33495654
b_2	0.23632823
b_3	0.13563357
b_4	0.064792661
b_5	0.026339254
b_6	0.0092739598
b_7	0.0028738026
b_8	0.0007933457
b_9	0.0001968836
b_{10}	0.0000445245

Table 1.

We can see that the coefficients decay very quickly, which indicates good convergence of the Chebyshev series to the exact solution.

Table 2 gives the difference between the exact and the approximate solution $d(x)$ at several points from the layer subinterval $[1-x_0, 1]$ for $\varepsilon^2 = 10^{-7}$. The size of the layer subinterval is evaluated depending on the chosen degree of the spectral approximation.

		$n = 10,\ x_0 = 0.003$	$n = 15,\ x_0 = 0.0047$
x	$y(x)$	$\lvert d(x)\rvert$	$\lvert d(x)\rvert$
0.999	0.958	$4 \cdot 10^{-6}$	$1 \cdot 10^{-7}$
0.9993	0.891	$1 \cdot 10^{-6}$	$2 \cdot 10^{-7}$
0.9996	0.718	$8 \cdot 10^{-6}$	$1 \cdot 10^{-8}$
0.9998	0.469	$4 \cdot 10^{-6}$	$4 \cdot 10^{-7}$
0.9999	0.272	$1 \cdot 10^{-5}$	$1 \cdot 10^{-7}$
0.99999	0.031	$8 \cdot 10^{-6}$	$6 \cdot 10^{-7}$
0.999999	0.003	$9 \cdot 10^{-7}$	$2 \cdot 10^{-8}$

Table 2.

References

[1] Adžić, N., On the Spectral Solution for Singularly Perturbed Problems, ZAMM 71 (1991) 6, T773-T776.

[2] Bicadze, A. and Samarski, A., On some simpler generalization of linear elliptic boundary value problems, DANN SSSR, 185, 4 (1969), 739-740.

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