NOVI SAD J. MATH.
VOL. 31, NO. 1, 2001, 65-73

ON THE GAUSSIAN AND MEAN CURVATURE OF CERTAIN SURFACES

Eberhard Malkowsky[1], **Vesna Veličković**[2]

Abstract. The Gaussian and mean curvatures of surfaces are real valued functions of two real variables. We apply our software for differential geometry [7], [1], [2], [5] and [6] to represent the Gaussian and mean curvatures of various types of surfaces.

AMS Mathematics Subject Classification (1991): 53A05, 68N05

Key words and phrases: Computer Graphics, Differential Geometry, Gaussian curvature, mean curvature

1. Introduction and notations

Throughout this paper we assume that $D \subset \mathbb{R}^2$ is a domain and surfaces are given by a parametric representation

$$(1.1) \quad \vec{x}(u^i) = (x^1(u^1,u^2), x^2(u^1,u^2), x^3(u^1,u^2)) \quad ((u^1,u^2) \in D)$$

where the component functions $x^j : D \to \mathbb{R}$ $(j = 1,2,3)$ have continuous partial derivatives of order $r \geq 1$, denoted as usual by $\vec{x} \in C^r(D)$, and the vectors $\vec{x}_k = \partial\vec{x}/\partial u^k$ $(k = 1,2)$ satisfy $\vec{x}_1 \times \vec{x}_2 \neq \vec{0}$. If we denote the *surface normal vectors*, the *first* and *second fundamental coefficients of a surface S* given by (1.1) by

$$\vec{N}(u^i) = \frac{\vec{x}_1(u^i) \times \vec{x}_2(u^i)}{\|\vec{x}_1(u^i) \times \vec{x}_2(u^i)\|}, \; g_{jk}(u^i) = \vec{x}_j(u^i) \bullet \vec{x}_k(u^i) \quad \text{and}$$

$$L_{jk}(u^i) = \vec{N}(u^i) \bullet \vec{x}_{jk} \quad \text{where } \vec{x}_{jk}(u^i) = \frac{\partial^2 \vec{x}}{\partial u^j \partial u^k} \quad \text{for } j,k = 1,2,$$

respectively, then the functions $K : D \to \mathbb{R}$ and $H : D \to \mathbb{R}$ with

$$K = \frac{L}{g} \text{ and } H = \frac{1}{2g}(L_{11}g_{22} - 2L_{12}g_{12} + L_{22}g_{11}),$$

where $g = \det(g_{jk})$ and $L = \det(L_{jk})$, are the *Gaussian curvature* and the *mean curvature* of S. We use our software to give a graphical representation of the Gaussian and mean curvatures of some interesting surfaces.

[1]Department of Mathematics, University of Giessen, Arndt– strasse 2, D–35392 Giessen, Germany, e-mail:eberhard.malkowsky@math.uni-giessen.de

[2]Department of Mathematics, Faculty of Science and Mathematics, University of Niš, Ćirila i Metodija 2, 18000 Niš, Yugoslavia, e-mail:vvesna@BankerInter.net

2. Pseudo–Spheres

Pseudo–spheres are surfaces of revolution with constant Gaussian curvature.

Let γ be a curve with parametric representation $\vec{x}(s) = (r(s), 0, h(s))$ and $r(s) > 0$ $(s \in I \subset \mathbb{R})$, where s is the arc length along γ, and RS be the surface of revolution generated by the rotation of γ about the x^3–axis. Putting $u^1 = s$ and writing u^2 for the angle of rotation, we obtain the following parametric representation for RS on $D = I \times (0, 2\pi)$

$$\vec{x}(u^i) = (r(u^1)\cos u^2, r(u^1)\sin u^2, h(u^1)) \ ((u^1, u^2) \in D) \tag{2.1}$$

Omitting the argument u^1, we find that the fundamental coefficients of RS are given by $g_{11} = (r')^2 + (h')^2 = 1$, since u^1 is the arc length along γ, $g_{12} = 0$, $g_{22} = r^2$, $L_{11} = r'h'' - r''h'$, $L_{12} = 0$ and $L_{22} = rh'$. So the Gaussian curvature of RS is given by $K = r^{-1}(r'h'' - r''h')$. Since $(r')^2 + (h')^2 = 1$ implies $r'r'' + h'h'' = 0$, we obtain $K = r^{-1}(r'h''h' - r''(h')^2) = -r^{-1}((r')^2 + (h')^2)r'' = -r''/r$ and consequently

$$r''(u^1) + K(u^1)r(u^1) = 0. \tag{2.2}$$

First, we assume $K = 0$. Then $r = c_1u^1 + c_2$ with the constants c_1 and c_2. If we choose $c_1 = 0$ then $h' = \pm 1$ implies $h = \pm u^1 + d$ with some constant d, and we obtain a circular cylinder. If $c_1 \neq 0$ then $(r')^2 + (h')^2 = 1$ implies $|c_1| \leq 1$. For $|c_1| = 1$, we have $h' \equiv 0$, hence $h \equiv const$, and we obtain a plane. For $0 < |c_1| < 1$ and a suitable choice of the coordinate system, we have $r = c_1u^1$ and $h = d_1u^1$ for some constant d_1 with $c_1^2 + d_1^2 = 1$, and we obtain a circular cone.

Let $K \neq 0$. Then we may assume $K = \pm 1$.

Let $K = 1$. Then the general solution of (2.2) is given by $r(u^1) = C \cdot \cos(u^1 + u^1_0)$. By a suitable choice of the arc length, we may assume that $C > 0$ and $u^1_0 = 0$. Now $(r')^2 + (h')^2 = 1$ implies

$$h(u^1) = \int \sqrt{1 - C^2\sin^2(u^1)}\, du^1. \tag{2.3}$$

The choice $C = 1$ yields the unit sphere. For $C \neq 1$, the integral in (2.3) is elliptic. It exists on $(-\pi/2, \pi/2)$ if $C < 1$, on $(-\arcsin(1/C), \arcsin(1/C))$ if $C > 1$.

Finally, let $K = -1$. Then the general solution of (2.3) is given by $r(u^1) = C_1\cosh u^1 + C_2\sinh u^1$. In the special case $C_1 = 1/2 = -C_2$, we obtain

$$r(u^1) = e^{-u^1} \text{ and } h(u^1) = \int \sqrt{1 - e^{-2u^1}}\, du^1 \text{ for } u^1 > 0.$$

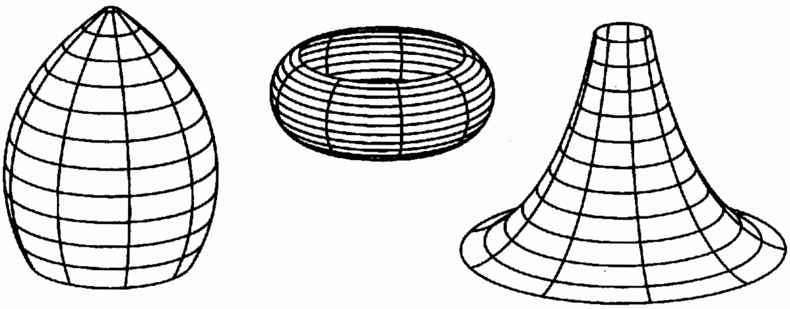

Figure 1: Pseudo–spheres $K = 1$, $C = 0.75$; $K = 1$, $C = 1.5$; and $K = -1$, $C_1 = 1/2 = -C_2$.

3. Exponential Cones

Let $h : \mathbb{C} \to \mathbb{C}$ be an analytic function and $f = |h| : \mathbb{R}^2 \to \mathbb{R}$. We write $z = u^1 + i \cdot u^2$. Then the function h generates an *explicit surface* with the parametric representation

$$\vec{x}(u^i) = (u^1, u^2, f(u^1, u^2)) \quad ((u^1, u^2) \in \mathbb{R}^2) \tag{3.1}$$

in a very natural way, and represents the modulus of h. A classification of surfaces of this kind with Gaussian curvature K of constant sign can be found in [8]. The surfaces generated by the function h defined by $h(z) = z^{\alpha+i\cdot\beta}$ for real constants α and β are called *exponential cones.* Here the cases $\alpha \geq 1$ and $\alpha \leq 1$ corrsepond to $K \geq 0$ and $K \leq 0$, respectively. Using the representation of complex numbers by polar coordinates $z = \rho e^{i\phi}$ for $\rho > 0$ and $\phi \in (0, 2\pi)$, we obtain $f(z) = \rho^\alpha e^{-\beta\phi}$. We put $u^1 = \rho$ and $u^2 = \phi$. Then exponential cones on $D = (0, \infty) \times (0, 2\pi)$ are given by

$$\vec{x}(u^i) = (u^1 \cos u^2, u^1 \sin u^2, (u^1)^\alpha e^{-\beta u^2}) \quad ((u^1, u^2) \in D);$$

are special cases of *screw surfaces* given by

$$\vec{x}(u^i) = (u^1 \cos u^2, u^1 \sin u^2, f(u^1, u^2)). \tag{3.2}$$

Since the first and second fundamental coefficients of exponential cones are

$$\begin{aligned}
g_{11} &= 1 + \alpha^2 (u^1)^{2\alpha-2} e^{-2\beta u^2}, \quad g_{12} = -\alpha\beta (u^1)^{2\alpha-1} e^{-2\beta u^2}, \\
g_{22} &= (u^1)^2 \left(1 + \beta^2 (u^1)^{2\alpha-2} e^{-2\beta u^2}\right), \\
g &= (u^1)^2 \left(1 + (\alpha^2 + \beta^2)^2 (u^1)^{2\alpha-2} e^{-2\beta u^2}\right), \\
L_{11} &= \frac{1}{\sqrt{g}} \alpha(\alpha - 1)(u^1)^{\alpha-1} e^{-\beta u^2}, \quad L_{12} = \frac{1}{\sqrt{g}} (1 - \alpha)\beta (u^1)^\alpha e^{-\beta u^2},
\end{aligned}$$

$$L_{22} = \frac{1}{\sqrt{g}}(\alpha+\beta^2)(u^1)^{\alpha+1}\mathrm{e}^{-\beta u^2},$$

we obtain, putting $\delta = \alpha^2 + \beta^2$ and $\gamma = (\alpha - 1)\delta$,

$$K(u^i) = (\alpha-1)\delta\frac{(u^1)^{2\alpha}\mathrm{e}^{-2\beta u^2}}{g^2} = \gamma\frac{(u^1)^{2\alpha-4}\mathrm{e}^{-2\beta u^2}}{(1+\delta(u^1)^{2\alpha-2}\mathrm{e}^{-2\beta u^2})^2}$$

and similarly

$$H(u^i) = \delta\frac{(u^1)^{\alpha-2}\mathrm{e}^{-\beta u^2}\left(1+\alpha(u^1)^{2\alpha-2}\mathrm{e}^{-2\beta u^2}\right)}{2(1+\delta(u^1)^{2\alpha-2}\mathrm{e}^{-2\beta u^2})^{3/2}}.$$

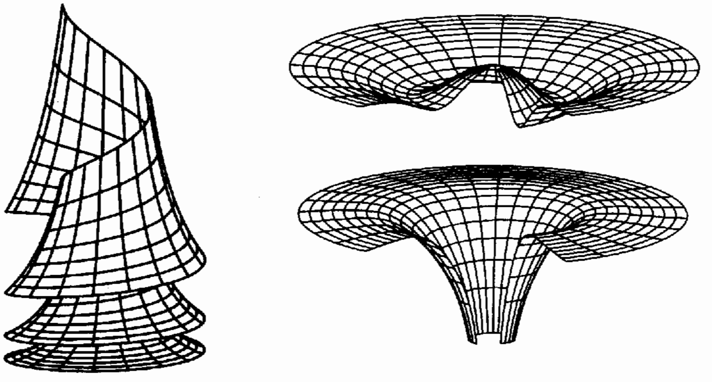

Figure 2: Exponential cone, $\alpha = -1$, $\beta = -0.1$ and its Gaussian and mean curvature.

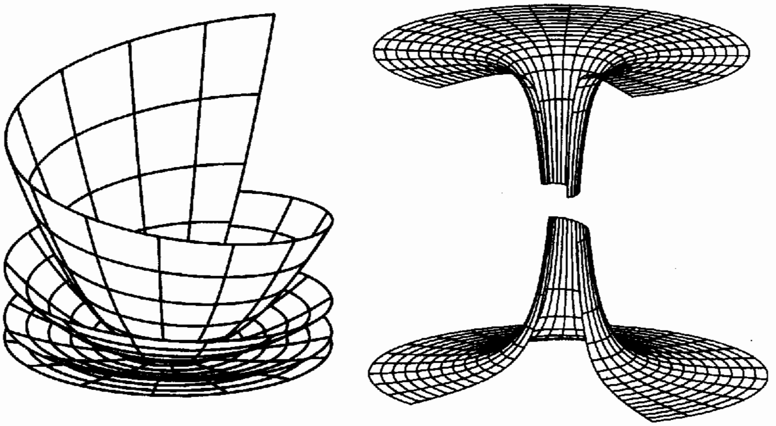

Figure 3: Exponential cone, $\alpha = 0.5$, $\beta = -0.05$ and its Gaussian and mean curvature.

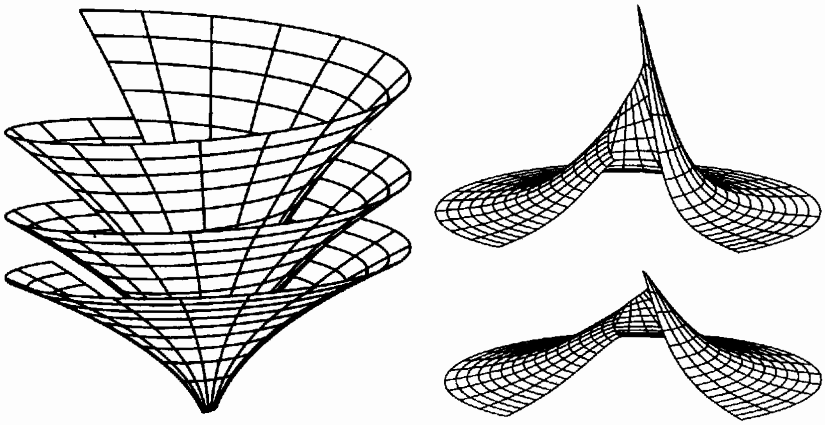

Figure 4: Exponential cone, $\alpha = 2$, $\beta = 0.15$ and its Gaussian and mean curvature.

4. Minimal surfaces

Surfaces with identically vanishing mean curvature are called *minimal surfaces.* It is well known (cf. e. g. [3, Satz 6.6, p. 60] that *if S is a surface the boundary of which is a closed curve such that the surface area of S is less than or equal to the surface area of any other "neighbouring" surface with the same boundary then S has identically vanishing mean curvature.*

The mean curvature of surfaces of revolution with parametric representation (2.1) is given by $H = (r^2(r'h''-r''h')+rh')/(2r^2)$. Now $H = 0$ is equivalent with $r(r'h''-r''h')+h' = 0$. If $h' = 0$, we obtain a plane. If $h' \neq 0$ then, multiplying by h' and using $h''h' = -r'r''$ and $(r')^2+(h')^2 = 1$, we obtain $r''r = (h')^2$. This yields $(r^2)'' = 2$, since $r''r = 1/2(r^2)''-(r')^2$ and $(r')^2+(h')^2 = 1$. By a suitable choice of the parameter u^1, we obtain $r(u^1) = \sqrt{(u^1)^2+c^2}$ $(u^1 \in \mathbb{R})$ where c is a constant. If $c = 0$, then $r(u^1) = u^1$ since $r(u^1) \geq 0$, and then $h'(u^1) = 0$, and we obtain a plane. If $c \neq 0$, then $r'(u^1) = u^1((u^1)^2+c^2)^{-1/2}$, and $(r')^2+(h')^2 = 1$ yields $(h')^2 = c^2((u^1)^2+c^2)^{-1}$, hence $h'(u^1) = |c|/\sqrt{(u^1)^2+c^2}$. Therefore $h(u^1) = c \cdot \operatorname{arsinh}(u^1/c)$ for a suitable choice of the coordinate system. Putting $u^{*1} = h(u^1)$ and $u^{*2} = u^2$, we obtain

$$\vec{x}(u^{*i}) = (|c|\cosh u^{*1}\cos u^{*2}, |c|\cosh u^{*1}\sin u^2, u^{*1}) \qquad ((u^{*1},u^{*2}) \in \mathbb{R}\times(0,2\pi)).$$

Thus the minimal surfaces of revolution are planes and catenoids.

Another minimal surface is *Scherk's surface*, given by a parametric representation

$$\vec{x}(u^i) = (u^1, u^2, \log(\frac{\cos u^2}{\cos u^1})) \quad ((u^1,u^2) \in R_{kj})$$

where, for $k, j \in \mathbb{Z}$ with $k + j \in 2 \cdot \mathbb{Z}$,

$$R_{kj} = I_k \times I_j = ((k - 1/2)\pi, (k + 1/2)\pi) \times ((j - 1/2)\pi, (j + 1/2)\pi).$$

It is easy to see that the Gaussian curvature of Scherk's minimal surface is given by $K(u^1, u^2) = -\cos^2 u^1 \cos^2 u^2 (1 - \sin^2 u^1 \sin^2 u^2)^{-2}$.

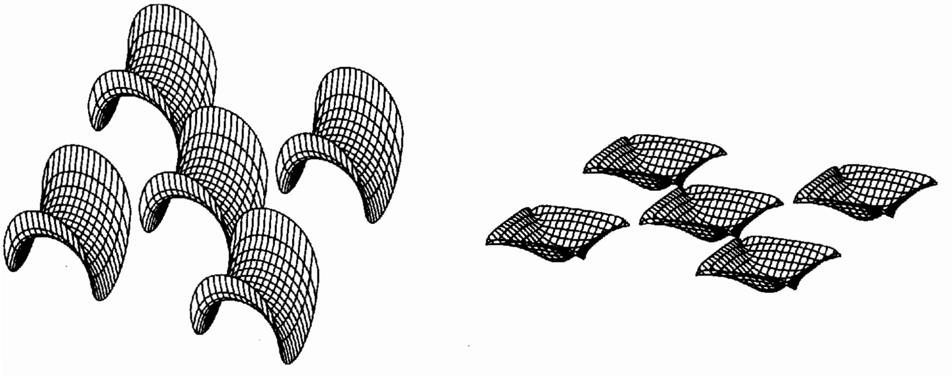

Figure 5: Scherk's minimal surface and its Gaussian curvature.

5. Surfaces generated by the modulus of analytic functions

The Gaussian and mean curvatures of surfaces with parametric representation (3.1), where $f = |h|$ and h is an analytic function, are given by

$$K = \frac{|h''|^2}{g^2} \left(\mathrm{Re} \left(\frac{(h')^2}{h''h} \right) - 1 \right) \text{ where } g = 1 + |h'|^2 \quad \text{and}$$

$$H = \frac{2}{g^{3/2}} |h| \left(\left| \frac{h'^2}{h} \right| g - |h''|^2 \mathrm{Re} \left(\frac{(h')^2}{h''h} \right) \right).$$

We consider the function h defined by $h(z) = 1/\sin \pi z$ $(z \notin \mathbb{Z})$, and put $w = w(z) = \cos 2\pi z$ and $\psi(z) = (h'(z))^2 (h''(z) h(z))^{-1}$. Since $2\cos^2 \pi z = 1 + \cos 2\pi z = 1 + w$ and $2 \sin^2 \pi z = 1 - \cos 2\pi z = 1 - w$, we obtain

$$\begin{aligned} h'(z) &= -\pi \frac{\cos \pi z}{\sin^2 \pi z}, \\ h''(z) &= \pi^2 \frac{\sin^2 \pi z + 2\cos^2 \pi z}{\sin^3 \pi z} = \frac{\pi^2}{2} \frac{w + 3}{\sin^3 \pi z}, \qquad \text{and so} \end{aligned}$$

$$\psi = \frac{2\pi^2 \cos^2 \pi z}{\pi^2 (w + 3)} = \frac{w + 1}{w + 3}.$$

Therefore

$$\begin{aligned}\mathrm{Re}(\psi(w)) - 1 &= \mathrm{Re}(\psi(w) - 1) = \mathrm{Re}\left(\frac{w+1}{w+3} - 1\right) = -2\mathrm{Re}\left(\frac{1}{w+3}\right) \\ &= -\left(\frac{1}{w+3} + \frac{1}{\bar{w}+1}\right) = -2\frac{\frac{1}{2}(\bar{w}+w)+3}{|w+3|^2} = -\frac{2(3+\mathrm{Re}(w))}{|w+3|^2}.\end{aligned}$$

Furthermore

$$|h''|^2 = \frac{\pi^4}{4}\frac{|w+3|^2}{|\sin^3 \pi z|^2}, \text{ and with } \phi(w) = |w-1|^2 + 2\pi^2|w+1|$$

$$\begin{aligned}g &= 1 + |h'|^2 = 1 + \frac{\pi^2}{2}\frac{|1+w|^2}{|\sin^2 \pi z|^2} \\ &= \frac{1}{4|\sin^2 \pi z|^2}\left(|2\sin^2 \pi z|^2 + 2\pi^2|w+1|\right) = \frac{1}{4|\sin^2 \pi z|^2}\phi(w),\end{aligned}$$

and so

$$K = \frac{|h''|^2}{g^2}\mathrm{Re}(\psi(w) - 1) = -\frac{4\pi^4|w-1|(3+\mathrm{Re}(w))}{\phi^2(w)}.$$

Finally putting

$$\begin{aligned}w_1(u^i) &= \mathrm{Re}(w) = \cosh 2\pi u^2 \cos 2\pi u^1, \\ w_2(u^i) &= |w-1| = \frac{1}{\sqrt{2}}(\cosh 4\pi u^2 + \cos 4\pi u^1 - 4w_1(u^i) + 2)^{1/2}, \\ w_3(u^i) &= |w+1| = \frac{1}{\sqrt{2}}(\cosh 4\pi u^2 + \cos 4\pi u^1 + 4w_1(u^i) + 2)^{1/2}\end{aligned}$$

and $w_4(u^i) = \phi(w) = (w_2(u^i))^2 + 2\pi^2 w_3(u^i)$, we have

$$K(u^i) = -\frac{4\pi^2 w_2(u^i)(3 + w_1(u^i))}{(w_4(u^i))^2}. \tag{5.1}$$

Similarly, putting

$$\begin{aligned}w_5(u^i) &= (w_3(u^i))^2, \; w_6(u^i) = w_3^2(u^i)w_4(u^i) - 2\pi^2 w_5(u^i) \quad \text{and} \\ f(u^i) &= |h(z)| = \frac{\sqrt{2}}{\sqrt{\cosh 2\pi u^2 - \cos 2\pi u^1}},\end{aligned}$$

we obtain

$$H(u^i) = \pi^2\frac{f(u^i)w_6(u^i)}{(w_4(u^i))^{3/2}}. \tag{5.2}$$

We represent the Gaussian and mean curvatures of exponential cones and explicit surfaces as screw surfaces and explicit surfaces by putting $f = K$ and $f = H$ in (3.2) and (3.1), respectively.

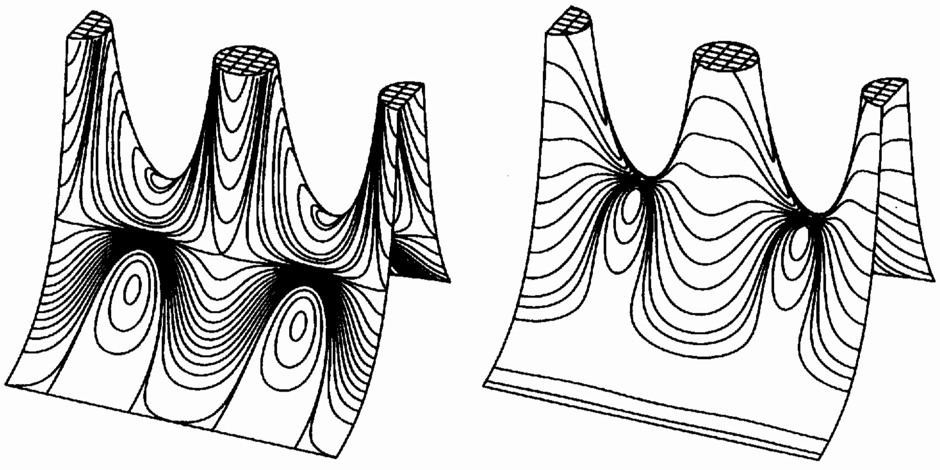

Figure 6: Lines of constant Gaussian and lines of constant mean curvature.

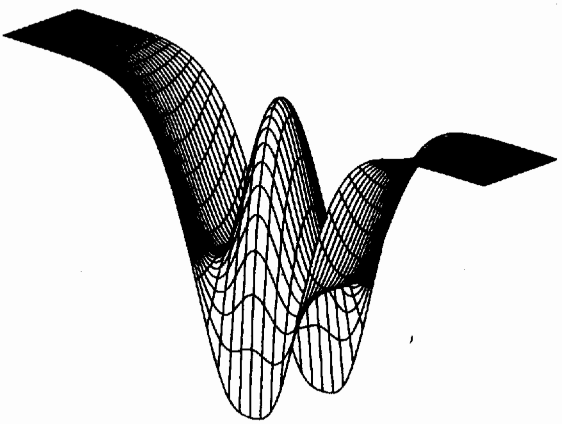

Figure 7: Gaussian curvature given by (5.1).

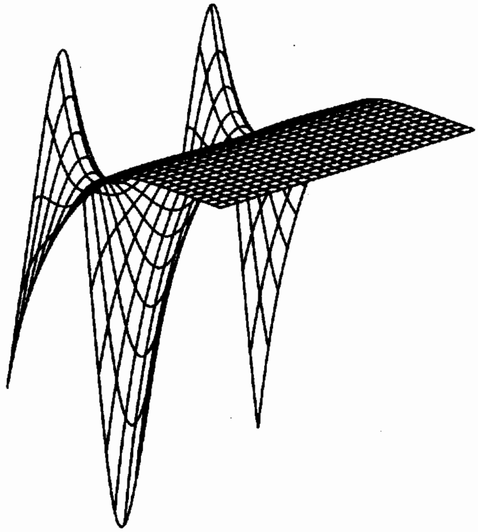

Figure 8: Mean curvature given by (5.2).

References

[1] Failing, M., Entwicklung numerischer Algorithmen zur computergrafischen Darstellung spezieller Probleme der Differentialgeometrie und Kristallographie, Ph.D. Thesis, Giessen, 1996, Shaker Verlag, Aachen,1996

[2] Failing, M., Malkowsky, E., Ein effizienter Nullstellenalgorithmus zur computergrafischen Darstellung spezieller Kurven und Flächen, Mitt. Math. Sem. Giessen, 229(1996), 11-25

[3] Laugwitz, D., Diferentialgeometrie, Teubner Verlag Stuttgart, 1977

[4] Kreyszig, E., Differentialgeometrie, Akademische Verlagsgesellschaft Leipzig, 1957

[5] Malkowsky, E., An open software in OOP for computer graphics and some applications in differential geometry, Proceedings of the 20th South African Symposium on Numerical Mathematics, (1994), 51-80

[6] Malkowsky, E., An open software in OOP for computer graphics in differential geometry, the basic concepts, ZAMM, 76, Suppl 1, (1996), 467-468

[7] Malkowsky, E., Nickel, W., Computergrafik in der Differentialgeometrie, Vieweg–Verlag Wiesbaden, Braunschweig, 1993

[8] Ullrich, E., Betragsflächen mit ausgezeichnetem Krümmungsverhalten, Math. Z., 54(1951)