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UNSTEADY INCOMPRESSIBLE BOUNDARY LAYER EQUATION IN FULL TWO AND TWO-ONCE LOCALIZED PARAMETRIC APPROXIMATION

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Abstract. The corresponding equations of unsteady boundary layer, by introducing the appropriate variable transformations, momentum and energy equations and one similarity parameter set, are transformed into generalized partial nonlinear differential equation. These parameters express the influence of the outer flow velocity, and the flow history in the boundary layer on the boundary layer characteristics. Since the equation contains the sums of terms equal to the number of parameters, it is necessary to limit the number of parameters for numerical integration. So it is very important that the chosen set of parameters possesses the following two properties:1. the first parameter is to be "strong" enough, so that the solutions lies close to the exact solution, and 2. the following parameters introduce in the solution small corrections only, and provide a sufficiently fast convergence. For this purpose, the modern parameter method has been developed to calculate boundary layers, known as generalized similarity method. The numerical integration of the generalized equation with boundary conditions has been performed by difference schemes and using Tridiagonal Algorithm Method with iterations in full two parametric approximation, where the first unsteady and dynamic parameters will remain, while all others will be let to be equal to zero, and in two-once localized parametric approximation, where also the first unsteady and dynamic parameters remain, while all others will be equal to zero and where the derivatives with respect to the first unsteady parameter will be considered equal to zero, while the derivatives with respect to the first porous parameter will be considered equal to zero. The obtained results show that for both the confuser and this diffuser regions as well as for both the accelerating and decelerating flows, there are differences between their values, especially close to the separation point of the boundary layer, and is very important for particular problem with laminar-turbulent transition region on the contour.

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1. Introduction

The multiparametric method known as generalized similarity method [1,2], is used to solve to the problem of unsteady incompressible plane boundary layer on the contour [3-6,8]. The method, regarding its own nature and properties, widely employed in the modern investigations concerning fluid mechanics. Similar solutions of the boundary layer equations play an important role in the investigation of the stability of hydrodynamic flows, developing semi-empirical criteria for the transition to turbulence, a wide application in technical practice i.e. especially in nuclear reactors, as well as in different devices in chemical technology as well as for flight control in aeronautics.

2. Mathematical problem

The mathematical model of the problem considered is described by the following equation:

$$\Psi_{ty} + \Psi_y \Psi_{xy} - \Psi_x \Psi_{yy} = U_t + UU_x + \nu \Psi_{yyy} \tag{1}$$

with the boundary and initial conditions:

$$\begin{aligned} &y = 0 : \ \Psi = \Psi_y = 0; \quad y \to \infty : \ \Psi_y \to U(x,t); \\ &t = t_0 : \ \Psi_y = u_1(x,y); \quad x = x_0 : \ \Psi_y = u_0(t,y), \end{aligned} \tag{2}$$

where: $\Psi(x,y,t)$ - stream function, $U(x,t)$ - free stream velocity; ν - kinematic viscosity; $u_1(x,y)$ - streamwise velocity distribution in the boundary layer at a given point of time $t = t_0$; $u_0(t,y)$ - streamwise velocity distribution in the boundary layer at the cross-section $x = x_0$; x - streamwise coordinate; y - cross-wise coordinate; t - time. For theoretical considerations of the given problem, it is necessary to solve equation (1), with the corresponding boundary and initial conditions (2).

3. Universal equation

Equation (1) can be solved using numerical method; in that case the function $U(x,t)$ must be known before applying the solution procedure. With such a solution of equation (1), one gets immediately the solutions of the concrete problems. The general similarity method [1,2], or method of "universalization", which has been developed for the problems of steady plane boundary layer, is also extended to unsteady boundary layer problems [3-6,8]. In this paper, the method of generalized similarity has been developed using also some ideas which are present in the appropriate method for steady problems [2,7].

According to the generalized similarity method [5,6,8], the new variables $x, t, \eta, \Phi(x,\eta,t)$ are introduced by the following relations:

$$x = x, \ \ t = t, \ \ \eta = yU^{b_0/2}\left(a_0 \nu \int_0^x U^{b_0 - 1} dx\right)^{-1/2},$$

$$\Phi(x,\eta,t) = \Psi U^{b_0/2-1}\left(a_0\nu\int_0^x U^{b_0-1}dx\right)^{-1/2} \tag{3}$$

where $a_0 = 0.4408$, $b_0 = 5.714$ [2], so that equation (1) can be reduced to the form:

$$\Phi_{\eta\eta\eta} + \left(a_0 U^{-b_0}\int_0^x U^{b_0-1}dx\right)U_x\left[1-\Phi_\eta^2 + (1-b_0/2)\Phi\Phi_{\eta\eta}\right] + \frac{a_0}{2}\Phi\Phi_\eta +$$

$$\left(a_0 U^{-(b_0-1)}\int_0^x U^{b_0-1}dx\right)U_t(1-\Phi_\eta) + \Phi_{\eta\eta}\frac{\eta}{2B^2}T^{**} = \tag{4}$$

$$\left(a_0 U^{-b_0}\int_o^x U^{b_0-1}dx\right)\Phi_{t\eta} + \frac{z^{**}}{B^3}\eta B_t\Phi_{\eta\eta} + \left(a_0 U^{1-b_0}\int_0^x U^{b_0-1}dx\right)(\Phi_\eta\Phi_{\eta x} - \Phi_x\Phi_{\eta\eta})$$

where the corresponding boundary conditions and are as follows:

$$\eta = 0: \ \Phi = \Phi_\eta = 0; \quad \eta\to\infty: \ \Phi_\eta \to 1, \tag{5}$$

$$(6) z^{**} = \left(a_0 U^{-b_0}\int_0^x U^{b_0-1}dx\right)B; \ \ B(x,t) = \int_0^\infty \Phi_\eta(1-\Phi_\eta)d\eta; \ \ T^{**} = z_t^{**}.$$

It can be noticed, that the conditions with respect to the variables x and t in (2), are not taken into account. These conditions are significant only in the calculation of the concrete problems, and thus they can be omitted here. By assuming that the the function $U(x,t)$ is analytical, one can introduce the set of parameters [5,6,8]:

$$f_{k,n} = U^{k-1}U^{(k+n)}_{x^{(k)}t^{(n)}}z^{**k+n} \quad (k,n = 1,2,3\ldots;\ k\vee n \neq 0) \tag{7}$$

as new independent variables, from which we have the first parameters:

$$f_{1,0} = z^{**}U_x; \quad f_{0,1} = U^{-1}z^{**}U_t \tag{8}$$

The parameters (7) express the influence of the outer flow velocity and the flow history in the boundary layer, on boundary layer characteristics.

Now, the already transformed equation (4) is transformed into a new form:

$$B^2\Phi_{\eta\eta\eta} + 0.5[a_0B^2 + (2-b_0)f_{1,0}]\Phi\Phi_{\eta\eta} + f_{1,0}(1-\Phi_\eta^2) + f_{0,1}(1-\Phi_\eta) + 0.5\eta T^{**}\Phi_{\eta\eta}$$

$$= \eta B^{-1}\sum_{\substack{k,n=0\\ k\vee n\neq 0}}^{\infty} C_{k,n}B_{f_{k,n}}\Phi_{\eta\eta} + \left\{\sum_{\substack{k,n=0\\ k\vee n\neq 0}}^{\infty}\left[C_{k,n}\Phi_{\eta f_{k,n}} + A_{k,n}(\Phi_\eta\Phi_{\eta f_{k,n}} - \Phi_{f_{k,n}}\Phi_{\eta\eta})\right]\right\}$$

(9)

with the corresponding boundary conditions:

$$\eta = 0: \Phi = \Phi_\eta; \ \ n\to\infty: \Phi_\eta \to 1; \ \ f_{k,n} = 0 \ \ (k = 0,1,..,;\ k\vee n \neq 0): \Phi = \Phi_0(\eta),$$

(10)

where $\Phi_0(\eta)$ is Blausius's solution for the problem of flat plate. In equation (9) the following notations have been used:

$$(11)\qquad \begin{aligned} A_{k,n} &= (k-1)f_{1,0}f_{k,n} + f_{k+1,n} + (k+n)f_{k,n}\overset{**}{\mathrm{F}};\\ C_{k,n} &= (k-1)f_{0,1}f_{k,n} + f_{k,n+1} + (k+n)f_{k,n}\overset{**}{\mathrm{T}};\quad \overset{**}{\mathrm{F}} = Uz_x^{**} \end{aligned}$$

In order to take equation (9) as universal, the multipliers $\overset{**}{\mathrm{F}}$ and $\overset{**}{\mathrm{T}}$ have to be expressed by means of quantities which are explicit functions only of parameters (7). In the determination of these functions, one can use the momentum equation:

$$(12)\qquad (U\delta^*)_t + (U^2\delta^{**})_x + UU_x\delta^* - \tau_w/\rho = 0;$$

and the equation of energy:

$$(13)\qquad (U^2\delta^{**})_t + U^3\delta_{1\,x}^{**} + U^2(\delta_t^* + 3\delta_1^{**}U_x - 2\nu e) = 0$$

where

$$\delta^* = L^{1/2}\int_0^\infty (1-\Phi_\eta)d\eta;\quad \tau_w = \rho\nu U^{b_0/2+1}L^{-1/2}(\Phi_{\eta\eta})_{\eta=0};$$

$$\delta_1^{**} = L^{1/2}\int_0^\infty \Phi_\eta(1-\Phi_\eta^2)d\eta;\quad e = L^{-1/2}\int_0^\infty \Phi_{\eta\eta}^2 d\eta;\quad L = a_0\nu U^{-b_0}\int_0^x U^{b_0-1}dx.$$

(14)

Introducing the quantities:

$$(15)\qquad \begin{aligned} &H^{**} = B^{-1}\int_0^\infty (1-\Phi_\eta)d\eta;\quad H_1^{**} = B^{-1}\int_0^\infty \Phi_\eta(1-\Phi_\eta^2)d\eta;\\ &\zeta = B(\Phi_{\eta\eta})_{\eta=0};\quad \alpha = b\int_0^\infty \Phi_{\eta\eta}^2 d\eta, \end{aligned}$$

and writting in developed form the derivative with respect to t, equation (13) reduces to a new form, from which the obtained expression for the function T^{**} depends only of the parameters (7):

$$(16)\qquad \begin{aligned} T^{**} = \;& \{2[2\sum_{\substack{k,n=0\\ k\vee n\neq 0}}^{\infty}(k+n)f_{k,n}H_{1\,f_{k,n}}^{**} + H_1^{**}][\zeta - 2f_{1,0} - H^{**}(f_{1,0}+f_{0,1})\\ & - \sum_{\substack{k,n=0\\ k\vee n\neq 0}}^{\infty}((k-1)f_{0,1}f_{k,n} + f_{k,n+1})H_{f_{k,n}}^{**}] + 2[\sum_{\substack{k,n=0\\ k\vee n\neq 0}}^{\infty}((k-1)f_{1,0}f_{k,n}\\ & + f_{k+1,n})H_{1\,f_{k,n}}^{**} + \sum_{\substack{k,n=0\\ k\vee n\neq 0}}^{\infty}((k-1)f_{0,1}f_{k,n} + f_{k,n+1})H_{f_{k,n}}^{**}\\ & + 2f_{0,1}] + 6H_1^{**}f_{1,0}] - 4\alpha\}\{[H^{**} + 2\sum_{\substack{k,n=0\\ k\vee n\neq 0}}^{\infty}(k+n)f_{k,n}H_{f_{k,n}}^{**}]\\ & \cdot[2\sum_{\substack{k,n=0\\ k\vee n\neq 0}}^{\infty}(k+n)f_{k,n}H_{1\,f_{k,n}}^{**} + H_1^{**} - 1] - 1\}^{-1}. \end{aligned}$$

Also, writting in a developed form the derivative with respect to x, equation (12) is reduced to a new form, from which the expression for the function F^{**} is obtained which depends only on the parameters (7):

$$
\begin{aligned}
F^{**} = \; & 2\{\zeta - 2f_{1,0} - H^{**}(f_{1,0} + f_{0,1} + 0.5T^{**}) \\
& - \sum_{\substack{k,n=0 \\ k \vee n \neq 0}}^{\infty} [(k-1)f_{0,1}f_{k,n} + f_{k,n+1} + (k+n)T^{**}f_{k,n}]H^{**}_{f_{k,n}}\}
\end{aligned}
\tag{17}
$$

By the above procedure the existence of the functions F^{**} and T^{**} is shown, and something more, their explicit forms are determined. In equation (9), the velocity at the outer border of boundary layer and its derivatives are not involved in explicit form, thus this equation can be called the generalized, i.e. universal equation. The universal boundary conditions have the form as (10).

4. Approximative universal equations

The numerical integration of equation (9), with the corresponding universal boundary conditions (10), can be performed "once and forever" only for its approximative form. This means that the solution of the universal equation in practice needs limitation of the number of the independent variables. It leads to the necessity of applying of the "segment" method, in which all variables from someone have to be equal to zero. In such a way, the approximative universal equation is obtained. Having the above procedure in mind, the parameters $f_{1,0}$, $f_{0,1}$ will be remained, while all others will be let to be equal to zero. Equation (9), in these full two-parametric approximation has the form:

$$
\begin{aligned}
& B^2\Phi_{\eta\eta\eta} + 0.5[a_0B^2 + (2-b_0)f_{1,0}]\Phi\Phi_{\eta\eta} + f_{1,0}(1-\Phi_\eta^2) + f_{0,1}(1-\Phi_\eta) \\
& \quad +0.5\eta T^{**}\Phi_{\eta\eta} = \eta B^{-1}[T^{**}(f_{1,0}B_{f_{1,0}} + f_{0,1}B_{f_{0,1}}) - f_{0,1}^2 B_{f_{0,1}}]\Phi_{\eta\eta} \\
& +[T^{**}(f_{1,0}\Phi_{\eta f_{0,1}} + f_{0,1}\Phi_{\eta f_{0,1}}) - f_{0,1}^2\Phi_{\eta f_{0,1}} + f_{1,0}F^{**}(\Phi_\eta\Phi_{\eta f_{1,0}} - \Phi_{f_{1,0}}\Phi_{\eta\eta}) \\
& \qquad + f_{0,1}(F^{**} - f_{1,0})(\Phi_\eta\Phi_{\eta f_{0,1}} - \Phi_{f_{0,1}}\Phi_{\eta\eta}),
\end{aligned}
\tag{18}
$$

and the corresponding boundary conditions (10) are reduced to the following:

$$
\eta = 0 : \Phi = \Phi_\eta = 0; \quad \eta \to \infty : \Phi_\eta \to 1; \quad f_{1,0} = f_{0,1} = 0 : \Phi = \Phi_0(\eta). \tag{19}
$$

where the functions T^{**} and F^{**} , after the same approximation in expressions (16) and (17), now have the following forms:

$$
\begin{aligned}
T^{**} = \; & \{2[2(f_{1,0}H^{**}_{1\,f_{1,0}} + f_{0,1}H^{**}_{1\,f_{0,1}}) + H_1^{**}][\zeta - 2f_{1,0} - H^{**}(f_{1,0} + f_{0,1}) \\
& + f_{0,1}^2 H^{**}_{f_{0,1}}] + 2[-f_{1,0}f_{0,1}H^{**}_{1\,f_{0,1}} - f_{0,1}^2 H^{**}_{f_{0,1}} + 2f_{0,1}] + 6H_1^{**}f_{1,0} - 4\alpha\} \\
& \{[H^{**} + 2(f_{1,0}H^{**}_{f_{1,0}} + f_{0,1}H^{**}_{f_{0,1}})][2(f_{1,0}H^{**}_{1\,f_{1,0}} + f_{0,1}H^{**}_{1\,f_{0,1}}) \\
& + H_1^{**} - 1] - 1\}^{-1};
\end{aligned}
$$

$$F^{**} = 2\{\zeta - 2f_{1,0} - H^{**}(f_{1,0} + f_{0,1} + 0.5T^{**}) + f_{0,1}^2 H^{**}_{f_{0,1}} - T^{**}(f_{1,0}H^{**}_{f_{1,0}} + f_{0,1}H^{**}_{f_{0,1}})\} \tag{20}$$

Also, we consider here approximation of equation (9) in two parametric "once localized" enviroment, and this is done when the first unsteady $f_{0,1}$ and dynamic $f_{1,0}$ parameters also remain, while all others will be equal to zero, and the derivatives with respect to the first unsteady $f_{0,1}$ parameter will be considered equal to zero. For this approximation equation (9) has the form:

$$\begin{aligned} &B^2\Phi_{\eta\eta\eta} + 0.5[a_0B^2 + (2 - b_0)f_{1,0}]\Phi\Phi_{\eta\eta} + f_{1,0}(1 - \Phi_\eta^2) \\ &+f_{0,1}(1 - \Phi_\eta) + 0.5\eta T^{**}\Phi_{\eta\eta} = \eta B^{-1}T^{**}f_{1,0}B_{f_{1,0}}\Phi_{\eta\eta} \\ &+[T^{**}f_{1,0}\Phi_{\eta f_{1,0}} + f_{1,0}F^{**}(\Phi_\eta\Phi_{\eta f_{1,0}} - \Phi_{f_{1,0}}\Phi_{\eta\eta})] \end{aligned} \tag{21}$$

and the corresponding boundary conditions (10) are the same as the conditions (19). The functions T^{**} and F^{**} after two-once localized approximation in expressions (16) and (17), have the following forms:

$$\begin{aligned} T^{**} = &\{2(2f_{1,0}H^{**}_{1\,f_{1,0}} + H_1^{**})[\zeta - 2f_{1,0} - H^{**}(f_{1,0} + f_{0,1})] + 6H_1^{**}f_{1,0} - 4\alpha\} \\ &\{(H^{**} + 2f_{1,0}H^{**}_{f_{1,0}})(2f_{1,0}H^{**}_{1\,f_{1,0}} + H_1^{**} - 1) - 1\}^{-1}; \end{aligned}$$

$$F^{**} = 2[\zeta - 2f_{1,0} - H^{**}(f_{1,0} + f_{0,1} + 0.5T^{**}) - T^{**}f_{1,0}H^{**}_{f_{1,0}}]. \tag{22}$$

The numerical integration of the generalized similarity equations (18) and (21), in full two and in two-once localized approximation with their boundary conditions (19) has been performed by means of the difference schemes and using Tridiagonal Algorithm method with iterations. The obtained results $\Phi_{\eta\eta_{\eta=0}}$, A and B of these equations, (18) and (21), can be used in the drawing general conclusions of boundary layer development and in calculation of particular problems. The results show that the unsteady parameter $f_{0,1}$ has a significiant influence on the friction distribution and especially on the location of separation point of the unsteady boundary layer. When this parameter is increasing, the friction value ζ is increasing and the separation point location is shifting toward the greater absolute values of the negative parameter $f_{1,0}$. This means, that the positive local acceleration leads to the postponing separation of the boundary layer in the diffuser region. The local deceleration favors the occurrence of separation of flow: in comparison with the steady flow, the separation is occurring at lower absolute values of the negative parameter $f_{1,0}$. For both the confuser and diffuser regions the accelerated flow around the contour increases the friction and postpones separation, and vice versa. Differences between the obtained solutions after numerical integrations of equations (18) and (21) are about 8-10% in the diffuser contour region, especially in the vicinity of the separation point of boundary layer for accelerating flow, while the differences for decelerating fluid flow in the same region is about 15%. In the conffuser region of the contour, there is no significant differences between the results obtained from (18) and (21), so one can use equation (18) to calcule the boundary layer characteristics. However, in the diffuser contour region is much better to use equation (21), because the results are more correct and there is laminar-turbulent transition of fluid flow.

5. Conclusion

This paper deals with the unsteady plane boundary layer of incompressible fluid flow on contour. A generalized similarity method is practically formed for studying of this problem. By applying this method, a universal partial nonlinear differential equation has been obtained. Finally, this equation has been numerically solved in approximative forms, i.e. in full two and two-once localized approximation. Beside the general conclusions of the analysis, the obtained results can be used to solve some particular problems of fluid dynamics.

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