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EQUIVALENCE CO-RELATIONS AND CO-CONGRUENCES OF CO-ALGEBRAS

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Abstract. This paper introduces the notions of equivalence co-relation and co-congruence of a co-algebra (understood in "the classical" sense). We show that equivalence co-relations correspond to equivalence relations (and therefore can be understood as encodings of the latter), while co-congruences of co-algebras correspond in the same way to bisimulation equivalences.

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1. Introduction

The notions of a co-operation and of a co-algebra understood as a set endowed with a set of co-operations were introduced in [2] as follows.

Let $X^{\sqcup n} := \underline{n} \times X$ denote the union of n disjoint copies of X, i.e. the *n-th co-power* of X (where $\underline{n} := \{1, 2, \dots, n\}$). An n-ary *co-operation* is any mapping $f : X \to X^{\sqcup n}$. We say that n is the arity of f and write $n = \mathsf{ar}(f)$. A *co-algebra* (in the "classical" sense) is a pair $\langle X, F\rangle$, where X is a nonempty set and F is a set of co-operations of various arities.

We say that a co-operation $f : X \to X^{\sqcup n}$ is *compatible* with a binary relation $\varrho \subseteq X^2$ if the following holds for every $\langle x, y\rangle \in \varrho$:

$$\text{if } f(x) = \langle j, p\rangle \text{ and } f(y) = \langle l, q\rangle \text{ then } j = l \text{ and } \langle p, q\rangle \in \varrho.$$

A binary relation $\varrho \subseteq X^2$ is a *bisimulation* on a co-algebra $\langle X, F\rangle$ (see [4]) if every co-operation in F is compatible with ϱ. *Bisimulation equivalence* is an equivalence relation on X that is a bisimulation on $\langle X, F\rangle$.

Objects called co-relations were introduced in [4] as the "relational counterpart" for clones of co-operations introduced in [1]. For a positive integer n, an *n-ary co-vector on (or colouring of) X* is any mapping $\mathbf{r} : X \to \underline{n}$. An *n-ary co-relation* is a set of n-ary co-vectors. Let $\mathbf{r} : X \to \underline{n}$ be a co-vector and let $A_i := \mathbf{r}^{-1}(i)$, $i \in \underline{n}$. Instead of $\mathbf{r}$ we shall often write $\langle A_1, \dots, A_n\rangle^{\nabla}$.

For a nonempty set A and mappings $g_1, \dots, g_n : X \to A$ let $[g_1, \dots, g_n]$ denote the *co-tupling* of $g_1, \dots, g_n$:

$$[g_1, \dots, g_n] : X^{\sqcup n} \to A : \langle i, x\rangle \mapsto g_i(x).$$

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Let f be an n-ary co-operation on X and let ϱ be a co-relation on X. We say that f *co-preserves* ϱ or that ϱ is *co-invariant* under f [4] if for every n co-vectors $\mathbf{r}^1, \ldots, \mathbf{r}^n \in \varrho$ we have $f \cdot [\mathbf{r}^1, \ldots, \mathbf{r}^n] \in \varrho$ (where the composition $f \cdot g$ of mappings $f : A \to B$ and $g : B \to C$ is taken to be $(f \cdot g)(x) := g(f(x))$, i.e. first f then g). Given a set F of co-operations on X, let $\mathsf{cInv}_X\, F$ denote the set of all co-relations (of all finite arities) that are co-invariant under every $f \in F$.

Bisimulation equivalences play the role of congruences in the structural theory of co-algebras. Thus, essentially covariant notion of a bisimulation equivalence is used in the analysis of essentially contravariant notion of co-algebra. The aim of this paper is to introduce equivalence co-relations (as some kind of a dual to equivalence relations) and co-congruences, and thus provide a contravariant tool in the analysis of co-algebras.

2. Equivalence co-relations

In the sequel we focus on ternary co-relations. Let us denote the set of all ternary co-relations on a set X by $\mathsf{cRel}_X^{(3)}$.

Note that one of the coordinates of a co-vector is superfluous. Therefore, by $\langle A, \bullet, C\rangle^\nabla$ we shall denote the co-vector $\langle A, X \setminus (A \cup C), C\rangle^\nabla$. For $x \in X$ and $A \subseteq X$, instead of $\langle \{x\}, \bullet, A\rangle^\nabla$ we shall simply write $\langle x, \bullet, A\rangle^\nabla$. So, for $x, y \in X$, $\langle x, \bullet, y\rangle^\nabla$ stands for $\langle \{x\}, X \setminus \{x, y\}, \{y\}\rangle^\nabla$.

The composition of co-relations and inversion of ternary co-relations were introduced in [3] as follows. For ternary co-relations σ, σ' we define σ^{-1} and $\sigma \circ \sigma'$ by:

$$\begin{aligned} \sigma^{-1} &:= \{\langle C, \bullet, A\rangle^\nabla \mid \langle A, \bullet, C\rangle^\nabla \in \sigma\} \\ \sigma \circ \sigma' &:= \{\langle A, \bullet, C'\rangle^\nabla \mid (\exists C, A' \subseteq X)(\langle A, \bullet, C\rangle^\nabla \in \sigma \wedge \\ &\qquad \langle A', \bullet, C'\rangle^\nabla \in \sigma' \wedge C = X \setminus A')\}. \end{aligned}$$

In [3] it is also shown that the set of ternary co-relations forms a monoid with respect to composition and, moreover, that the semigroup of reflexive binary relations can be embedded into the semigroup reduct of the monoid of ternary co-relations (for proof, see [3, Proposition 3.58]):

Proposition 2.1. *Let $\Delta_X := \{\langle x, x\rangle \mid x \in X\}$, and let $R_X := \{\varrho \subseteq X^2 \mid \Delta_X \subseteq \varrho\}$ denote the set of all reflexive binary relations on X. Further, let $\mathbf{S}_X := \langle R_X, \circ, ^{-1}\rangle$ denote the semigroup of reflexive binary relations with the usual inversion. Then the mapping $\varphi : R_X \to \mathsf{cRel}_X^{(3)}$ defined by*

$$\varphi(\varrho) = \{\langle A, \bullet, B\rangle^\nabla \mid \varrho \cap (A \times B) = \emptyset\}.$$

is an embedding of $\mathbf{S}_X$ into $\langle \mathsf{cRel}_X^{(3)}, \circ, ^{-1}\rangle$.

In order to introduce equivalence co-relations, we shall first introduce *full* co-relations, which play the role of reflexive relations.

Definition 2.2. *A co-relation* $\sigma \in \mathsf{cRel}_X^{(3)}$ *is said to be* full *if the following implication holds for all* $A, C \subseteq X$*:*

$$\Big(A \cap C = \emptyset \text{ and } A \times C \subseteq \bigcup\nolimits_{\langle P,\bullet,R\rangle^\nabla \in \sigma} (P \times R)\Big) \Rightarrow \langle A, \bullet, C\rangle^\nabla \in \sigma.$$

Let us mention some elementary properties of full ternary co-relations which we shall need later.

Lemma 2.3. *Let* σ *be a full ternary co-relation.*

(i) $\langle X, \bullet, \emptyset\rangle^\nabla, \langle \emptyset, \bullet, X\rangle^\nabla \in \sigma$.
(ii) *If* $\langle A, \bullet, C\rangle^\nabla \in \sigma$ *and* $x \in A$, $y \in C$ *then* $\langle x, \bullet, y\rangle^\nabla \in \sigma$.

Definition 2.4 *A co-relation* $\sigma \in \mathsf{cRel}_X^{(3)}$ *is called an* equivalence co-relation *if*

- σ *is full,*
- σ *is symmetric:* $\sigma^{-1} = \sigma$, *and*
- σ *is transitive:* $\sigma \circ \sigma = \sigma$.

Let ϱ be an equivalence relation on X and let $E_\varrho := \{\{x,y\} \mid \langle x,y\rangle \in \varrho\}$. Connectedness components of $G_\varrho := \langle X, E_\varrho\rangle$, the graph of ϱ, are complete graphs with loops. More precisely, $G_\varrho = K^\circ_{n_1} \oplus \ldots \oplus K^\circ_{n_l}$, where $\oplus$ denotes the disjoint union of graphs, and K°_n is the complete graph on n vertices with loops. Components of G_ϱ are generally referred to as blocks of ϱ.

We are going to show that the situation concerning equivalence co-relations is dual. As we shall see in the next proposition, the graph G_σ assigned to an equivalence co-relation σ in the similar fashion turns out to be a complete multipartite graph.

Proposition 2.5. *Let* $\sigma \in \mathsf{cRel}_X^{(3)}$ *be an equivalence co-relation and let* $E_\sigma := \{\{x,y\} \mid \langle x, \bullet, y\rangle^\nabla \in \sigma\}$. *Then* $G_\sigma := \langle X, E_\sigma\rangle$ *is a complete multipartite graph.*

Proof. Fix an equivalence co-relation σ. By $u \sim v$ we denote that the vertices $u, v \in X$ are adjacent in G_σ. For $u \in X$ let $N(u) := \{v \in X \mid v \sim u\}$ stand for the neighbour set of u in G_σ. To show that G_σ is a complete multipartite graph it suffices to show that $u \not\sim v \Rightarrow N(u) = N(v)$.

Suppose $u \not\sim v$ and take any $w \in N(u)$. Then $\langle u, \bullet, w\rangle^\nabla \in \sigma$. Since $\sigma = \sigma \circ \sigma$, there exists an $A \subseteq X$ such that $\langle u, \bullet, A\rangle^\nabla \in \sigma$ and $\langle X \setminus A, \bullet, w\rangle^\nabla \in \sigma$. Now $u \not\sim v$ and Lemma 2.3, (ii), imply $v \notin A$. So, $v \in X \setminus A$. Applying Lemma 2.3, (ii), again we obtain $\langle v, \bullet, w\rangle^\nabla \in \sigma$ i.e. $w \in N(v)$. This proves $N(u) \subseteq N(v)$. The other inclusion follows analogously. □

Every complete multipartite graph induces a partition on the set of its vertices. Blocks in the partition induced by G_σ shall be referred to as blocks of σ, as stipulated by the next definition.

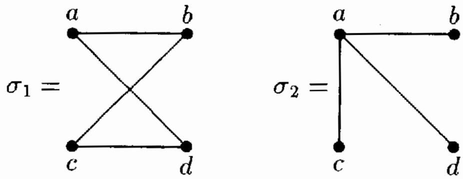

Figure 1: Two equivalence co-relations on $\{a, b, c, d\}$

Definition 2.6. *Let $\sigma \in \mathsf{cRel}_X^{(3)}$ be an equivalence co-relation and let $\mathcal{B} := \{B_1, \ldots, B_k\}$ be the partition of X induced by G_σ. We say that $\mathcal{B}$ is the* partition induced by σ *and refer to B_j's as* blocks of σ.

In analogy with equivalence relations, partition $\mathcal{B}$ shall be denoted by X/σ. For $x \in X$, the unique block of σ containing x shall be denoted by x/σ.

The partition X/σ induces an equivalence relation on X which shall be denoted by $\mathsf{rel}(\sigma)$.

The operator rel has some important properties which we list in the following proposition.

Proposition 2.7. *Let σ be an equivalence co-relation. Then:*

(i) *for all $x \in X$ we have $x/\sigma = x/\mathsf{rel}(\sigma)$;*

(ii) $\mathsf{rel}(\sigma) = X^2 \setminus \bigcup_{\langle P, \bullet, R\rangle^\nabla \in \sigma} (P \times R)$.

Also, if ϱ is a binary reflexive relation then $\varphi(\varrho)$ is full, and $\mathsf{rel}\,\varphi(\varrho) = \varrho$.

Remarks 2.8. (i) $\varphi(\mathbf{S}_X)$ (see Proposition 2.1) is exactly the set of all full ternary co-relations.

(ii) If we denote by $\mathsf{Eq}(X)$ the set of all binary equivalence relations on X and by $\mathsf{cEq}(X)$ the set of all ternary equivalence co-relations on X, then φ is an isomorphism between the partially ordered sets $\langle \mathsf{Eq}(X), \subseteq\rangle$ and $\langle \mathsf{cEq}(X), \supseteq\rangle$.

(iii) The set of all ternary equivalence co-relations is a bounded partially ordered set with respect to $\subseteq$, $\mathsf{c}\Delta_X := \{\langle A, \bullet, B\rangle^\nabla \mid A = \emptyset \text{ or } B = \emptyset\}$ and $\mathsf{c}\nabla_X := \{\langle A, \bullet, B\rangle^\nabla \mid A \cap B = \emptyset\}$ being the least and the the greatest element, respectively.

(iv) The intersection and the union of two equivalence co-relations need not be an equivalence co-relation. Consider the co-relations σ_1 and σ_2 depicted in Fig. 1 via the corresponding (complete multipartite) graphs. $\sigma_1 \cup \sigma_2$ is not full, while $\sigma_1 \cap \sigma_2$ is not transitive. Therefore, neither is an equivalence co-relation.

3. Equivalence co-relations as co-congruences of co-algebras

A congruence of a universal algebra $\langle A, F\rangle$ is an equivalence relation invariant under all the fundamental operations. This motivates the following definition.

Definition 3.1. *Let* $\mathbf{A} := \langle X, F\rangle$ *be a co-algebra.* $\sigma \in \mathsf{cRel}_X^{(3)}$ *is said to be a* co-congruence *of* $\mathbf{A}$ *if*

- σ *is an equivalence co-relation, and*
- $\sigma \in \mathsf{cInv}_X F$.

Denote by $\mathsf{cCon}\,\mathbf{A}$ *the set of all co-congruences of* $\mathbf{A}$.

Having the notion of co-congruence at hand, we would now like to show how to factor co-algebras by co-congruences and then to show that factoring by co-congruences corresponds to factoring by bisimulation equivalences.

Lemma 3.2. *Let* σ *be a co-congruence of the co-algebra* $\langle X, F\rangle$ *and let* $f \in F$. *For all* $x, x', y, y' \in X$ *and* $j, j' \in \mathsf{ar}(f)$ *the following holds: if* $x/\sigma = x'/\sigma$, $f(x) = \langle j, y\rangle$ *and* $f(x') = \langle j', y'\rangle$, *then* $j = j'$ *and* $y/\sigma = y'/\sigma$.

Proof. For a co-operation $f : X \to X^{\sqcup n}$ and $j \in \underline{n}$, denote by f^j the partial mapping $X \multimap X$ defined by $f^j(x) = y \iff f(x) = \langle j, y\rangle$.

Suppose first that $j \neq j'$, say, $j = 1$ and $j' = 2$. Let $\mathbf{r} := \langle X, \bullet, \emptyset\rangle^\nabla$, $\mathbf{s} := \langle \emptyset, \bullet, X\rangle^\nabla$ and $\mathbf{t} := f \cdot [\mathbf{r}, \mathbf{s}, \ldots, \mathbf{s}]$. Since $\mathbf{r}, \mathbf{s} \in \sigma$ (Lemma 2.3, (i)) and $\sigma \in \mathsf{cInv}_X\{f\}$, we have $\mathbf{t} \in \sigma$. Let $\mathbf{t} =: \langle P, \bullet, Q\rangle^\nabla$. Then $y \in X$ implies $(f^1)^{-1}(y) \subseteq P$. So, $x \in P$ because $f^1(x) = y$. Similarly, $y' \in X$ implies $(f^2)^{-1}(y') \subseteq Q$ and thus, $x' \in Q$. Now, $x \in P$, $x' \in Q$ and $\langle P, \bullet, Q\rangle^\nabla \in \sigma$, so, by Lemma 2.3, (ii), we have $\langle x, \bullet, x'\rangle^\nabla \in \sigma$. But $\langle x, \bullet, x'\rangle^\nabla \in \sigma$ means that $x/\sigma \neq x'/\sigma$. Contradiction. Therefore, $j = j'$.

In order to show the other part of the statement, suppose to the contrary that $y/\sigma \neq y'/\sigma$. Then $\langle y, \bullet, y'\rangle^\nabla \in \sigma$. As above, let $\mathbf{r} := \langle X, \bullet, \emptyset\rangle^\nabla$ and let $\mathbf{t} := \langle P, \bullet, Q\rangle^\nabla := f \cdot [\mathbf{r}, \mathbf{r}, \ldots, \mathbf{r}]$. Since $\mathbf{r} \in \sigma$ and $\sigma \in \mathsf{cInv}_X\{f\}$, we have $\mathbf{t} \in \sigma$. But, $x \in P$, $x' \in Q$, $\mathbf{t} \in \sigma$ and Lemma 2.3, (ii), imply $\langle x, \bullet, x'\rangle^\nabla \in \sigma$ i.e. $x/\sigma \neq x'/\sigma$. Contradiction. □

Definition 3.3. *Let* $\mathbf{A} = \langle X, F\rangle$ *be a co-algebra and let* $\sigma \in \mathsf{cCon}\,\mathbf{A}$. *For* $f \in F$, $\mathsf{ar}(f) = n$, *define* $\overline{f} : X/\sigma \to (X/\sigma)^{\sqcup n}$ *by:* $\overline{f}(x/\sigma) := \langle j, y/\sigma\rangle$ *where* $\langle j, y\rangle := f(x)$. *Let* $\overline{F} := \{\overline{f} \mid f \in F\}$. *The co-algebra* $\mathbf{A}/\sigma := \langle X/\sigma, \overline{F}\rangle$ *is referred to as the* factor co-algebra *of* $\mathbf{A}$.

Let us note that Lemma 3.2 ensures that the above definition is correct. The following theorem, which follows immediately from Lemma 3.2, settles the question of the relationship of bisimulation equivalences and co-congruences of a co-algebra.

Theorem 3.4. *Let* $\mathbf{A} := \langle X, F\rangle$ *be a co-algebra and let* $\sigma \in \mathsf{cCon}\,\mathbf{A}$. *Then:*

(i) $\mathsf{rel}(\sigma)$ *is a bisimulation equivalence of the transition system* $\langle X, F\rangle$.

(ii) *If* R *is a bisimulation equivalence of the transition system* $\langle X, F\rangle$ *and* $\sigma := \varphi(R)$ *(see Proposition 2.1), then* $\sigma = {}^{\bullet}R$ *up to a permutation of coordinates (for the definition of* ${}^{\bullet}R$ *see Definition 6.5 in [4]).*

(iii) *The poset* $\langle \mathsf{cCon}\,\mathbf{A}, \subseteq\rangle$ *is isomorphic to the poset* $\langle \mathsf{BsEq}\,\mathbf{A}, \supseteq\rangle$, *where* $\mathsf{BsEq}\,\mathbf{A}$ *denotes the set of all of bisimulation equivalences of the transition system* $\mathbf{A}$.

Finally, let us remark that co-congruences exibit quite a peculiar behaviour. We say that a co-operation $f : X \to X^{\sqcup n}$ depends essentially on exactly one argument if there exists a $j \in \underline{n}$ and a function $g : X \to X$ such that for all $x \in X$, $f(x) = \langle j, g(x)\rangle$. We say that a co-operation $f : X \to X^{\sqcup n}$ depends on at least two arguments if it is not true that f depends essentially on exactly one argument.

Proposition 3.5 *Let* $\mathbf{A} := \langle X, F\rangle$ *be a co-algebra.* $\mathsf{c}\Delta_X \in \mathsf{cCon}\,\mathbf{A}$ *if and only if every* $f \in F$ *depends essentially on exactly one argument.*

Proof. $\Leftarrow$: Obvious.

$\Rightarrow$: Suppose that there exists an $f \in F$ which depends on at least two arguments, say, the first two, and suppose that $\mathsf{c}\Delta_X \in \mathsf{cCon}\,\mathbf{A}$. Let $\mathbf{r} := \langle X, \bullet, \emptyset\rangle^{\nabla}$, $\mathbf{s} := \langle \emptyset, \bullet, X\rangle^{\nabla}$, and $\langle P, \bullet, Q\rangle^{\nabla} := f \cdot [\mathbf{r}, \mathbf{s}, \ldots, \mathbf{s}]$. Then $\langle P, \bullet, Q\rangle^{\nabla} \in \mathsf{c}\Delta_X$. But, both P and Q are nonempty. Contradiction. □

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