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FINITE DIFFERENCE SCHEMES FOR BOUNDARY VALUE PROBLEMS WITH GENERALIZED SOLUTIONS[1]

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Abstract

A survey of results concerning the convergence of finite difference schemes for boundary value problems with generalized solutions from Sobolev space is presented.

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1. Introduction

Finite difference schemes (FDSs) are often used for approximation of boundary value problems (BVPs) with generalized solutions. In such cases it is preferable to have the convergence result for the minimal smoothness of input data. This leads to several problems as: the right-hand side of the equation and the solution may be discontinuous functions; small smoothness of the solution requires the convergence rate estimate in the weak norm; coefficients of equation do not belong to standard Sobolev spaces, etc.

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In this paper we give a survey of techniques for overcoming these problems. Special attention is paid to deriving convergence rate estimates consistent with the smoothness of input data.

2. Model Problem

As a model problem we consider the Dirichlet BVD for the Poisson equation in the square $\Omega = (0,\,1)^2$:

$$(1)\qquad -\Delta\, u = f(x)\,, \quad x = (x_1,\, x_2) \in \Omega\,; \qquad u(x) = 0\,, \quad x \in \Gamma = \partial\Omega\,.$$

We assume that the solution of BVP (1) is sufficiently smooth, that is, the function $f(x)$ satisfies all the necessery conditions for that.

Let $\overline{\omega}_h$ be the uniform mesh in $\overline{\Omega}$ with the step size h, $\omega_h = \overline{\omega}_h \cap \Omega$ and $\gamma_h = \overline{\omega}_h \cap \Gamma$. We define finite differences in the usual way [18]: $v_{x_i} = (v^{+i} - v)/h\,,\ v_{\bar{x}_i} = (v - v^{-i})/h\,,$ where $v^{\pm i}(x) = v(x \pm h r_i)\,,$ and r_i denotes the unit vector of the x_i axis. With $\|v\|^2_{L_2(\omega_h)} = h^2 \sum_{x\in\omega_h} v^2(x)$ we denote a discrete L_2-norm in ω_h. We also introduce discrete Sobolev norms $\|v\|_{W_2^k(\omega_h)}$ $(k = 1,\, 2,\, ...)$.

We approxsimate (1) with the standard five–point FDS:

$$(2)\qquad -\Delta_h\, v = f\,, \quad x \in \omega_h\,; \qquad v = 0\,, \quad x \in \gamma_h\,.$$

The error $z = u - v$ satisfies the conditions:

$$(3)\qquad -\Delta_h\, z = \psi\,, \quad x \in \omega_h\,; \qquad z = 0\,, \quad x \in \gamma_h\,,$$

where $\psi = \Delta\, u - \Delta_h\, u = \left(\frac{\partial^2 u}{\partial x_1^2} - u_{x_1\bar{x}_1}\right) + \left(\frac{\partial^2 u}{\partial x_2^2} - u_{x_2\bar{x}_2}\right) = \psi_1 + \psi_2\,.$

From relation [19] $\|\Delta_h\, z\|_{L_2(\omega_h)} \geq C_0\, \|z\|_{W_2^2(\omega_h)}$ immediatelly follows a priori estimate

$$(4)\qquad \|z\|_{W_2^2(\omega_h)} \leq C\, \|\psi\|_{L_2(\omega_h)}\,.$$

In the following C denotes a positive generic constant which may take different values in different formulas. In such a way, to prove the convergence of FDS (2) we must estimate ψ. From Taylor's formula follows: $\psi_i(x) = \frac{h^2}{12}\,\frac{\partial^4 u(\tilde{x})}{\partial x_i^4}\,,$ where $\tilde{x}$ is some midpoint. From here one, immediately obtains:

$$\|z\|_{W_2^2(\omega_h)} \leq C\, h^2\, \|u\|_{C^4(\bar{\Omega})}\,.$$

More precise estimate may be obtained using integral representation of the residual. We have

$$\psi_1(x) = \frac{1}{h^2}\int_{x_1-h}^{x_1+h}\int_{x_2-h}^{x_2+h}\left(1-\frac{|x_1'-x_1|}{h}\right)\left(1-\frac{|x_2'-x_2|}{h}\right)\times$$

$$\times\left(\int_{x_1'}^{x_1}\int_0^{x_1''}\frac{\partial^4 u(x_1''', x_2')}{\partial x_1^4}\,dx_1'''\,dx_1'' + \int_{x_1'}^{x_1}\int_{x_2'}^{x_2}\frac{\partial^4 u(x_1'', x_2'')}{\partial x_1^3\,\partial x_2}\,dx_2''\,dx_1''\right)dx_2'\,dx_1'$$

and an analogous formula for ψ_2. It follows

$$|\psi(x)| \le C\,h\,\|u\|_{W_2^4(e)} \qquad \text{where} \qquad e = (x_1-h,\, x_1+h)\times(x_2-h,\, x_2+h)\,.$$

Summing over the mesh ω_h one obtains $\|\psi\|_{L_2(\omega_h)} \le C\,h^2\,\|u\|_{W_2^4(\Omega)}$, wherefrom follows

$$\|z\|_{W_2^2(\omega_h)} \le C\,h^2\,\|u\|_{W_2^4(\Omega)}\,. \tag{5}$$

The estimate (5) can be obtained also by applying of the Bramble–Hilbert lemma (see [1], [4]). Moreover, as the value of ψ at the node $x \in \omega$ is a bounded linear functional of $u \in W_2^s(e)$, for $s > 3$, which vanishes on polynomials of third degree, applying the Dupont–Scott lemma [4] we obtain a more general result:

$$\|z\|_{W_2^2(\omega_h)} \le C\,h^{s-2}\,|u|_{W_2^s(\Omega)}\,, \qquad 3 < s \le 4\,. \tag{6}$$

Convergence rate estimates of the form

$$\|z\|_{W_2^k(\omega_h)} \le C\,h^{s-k}\,\|u\|_{W_2^s(\Omega)}\,, \qquad s \ge k\,,$$

are called consistent with the smoothness of the solution of BVP (1) (see [14]). Such estimates are obtained in [21], [13], [5], etc. An extensive bibliography can be found in [10]. Note that similar estimates

$$\|z\|_{W_2^k(\Omega)} \le C\,h^{s-k}\,\|u\|_{W_2^s(\Omega)}\,, \qquad s \ge k\,,$$

are characteristic for the finite element method (see [2]).

3. Boundary Value Problems with Weak Solutions

For $s \le 3$ the right-hand side of (1) and (2) may be a discontinuous function, and consequently, FDS (2) is not well defined. To obtain a well-defined FDS we average $f(x)$ using the Steklov averaging operators:

$$T_i f(x) = T_i^- f(x + 0.5\,h\,r_i) = T_i^+ f(x - 0.5\,h\,r_i) = \int_{-1/2}^{1/2} f(x + h\,y\,r_i)\,dy\,.$$

These operators commute and satisfy the following relations

$$T_i^+ T_i^- = T_i^2\,, \quad T_i^- \frac{\partial f}{\partial x_i} = f_{\bar{x}_i}\,, \quad T_i^+ \frac{\partial f}{\partial x_i} = f_{x_i}\,, \quad T_i^2 \frac{\partial^2 f}{\partial x_i^2} = f_{x_i \bar{x}_i}\,.$$

For $s < 2$, the convergence of FDS (2) does not follow from (6). Consequently, the weaker norms must be used to prove the convergence. The following assertion is valid [19], [5].

Lemma 1. *If in (3)* $\psi = \eta_{1,\bar{x}_1} + \eta_{2,\bar{x}_2}$ *, then*

(7) $$\|z\|_{W_2^1(\omega_h)} \le C\left(\|\eta_1\|_{L_2(\omega_h)} + \|\eta_2\|_{L_2(\omega_h)}\right).$$

If $\psi = \zeta_{1,x_1\bar{x}_1} + \zeta_{2,x_2\bar{x}_2}$ *and* $\zeta_i = 0$ *for* $x_i = 0$ *then*

(8) $$\|z\|_{L_2(\omega_h)} \le C\left(\|\zeta_1\|_{L_2(\omega_h)} + \|\zeta_2\|_{L_2(\omega_h)}\right).$$

Let us consider FDS with averaged right-hand side [5]:

(9) $$-\Delta_h v = T_1^2 T_2^2 f\,, \quad x \in \omega_h\,; \qquad v = 0\,, \quad x \in \gamma_h\,.$$

The error $z = u - v$ satisfies the conditions (3), where: $\psi = \psi_1 + \psi_2$, $\psi_i = \zeta_{i,x_i\bar{x}_i}$, $\zeta_i = T_{3-i}^2 u - u$, $i = 1, 2$. By lemma 1 one obtains a priori estimates (4), (8) and

$$\|z\|_{W_2^1(\omega_h)} \le C\left(\|\zeta_{1,x_1}\|_{L_2(\omega_h)} + \|\zeta_{2,x_2}\|_{L_2(\omega_h)}\right).$$

Using the Dupont–Scott lemma, analogously as in the previous case, one obtains the following convergence rate estimates: (6) for $2 \le s \le 4$,

$$\|z\|_{W_2^1(\omega_h)} \le C\,h^{s-1}\,\|u\|_{W_2^s(\Omega)}\,, \qquad 1 < s \le 3\,, \qquad \text{and}$$

$$(10) \qquad \|z\|_{L_2(\omega_h)} \le C\, h^s \, \|u\|_{W_2^s(\Omega)} \,, \qquad 1 < s \le 2 \,.$$

In the case $0 < s \le 1$ the solution of (1) may be a non–continuous function. Let us consider FDS

$$-\Delta_h \, u = T_1^3 T_2^3 f \,, \quad x \in \omega_h \,; \qquad v = 0 \,, \quad x \in \gamma_h$$

and define the error in the following manner: $z = T_1 T_2 \, u - v$. Similarly as in the previous cases one obtains estimate (10) for $0 < s \le 2$ [5].

For the problems with solutions from $W_p^s(\Omega)$, $(1 < p < \infty \,, \ p \ne 2)$ analogous results hold (see [20], [3]). Analogs of a priori estimates (7) and (8) in discrete W_p^1 and L_p norms may be obtained using theory of discrete Fourier multipliers [17]. Convergence rate estimates have the form

$$\|z\|_{W_p^k(\omega_h)} \le C\, h^{s-k} \, \|u\|_{W_p^s(\Omega)} \,, \qquad s \ge k \,,$$

and may be obtained using the Dupont–Scott lemma.

4. Alternative Technique

As we have been seen, for integer values of s convergence rate estimates can be constructed in an "elementary" way, without the Bramble–Hilbert lemma. Using such estimates and the interpolation theory of Hilbert spaces [15] one easily obtains the corresponding estimates for non–integer s.

Let X and Y be two Hilbert spaces and let X be continuously imbeded in Y. Let $0 < \theta < 1$ and let $[X,\, Y]_\theta$ denotes the intermediate space obtained by interpolation [15]. Then $X \subset [X,\, Y]_\theta \subset Y$ and for every $u \in X$ the inequality $\|u\|_{[X,\,Y]_\theta} \le C_\theta \, \|u\|_X^{1-\theta} \, \|u\|_Y^\theta$ holds.

In particular, Sobolev spaces are interpolation spaces and

$[W_2^{s_1}(\Omega),\, W_2^{s_2}(\Omega)]_\theta = W_2^{(1-\theta)\, s_1 + \theta\, s_2}(\Omega)$.

Lemma 2. [15] *Let A be a bounded linear operator from X_i into Y_i $(i = 0,\, 1)$. Then A is also a bounded linear operator from $[X_0,\, X_1]_\theta$ into $[Y_0,\, Y_1]_\theta$ and the following relation holds*

$$\|A\|_{[X_0,\,X_1]_\theta \to [Y_0,\,Y_1]_\theta} \le C_\theta \, \|A\|_{X_0 \to X_1}^{1-\theta} \, \|A\|_{Y_0 \to Y_1}^{\theta} \,.$$

Let us consider again FDS (9). Similarly as in Section 2, one easily shows that

$$\|z\|_{W_2^2(\omega_h)} \le C\, h^2\, \|u\|_{W_2^4(\Omega)} \qquad \text{and} \qquad \|z\|_{W_2^2(\omega_h)} \le C\, \|u\|_{W_2^2(\Omega)} \,,$$

wherefrom, using Lemma 2, one immediately obtains estimate (6) for $2 \le s \le 4$. In an analogous manner one obtains convergence rate estimates in other discrete norms (see [11], [12]).

5. Equations with Variable Coefficients

Let us now consider elliptic equation with variable coefficients:

$$(11) \qquad \mathcal{L}\, u \equiv -\sum_{i,j=1}^{2} \frac{\partial}{\partial x_i}\left(a_{ij}\, \frac{\partial u}{\partial x_j}\right) = f(x)\,, \quad x \in \Omega$$

with homogeneous Dirichlet boundary condition. We assume that $u \in W_2^s(\Omega)$ and $f \in W_2^{s-2}(\Omega)$.

Let V and W be two function spaces in the same domain. The space of multipliers $M(V,\, W)$ is defined by [16]: $M(V,\, W) = \{a(x) \,:\, a(x)\, v(x) \in W\,,\ \forall v(x) \in V\}$, $M(V) = M(V,\, V)$. It is easy to see that coeficients a_{ij} of equation (11) belong to the space of multipliers $M(W_2^{s-1}(\Omega))$.

The following relations are valid [10]:

$$W_2^{|s-1|}(\Omega) = M(W_2^{s-1}(\Omega))\,, \qquad |s-1| > 1\,,$$

$$W_{2/|s-1|}^{|s-1|+\varepsilon}(\Omega) \subset M(W_2^{s-1}(\Omega))\,, \quad \varepsilon > 0, \qquad 0 < |s-1| < 1\,,$$

$$L_\infty(\Omega) = M(L_2(\Omega)) = M(W_2^{s-1}(\Omega))\,, \qquad s = 1\,.$$

Let us consider FDS

$$(12) \qquad \mathcal{L}_h\, v \equiv -\frac{1}{2} \sum_{i,j=1}^{2} \left[(a_{ij}\, v_{x_j})_{\bar{x}_i} + (a_{ij}\, v_{\bar{x}_j})_{x_i}\right] = T_1^2 T_2^2 f\,, \quad x \in \omega_h$$

with the previous boundary condition. The error $z = u - v$ satisfies conditions

$$\mathcal{L}_h\, z = \sum_{i,j=1}^{2} \eta_{ij,\bar{x}_i}\,, \quad x \in \omega_h\,; \qquad z = 0\,, \quad x \in \gamma_h$$

where $\eta_{ij,\bar{x}_i} = T_i^+ T_{3-i}^2 \Big(a_{ij} \frac{\partial u}{\partial x_j} \Big) - \frac{1}{2} \, (a_{ij} \, u_{x_j} + a_{ij}^{+i} \, u_{\bar{x}_j}^{+i})$. The following a priori estimates hold

(13) $$\|z\|_{W_2^2(\omega_h)} \le C \sum_{i,j=1}^{2} \|\eta_{ij,\bar{x}_i}\|_{L_2(\omega_h)} \qquad \text{and}$$

(14) $$\|z\|_{W_2^1(\omega_h)} \le C \sum_{i,j=1}^{2} \|\eta_{ij}\|_{L_2(\omega_h)}$$

Using bilinear version of the Bramble–Hilbert lemma or interpolatory properties of bounded bilinear operators from (13) and (14) one obtains convergence rate estimates in the form (see [10], [12])

$$\|z\|_{W_2^2(\omega_h)} \le C \, h^{s-2} \max_{i,j} \|a_{ij}\|_{W_2^{s-1}(\Omega)} \|u\|_{W_2^s(\Omega)}, \qquad 2 < s \le 4,$$

$$\|z\|_{W_2^1(\omega_h)} \le C \, h^{s-1} \max_{i,j} \|a_{ij}\|_{W_2^{s-1}(\Omega)} \|u\|_{W_2^s(\Omega)}, \qquad 2 < s \le 3, \qquad \text{and}$$

$$\|z\|_{W_2^1(\omega_h)} \le C \, h^{s-1} \max_{i,j} \|a_{ij}\|_{W_{2/(s-1)}^{s-1+\varepsilon}(\Omega)} \|u\|_{W_2^s(\Omega)}, \qquad 1 < s \le 2.$$

In a multidimensional case ($n > 2$) there arise additional problems caused by the discontinuity of right-hand side of equation ($f \in W_2^{s-2}(\Omega) \not\subset C(\bar{\Omega})$ for $s \le 2 + n/2$) or its solution ($u \in W_2^s(\Omega) \not\subset C(\bar{\Omega})$ for $s \le n/2$). These problems may be resolved by convenient averaging. Note also that $M(W_2^{s-1}(\Omega)) \ne W_2^{s-1}(\Omega)$ for $s \le 1 + n/2$.

6. Parabolic Problems

In parabolic case analogous results hold as in the elliptic case. Let us consider the following initial–boundary value problem (IBVP)

(15) $$\frac{\partial u}{\partial t} + \mathcal{L} u = f(x, t) \qquad \text{in} \qquad Q = \Omega \times (0, T),$$

$$u(x, 0) = u_0(x), \qquad u(x, t) = 0 \quad \text{on} \quad \Gamma \times (0, T).$$

Let us introduce the mesh $Q_{h\tau} = \omega_h \times \omega_\tau$, where ω_τ is a uniform mesh with the step size τ in $(0, T)$. We also define the discrete L_2-norm

$\|v\|^2_{L_2(Q_{h\tau})} = h^2\,\tau\sum_{(x,t)\in Q_{h\tau}} v(x,\,t)^2$, and the discrete Sobolev norms $\|v\|_{W_2^{k,\,k/2}(Q_{h\tau})}$.

We consider implicit FDS

$$v_{\bar t} + \mathcal{L}_h\, v = T_1^2 T_2^2 T_t^-\, f\,,$$

with the corresponding boundary and initial conditions, where T_t^- is the Steklov averaging operator on t. The error $z = u - v$ satisfies the equation

$$z_{\bar t} + \mathcal{L}_h\, z = \varphi_{\bar t} + \sum_{i,j=1}^{2} \eta_{ij,\,\bar x_i}$$

and homogeneous boundary and initial conditions. Here

$$\varphi = u - T_1^2 T_2^2 u\,, \qquad \eta_{ij,\,\bar x_i} = T_i^+ T_{3-i}^2 T_t^- \Big(a_{ij}\,\frac{\partial u}{\partial x_j}\Big) - \frac{1}{2}\,(a_{ij}\,u_{x_j} + a_{ij}^{+i}\,u_{\bar x_j}^{+i})\,.$$

The following a priori estimates are valid

$$\|z\|_{W_2^{2,\,1}(Q_{h\tau})} \le C\Big(\|\varphi_{\bar t}\|_{L_2(Q_{h\tau})} + \sum_{i,j=1}^{2} \|\eta_{ij,\,\bar x_i}\|_{L_2(Q_{h\tau})}\Big) \qquad \text{and}$$

$$\|z\|_{W_2^{1,\,1/2}(Q_{h\tau})} \le C\Big([\varphi]_{1/2} + \sum_{i,j=1}^{2} \|\eta_{ij}\|_{L_2(Q_{h\tau})}\Big)\,,$$

where $[\varphi]_{1/2}^2 = h^2 \sum_{x\in\omega_h} \tau^2 \sum_{t,t'\in\overline{\omega}_\tau,\,t\ne t'} \frac{|\varphi(x,t)-\varphi(x,t')|^2}{|t-t'|^2}$. From here, in a similar manner as in the elliptic case, for $\tau \asymp h^2$, one obtains convergence rate estimates [6], [7]

$$\|z\|_{W_2^{2,\,1}(Q_{h\tau})} \le C\,h^{s-2}\,(\max_{i,j}\,\|a_{ij}\|_{W_2^{s-1}(\Omega)} + 1)\,\|u\|_{W_2^{s,\,s/2}(Q)}\,,$$

for $2 < s \le 4$, and

$$\|z\|_{W_2^{1,\,1/2}(Q_{h\tau})} \le C\,h^{s-1}\,(\max_{i,j}\,\|a_{ij}\|_{W_2^{s-1}(\Omega)} +$$
$$+ \sqrt{\ln{(1/h)}})\,\|u\|_{W_2^{s,\,s/2}(Q)}\,, \quad \text{for} \quad 2 < s \le 3$$

For $1 \le s \le 2$ the solution of IBVP (15) may be discontinuous. In that case the error may be defined as: $z = T_1 T_2 u - v$.

7. Hyperbolic Problems

Convergence rate estimates for hyperbolic IBVPs, contrary to the case of elliptic and parabolic problems, usually are nonconsistent with the smoothness of data. Let us consider the following IBVP

(16) $$\frac{\partial^2 u}{\partial t^2} + \mathcal{L}\, u = f(x,\, t) \qquad \text{in} \qquad Q = \Omega \times (0,\, T)\,;$$

$$u(x,\, 0) = u_0(x)\,,\ \frac{\partial u(x,\, 0)}{\partial t} = u_1(x)\,;\quad u(x,\, t) = 0 \ \text{ on } \ \Gamma \times (0,\, T)\,.$$

We introduce the mesh in the same manner as in the previous case and define the norm

$$\|v\|_{C_\tau(W_2^1(\omega_h))} = \max_{t\in\omega_\tau} \left(\|z_t\|^2_{L_2(\omega_h)} + \sum_{i=1}^{2} \left\| \left(\frac{z^+ + z}{2}\right)_{x_i} \right\|^2_{L_2(\omega_h)} \right)^{1/2}$$

Consider FDS

(17) $$v_{t\bar t} + \frac{1}{4}\, \mathcal{L}_h\, (v^+ + 2\, v + v^-) = T_1 T_2 T_t\, f\,,$$

with the corresponding initial and boundary conditions, where $v^\pm = v(x,\, t\pm \tau)$. For $\tau \asymp h$ the following convergence rate estimate is valid [8]

(18) $$\|z\|_{C_\tau(W_2^1(\omega_h))} \le C\, h^{s-2}\, (\max_{i,j} \|a_{ij}\|_{W_2^{s-1}(\Omega)} + 1)\, \|u\|_{W_2^s(Q)}$$

for $2 < s \le 4$.

In some cases by interpolation technique one can obtain estimates which garantee a faster convergence on weaker solutions (see [22]). Let us consider the following model problem

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} \qquad \text{in} \qquad Q = (0,\, 1) \times (0,\, T)\,;$$

$$u(x,\, 0) = u_0(x)\,,\ \frac{\partial u(x,\, 0)}{\partial t} = 0\,;\ \ u(x,\, t) = 0 \text{ on } \{0,\, 1\} \times (0,\, T)$$

and approximate it by FDS of the form (17). Using integral representation of the residual, one easily obtains the estimates

$$\|z\|_{C_\tau(W_2^1(\omega_h))} \le C\, (h + \tau)^2\, \|u_0\|_{W_2^4(0,\,1)} \qquad \text{and}$$

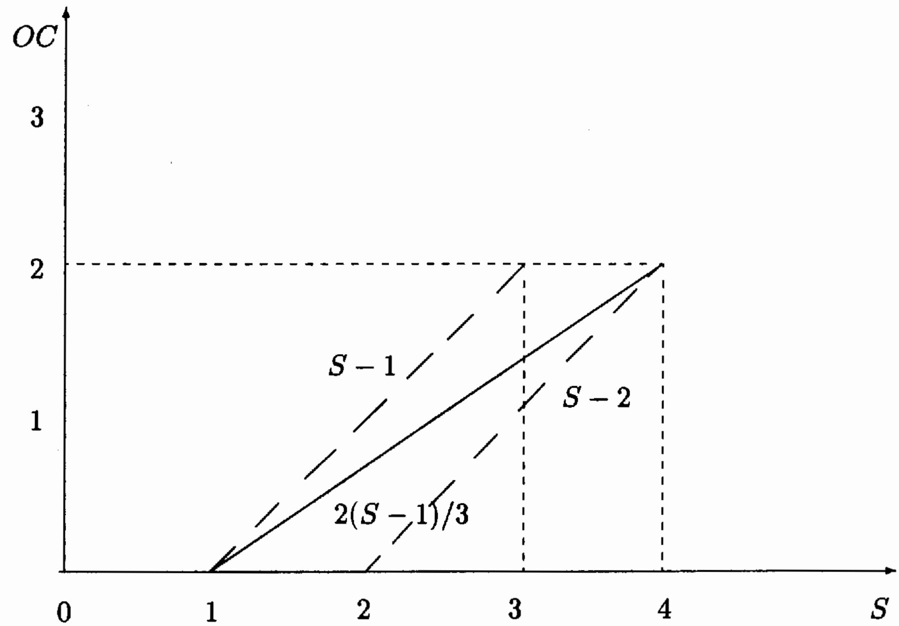
$$\|z\|_{C_\tau(W_2^1(\omega_h))} \le C\,\|u_0\|_{W_2^1(0,1)}$$

From these estimates one obtains by interpolation [9]:

$$\|z\|_{C_\tau(W_2^1(\omega_h))} \le C\,(h+\tau)^{\frac{2}{3}(s-1)}\|u_0\|_{W_2^s(0,1)}\,, \qquad 1 \le s \le 4 \tag{19}$$

Contrary to (18), estimate (19) guarantees convergence even for $1 < s \le 2$.

In the following diagram the relation between smoothness s and order of convergence (O.C.) is presented for the case of estimates (18) and (19).



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