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# A MAXIMAL RIGHT ZERO BAND COMPATIBLE COEQUALITY RELATION ON SEMIGROUP WITH APARTNESS

**Daniel Abraham Romano**
Faculty of Sciences, Department of Mathematics and Informatics
78000 Banja Luka, 2 Mladen Stojanović Street
Republic of Srpska - Bosnia and Herzegovina
e-mail: daniel@urc.bl.ac.yu

**Abstract**

Band congruences of semigroup were first defined and studied by T.Tamura and N.Kimura in 1955. First general results concerning left zero bands and right zero bands were obtained by P.Dubriel in 1951 and G.Therrin in 1956.
This study is concerned with the constructive algebra in the sense of Bishop, van Dalen, Richman and Ruitenburg. The main subject is the characterization of the maximal right zero band coequality relation compatible with the semigroup operation by means of special right consistent left ideal of the semigroup $(S, =, \neq, 1, \cdot)$ with apartness. Also, we give a construction of the maximal strongly extensional right consistent left ideals $A(a)$ of $S$ generated by an element $a$ in $S$ such that $a \# A(a)$.

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## 1

Let $S = (S, =, \neq, 1, \cdot)$ be a semigroup with apartness in the sense of the books [1,7,11,13] and the paper [9]. Throughout this paper $Sa$ will denote a principal left ideal of a semigroup $S$ generated by the element $a$. Let $Y$ be a subset of $S$ and $a \in S$. By $a \# Y$ we denote $(\forall y \in Y)(y \neq a)$ and by $Y'$ the subset $\{s \in S : \ s \# Y\}$. A subset $A$ of $S$ is a *right consistent subset* ([2]-[5]) iff

$$(\forall x, y \in S)(xy \in A \Rightarrow y \in A).$$

The relation $q$ on $S$ is a coequality relation on $S$ iff it is a consistent, symmetric and cotransitive relation ([9]). It holds $q \subseteq \neq$ . A coequality relation is a generalization of the apartness. Apartness studied Bishop ([1]), Mines ([7]), Ruitenburg ([11]) and Troelstra ([13]); the coequality relation was studied by the author ([9],[10]). If $q$ is a coequality relation on $S$ such that

$$(\forall a, b, x, y \in S)((ax, by) \in q \Rightarrow (a, b) \in q \vee (x, y) \in q)$$

then we say that it is a *cocongruence* ([9]) on $S$ or it is *compatible with the semigroup operation.* If $q$ is a cocongruence on a semigroup $S$ then the relation $q'$ is a congruence on $S$ and we can construct the quotient-semigroups $S/(q', q)$ and $S/q$ where

$$aq' = bq' \Leftrightarrow (a, b) \# q, aq' \neq bq' \Leftrightarrow (a, b) \in q, aq' \cdot bq' = abq'$$

$$aq = bq \Leftrightarrow (a, b) \# q, aq \neq bq \Leftrightarrow (a, b) \in q, aq \cdot bq = abq.$$

There exists a strongly extensional and embedding isomorphism $S/(q', q) \rightarrow S/q$. An element $a$ of $S$ is an idempotent of $S$ iff $a^2 = a$. Let $E(S) = \{a \in S : a^2 = a\}$. If $E(S) = S$, then $S$ is a *band.* A band congruence of semigroup was first defined and studied by T.Tamura and N.Kimura 1955. A band $S$ is a *right zero band* iff $(\forall a, b \in S)(ab = b)$. A cocongruence $q$ on $S$ is a *right zero band cocongruence* on $S$ iff $S/(q', q)$ is a right zero band. The first general result on zero band congruence is due to Dubriel ([6]) and Therrien ([12]).

For undefined notions and notations in the semigroup theory we refer to the works [2-6,8,12] and in the constructive mathematics to the books [1,7,11,13] and the paper [9].

Algebraic structures with apartness were first defined and studied by A.Heyting in 1936. After that, several authors have worked on this very important topic as for example Troelstra and van Dalen ([13), Johnstone (1977), Mulvey (1974), W.Ruitenburg (1982) and the author in several of his papers. There are more general problems on semigroups in constructive mathematics. In this paper we shall address only these two issues:

(i) *Is there a maximal right zero band coequality relation $q$ on $S$ compatible with the semigroup operation on $S$ such that $q \subseteq \neq$ ?*

(ii) *If such $q$ exists, how to describe its classes.*

If $q$ is a right zero band cocongruence on $S$, then the class $aq$ of $q$ generated by an element $a$ is a strongly extensional right consistent left ideal of $S$ such that $a \# aq$. We will give the construction of a maximal right zero band cocongruence on the semigroup $S$ with apartness by using the principal left consistent subsets of semigroup $S$ and we will prove that its classes are the maximal strongly extensional right consistent left ideals of $S$.

## 2

Now, we introduce the notion of the principal right consistent subset of semigroup with apartness, the relation defined by these sets, and describe their basic characteristics. We start with the following theorem without proof.

**Theorem 1.** *Let $a$ be an element of $S$. Then the set $L_{(a)} = \{x \in S : x \# Sa\}$ is a right consistent subset of $S$ such that:*

*(1.1)* $a \# L_{(a)}$;

*(1.2)* $L_{(a)} \neq \emptyset \Rightarrow 1 \in L_{(a)}$;

*(1.3) Let $a$ be an inevitable element of $S$. Then $L_{(a)} = \emptyset$.*

*(1.4)* $(\forall x \in S)(L_{(a)} \subseteq L_{(xa)})$;

*(1.5)* $(\forall n \in N)(L_{(a)} \subseteq L_{(a^n)})$;

*(1.6)* $(\forall x \in S)\neg(xa \in L_{(a)})$;

*(1.7)* $(\forall n \in N)\neg(a^n \in L_{(a)})$.

We introduce the following relation

$$(a, b) \in p \Leftrightarrow b \in L_{(a)} \wedge a \in L_{(b)}.$$

This relation has the following properties:

**Theorem 2.** *The relation $p$ is a consistent symmetric relation on $S$ such that*

$$(\forall a, b, y \in S)((ay, by) \in p \Rightarrow (a, b) \in p).$$

Let $c(p) = \bigcap_{n \in N} {}^{n}p$ be the cotransitive closure ([9]) of $p$. Then we will give the first of the main results of this paper in the following statement:

**Theorem 3.** *The relation $q = c(p)$ is a right zero band cocongruence on $S$.*

*Proof.*

*(i)* It is clear that the relation $q$ is a coequality relation on semigroup $S$.

*(ii)* Let $(u, v)$ be an arbitrary element of $q$ and let $a$ be an element of $S$. Then

$$\begin{aligned} (u, v) \in q &\Rightarrow (u, a^2) \in q \vee (a^2, a) \in q \vee (a, v) \in q \\ &\Rightarrow u \neq a^2 \vee (a^2, a) \in p \vee a \neq v \\ &\Rightarrow (u, v) \neq (a^2, a) \vee (a^2 \in L_{(a)} \wedge a \in L_{(a^2)}) \\ &\Rightarrow (u, v) \neq (a^2, a) \qquad \text{(by (1.7))}. \end{aligned}$$

*(iii)* Let $(u, v)$ be an arbitrary element of $q$ and let $a$ and $b$ be elements of $S$. Then

$$\begin{aligned} (u, v) \in q &\Rightarrow (u, ab) \in q \vee (ab, b) \in q \vee (b, v) \in q \\ &\Rightarrow u \neq ab \vee (ab, b) \in p \vee b \neq v \\ &\Rightarrow (u, v) \neq (ab, b) \vee (ab \in L_{(b)} \wedge b \in L_{(ab)}) \\ &\Rightarrow (u, v) \neq (ab, b) \qquad \text{(by (1.6))}. \end{aligned}$$

*(iv)*

$$\begin{aligned}(ay,by)\in q &\Rightarrow (ay,by)\in p\\ &\Rightarrow (a,b)\in p.\end{aligned}$$

Suppose that for $n\in N$ holds

$$(ay,by)\in {}^{n+1}p \Rightarrow (a,b)\in {}^{n}p.$$

Then

$$\begin{aligned}(ay,by)\in q &\Rightarrow (ay,by)\in {}^{n+1}p\\ &\Rightarrow (\forall s\in S)((ay,sy)\in {}^{n}p \vee (sy,by)\in p)\\ &\Rightarrow (\forall s\in S)((a,s)\in {}^{n}p \vee (s,b)\in p)\\ &\Rightarrow (a,b)\in {}^{n+1}p.\end{aligned}$$

Thus by induction we have

$$(ay,by)\in q \Rightarrow (a,b)\in q.$$

*(v)*

$$\begin{aligned}(xa,xb)\in q &\Rightarrow (xa,a)\in q \vee (a,b)\in q \vee (b,xb)\in q\\ &\Rightarrow (xa,a)\in p \vee (a,b)\in q \vee (b,xb)\in p\\ &\Rightarrow \left(\begin{array}{l}(xa\in L_{(a)} \wedge a\in L_{(xa)}) \vee (a,b)\in q\\ \vee(b\in L_{(xb)} \wedge xb\in L_{(b)})\end{array}\right)\\ &\Rightarrow (a,b)\in q \qquad \text{by (1.6))}\Box\end{aligned}$$

## 3

For an element $a$ of a semigroup $S$ and for $n\in N$ we introduce the following notations

$$A_n(a)=\{x\in S:\ (a,x)\in {}^{n}p\},\ A(a)=\{x\in S:\ (a,x)\in c(p)\}.$$

By the following results we will present some basic characteristics of $q$-class $aq=A(a)$ generated by the element $a$.

**Theorem 4.** *Let $a$ be an element of $S$ and $n \in N$. Then:*

*(4.1)* $A_1(a) = \{x \in L_{(a)} : a \in L_{(x)}\}$.

*(4.2)* $A_{n+1}(a) \subseteq A_n(a)$.

*(4.3)* $x \in A_{n+1}(a) \Leftrightarrow S = A_n(a) \bigcup A_1(x)$.

*(4.4)* $A(a) = \bigcap_{n \in N} A_n(a)$.

*(4.5)* $a \# A(a)$.

*(4.6)* $A(a^2) = A(a)$.

*(4.7)* $(\forall b \in S)(A(ab) = A(b))$.

*Proof.*

(3)

$$\begin{aligned} x \in A_{n+1}(a) &\Leftrightarrow (a, x) \in {}^{n+1}p \\ &\Leftrightarrow (\forall s \in S)((a, s) \in {}^{n}p \vee (s, x) \in p) \\ &\Leftrightarrow (\forall s \in S)(s \in A_n(a) \vee s \in A_1(x)) \\ &\Leftrightarrow S = A_n(a) \cup A_1(x) \end{aligned}$$

(6)

$$\begin{aligned} x \in A(a^2) &\Leftrightarrow (a^2, x) \in q \\ &\Rightarrow (a^2, a) \in q \vee (a, x) \in q \\ &\Rightarrow x \in A(a) \\ &\Leftrightarrow (a, x) \in q \\ &\Rightarrow (a, a^2) \in q \vee (a^2, x) \in q \\ &\Rightarrow (a^2, x) \in q \\ &\Leftrightarrow x \in A(a^2). \end{aligned}$$

(7)

$$x \in A(ab) \Leftrightarrow (ab, x) \in q$$

$$\begin{aligned}
&\Rightarrow && (ab, b) \in q \vee (b, x) \in q \\
&\Rightarrow && x \in A(b) \\
&\Leftrightarrow && (b, x) \in q \\
&\Rightarrow && (b, ab) \in q \vee (ab, x) \in q \\
&\Rightarrow && (ab, x) \in q \\
&\Leftrightarrow && x \in A(ab). \qquad \square.
\end{aligned}$$

In the following theorem we will give the third of the main results of this paper.

**Theorem 5.** *The set $A(a)$ is a strongly extensional maximal right consistent left ideal of $S$.*

*Proof.*

*(i)* It is easy to see that $A(a)$ is a strongly extensional subset of $S$.

*(ii)* Let $xy \in A(a)$, i.e. let $(a, xy) \in q$. Then, by cotransitivity of $q$, $(a, xa) \in q \vee (xa, xy) \in q$. If $(a, xa) \in q$ we will have $(a, xa) \in p$, which means $xa \in L_{(a)}$ and $a \in L_{(xa)}$. The last is impossible by (1.6). So, it must be $(xa, xy) \in q$. Hence, $(a, y) \in q$, i.e. $y \in A(a)$. Therefore, the set $A(a)$ is a right consistent subset of $S$.

*(iii)* Suppose that $y \in A(a)$, i.e. $(a, y) \in q$. Thus $(a, xy) \in q$ or $(xy, y) \in q$. As the last is impossible, we have $xy \in A(a)$. So, the set $A(a)$ is a left ideal of $S$.

*(iv)* Let $T$ be a strongly extensional right consistent left ideal of $S$ such that a $\# T$. Let $t$ be an arbitrary element of $T$ and let $u, v \in S$. Then $t \neq ua \vee ua \in T$. As $ua \in T$ is impossible because $T$ is a right consistent subset of $S$, we have $t \# Sa$. So, $T \subseteq \{x \in S : x \# Sa\} = L_{(a)}$. Besides, as $T$ is a left ideal of $S$, we have $a \# T \supseteq ST$. Thus $a \in L_{(t)}$ for every $t$ in $T$. Therefore, we have $T \subseteq A_1(a)$. Assume that $T \subseteq A_n(a)$. Let $t$ be an arbitrary element of $T$ and let $z$ be an arbitrary element of $S$. Then $uz \in T$ or $uz \neq t$ for every $u$ in $S$ because $T$ is a strongly extensional subset of $S$. Thus $z \in A_n(a)$ or $t \in L_{(z)}$ because $T$ is a right consistent subset of $S$ and $A_n(a) \supseteq T$. Similarly, we have $z \in T \vee z \neq ut$ for every $u$ in $S$. Thus $z \in A_{(n)}(a)$ or $z \in L_{(t)}$. Hence,

we have $z \in A_n(a) \vee z \in A_1(t)$. So, $S = A_n(a) \cup A_1(t)$, i.e. $t \in A_{n+1}(a)$. By induction, we have $T \subseteq A(a)$. Therefore, the set $A(a)$ is a maximal strongly extensional right consistent left ideal of $S$ such that $a \# A(a)$. □

As a corollary of the above theorem we have the fourth main result of this paper

**Theorem 6.** *The relation $q$ is the maximal right zero band coequality relation compatible with the semigroup operation on $S$.*

*Proof.* Let $g$ be a right zero band cocongruence on $S$ and let $a$ be an arbitrary element of $S$. Then $ag$ is a strongly extensional right consistent left ideal of $S$ such that $a \# ag$. So, $ag \subseteq A(a) = aq$ because $A(a)$ is a maximal strongly extensional right consistent left ideal of $S$ such that $a \# A(a)$. Thus, by theorem 2.3 in [9], $g \subseteq q$. Therefore, the relation $q$ is a maximal right zero band cocongruence on $S$. □

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