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NUMERICAL EXPERIMENTS WITH DIFFERENT SCHEMES FOR A SINGULARLY PERTURBED PROBLEM

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Abstract. A number of different types of meshes and finite difference schemes for solving singularly perturbed boundary value problems are considered. All of the schemes are uniformly convergent in the perturbation parameter. Numerical experiments show how the different distribution of mesh points within the considered interval, namely the different number of mesh points in the boundary layers, influences the accuracy of the methods.

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1. Introduction

We consider the problem

$$(1)\qquad \begin{aligned} -\varepsilon u'' + c(x,u) &= 0, \quad x \in I = (0,1) \\ u(0) = u(1) &= 0, \end{aligned}$$

where $\varepsilon \in (0, \varepsilon_0)$, $\varepsilon_0 \ll 1$, is a small perturbation parameter, assuming that the following conditions are satisfied:

$$(2)\qquad c \in C^\infty (I \times R),$$

$$(3)\qquad c_u(x,u) \geq \gamma^2 > 0, \quad (x,u) \in (I \times R).$$

The condition (3) is the standard stability condition, which implies that both (2) and the reduced problem, $c(x,u) = 0$ have unique smooth solutions u_ε and u_0, respectively.

The main objective of this paper is to consider different uniform difference schemes for the boundary value problem (2). One class of numerical methods, polynomial-based schemes on special non-equidistant meshes are analyzed by Herceg and Vulanović [3], [4], [7] and Sun and Stynes [10], [11]. Another class is

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cubic spline difference schemes combined with different meshes studied by Surla and Vukoslavčević, [13], [12].

Two types of non-equidistant meshes, Bakvalov's and Shishkin's, are being used. An explicit mesh construction method to solve a singularly perturbed problem of the type (2) was used first by Bakhvalov [1], where he obtained the special discretization mesh $I_h = \{x_i = \lambda(i/n) : i = 0, 1, \ldots, n\}$, $n \in N$, $h = 1/n$, whereby λ is the mesh generating function that consists of three parts: λ_1, λ_2 and λ_3. Functions λ_1 and λ_3 generate mesh points in the boundary layers in the neighborhood of $x = 0$ and $x = 1$, respectively. Function λ_2 generates mesh points outside the boundary layers and it is a tangent line to both λ_1 and λ_3, and $\lambda_2(0.5) = 0.5$.

A much simpler mesh was constructed by Shishkin (see [2]), but many difference schemes applied to Bakhvalov's mesh show better results.

In order to simplify Bakhvalov's mesh, but also to increase the density of mesh points in the boundary layers, according to Shishkin's mesh, Herceg and Vulanović modified the former mesh-generating function [3], [4], [8], [5].

In this paper, a new mesh constructed by Herceg will be presented, being another modification of Bakhvalov's mesh. Numerical examples demonstrate the effectiveness of the methods arising from a combination of this mesh and the schemes mentioned above.

There are many interesting and relevant boundary value problems of the form (2), for which the condition (3) is not satisfied (see [18]). However, by applying the considered discretization on Bakhvalov's type of meshes, we have obtained good results.

2. Meshes

For a given positive integer n, we denote by I_h a mesh where

$$I_h : 0 = x_0 < x_1 < \cdots < x_{n-1} < x_n = 1,$$

with $x_i = \lambda(i/n), i = 0, 1, \ldots, n$. For simplicity, we take n to be an even number.

We present all meshes only by their mesh-generating functions.

Bakhvalov's generating function is given by

$$\lambda(t) = \begin{cases} \phi(t) := a\sqrt{\varepsilon}\ln \dfrac{q}{q-t}, & t \in [0, \alpha], \\ \phi(\alpha) + \phi'(\alpha)(t-\alpha), & t \in [\alpha, 0.5], \\ 1 - \lambda(1-t), & t \in [0.5, 1], \end{cases} \tag{4}$$

where a and q are constants, independent of ε, such that

$$q \in (0, 0.5), \quad a \in (0, q/\sqrt{\varepsilon}), \tag{5}$$

and additionally $a\gamma \geq 2$.
Here α is the abscissa of the contact point of the tangent line from $(0.5, 0.5)$ to $\phi(t)$. The generated mesh will be called B-mesh.

Shishkin's mesh (S-mesh) is a piecewise equidistant and consequently much simpler than the mesh above. The generating function for this mesh is

$$\lambda(t) = \begin{cases} 4\alpha t & t \in [0, \alpha], \\ \alpha + 2(1 - 2\alpha)(t - 0.25), & t \in [\alpha, 0.5], \\ 1 - \lambda(1 - t), & t \in [0.5, 1], \end{cases} \tag{6}$$

with

$$\alpha = \min\{1/4, 4\gamma_0^{-1}\sqrt{\varepsilon}\ln n\}, \quad \gamma_0 = \min\{\gamma, 1\}. \tag{7}$$

In [8], [15], Vulanović has shown that λ_1 need not be a logarithmic function. A class of suitable mesh generating functions was given and it includes functions of a much simpler rational form. Out of those functions we select the following two:

$$\lambda(t) = \begin{cases} \mu(t) := \dfrac{a\sqrt{\varepsilon}\,t}{q - t}, & t \in [0, \alpha], \\ \mu(\alpha) + \mu'(\alpha)(t - \alpha), & t \in [\alpha, 0.5], \\ 1 - \lambda(1 - t), & t \in [0.5, 1], \end{cases} \tag{8}$$

with q and a satisfying the conditions (5) and the parameter α has the same meaning as in (4), but here it can be explicitly calculated:

$$\alpha = \frac{q - \sqrt{aq\sqrt{\varepsilon}\,(1 - 2q + 2a\sqrt{\varepsilon})}}{1 + 2a\sqrt{\varepsilon}}.$$

The mesh will be called R-mesh.

The other generating function to be mentioned is:

$$\lambda(t) = \begin{cases} \mu(t) := \dfrac{a\sqrt{\varepsilon}\,t}{q - t}, & t \in [0, \alpha], \\ \pi(t), & t \in [\alpha, 0.5], \\ 1 - \lambda(1 - t), & t \in [0.5, 1], \end{cases} \tag{9}$$

with q and a satisfying the conditions (5),

$$\pi(t) = \omega(t - \alpha)^3 + 0.5\mu''(\alpha)(t - \alpha)^2 + \mu'(\alpha)(t - \alpha) + \mu(\alpha),$$

$$\omega = \frac{1}{(0.5 - \alpha)^3}(0.5 - a(q - (0.5 - \alpha)^2 + q(0.5 - \alpha)\sqrt[6]{\varepsilon} + \alpha\sqrt[3]{\varepsilon}))$$

and $\alpha = q - \sqrt[6]{\varepsilon}$. The generated mesh will be called V-mesh.

Starting from the analysis of these meshes Herceg has constructed a new mesh (H-mesh), generated by the following function:

$$\lambda(t) = \begin{cases} \dfrac{a\sqrt{\varepsilon}}{q-\beta}(\beta + q\dfrac{t-\beta}{q-\beta}) + \delta, & t \in [0,\beta], \\ \mu(t) := \delta + \dfrac{a\sqrt{\varepsilon}}{q-t}, & t \in [\beta,\alpha], \\ \dfrac{a\sqrt{\varepsilon}}{q-\beta}(\alpha + q\dfrac{t-\alpha}{q-\alpha} + \delta), & t \in [\alpha, 0.5], \\ 1 - \lambda(1-t), & t \in [0.5, 1], \end{cases} \tag{10}$$

where

$$\alpha = \frac{-q + 2q\delta + \sqrt{a}\sqrt{q}\sqrt[4]{\varepsilon}\sqrt{1 - 2q - 2\delta + 4q\delta + 2a\sqrt{\varepsilon}}}{-1 + 2\delta - 2a\varepsilon}, \qquad \beta = \frac{q\sqrt{\delta}}{\sqrt{\delta} + \sqrt{a}\sqrt[4]{\varepsilon}}.$$

Here α and β are the abscissas of the contact points of the tangent line from $(0.5, 0.5)$ to $\mu(t)$, and from $(0,0)$ to $\mu(t)$, respectively. The parameters a and q satisfy the conditions (5), and δ can be chosen from the interval $[0, (\sqrt{q} - \sqrt{a}\sqrt[4]{\varepsilon})^2]$. In the extreme case, that is when $\delta = 0$, the considered generating mesh becomes (8), and when $\delta = (\sqrt{q} - \sqrt{a}\sqrt[4]{\varepsilon})^2$, the mesh becomes equidistant. So, by changing the parameter δ, we can get a different number of mesh points in the boundary layers.

In order to emphasize the differences between these meshes we present Table 1, where the percentage of the number of mesh points in the boundary layers, i.e. in $[0, \sqrt{\varepsilon}] \bigcup [1 - \sqrt{\varepsilon}, 1]$, is given, with $q = 0.4$, $a = 1$ and $\varepsilon = 2^{-8}$.

Table 1: Points in the boundary layers (%)

	n		
Mesh	8	64	512
Mesh R	22.22	36.92	39.77
Mesh V	22.22	36.92	38.21
Mesh B	44.44	49.23	50.29
Mesh S	0	9.23	12.08
Mesh H ($\delta = \sqrt{\varepsilon}$)	0	18.46	19.88
Mesh H ($\delta = \varepsilon$)	22.22	36.92	38.60

3. Schemes

In order to obtain discretization of the problem (2) we approximate the differential equation (2) by the difference formulas. The discrete analogue is formed as:

$$
(11)\qquad \begin{aligned}
&F_0 u := u_0 = 0,\\
&F_i u := \varepsilon(a_1(i)u_{i-1} + a_0(i)u_i + a_2(i)u_{i+1})\\
&\qquad +b_1(i)c(x_{i-1}, u_{i-1}) + b_0(i)c(x_i, u_i) + b_2(i)c(x_{i+1}, u_{i+1})\\
&\qquad\qquad i = 1, 2, \ldots, n-1\\
&F_n u := u_n = 0,
\end{aligned}
$$

The coefficients of these formulas depend on the mesh points x_{i-1}, x_i, x_{i+1}, so we denote by $h_i = x_i - x_{i-1}$.
The solution $u = [u_0, u_1, \ldots, u_n]^T$ to (11), i.e. to $Fu = 0$, where $F = (F_0, F_1, \ldots, F_n)$ is the approximation to the exact solution u_ε of (2).

First, we shall introduce the central second rate difference scheme (C-scheme) analyzed in [11], [15]. The coefficients of this scheme are:

$$
a_1(i) = -\frac{2}{h_i\,(h_i + h_{i+1})}, \qquad a_0(i) = \frac{2}{h_i h_{i+1}}, \qquad a_2(i) = -\frac{2}{h_{i+1}\,(h_i + h_{i+1})},
$$

$$
b_1(i) = 0, \qquad b_0(i) = 1, \qquad b_2(i) = 0.
$$

The following two fourth-order difference schemes are obtained by Herceg (H-scheme) and Vulanović (V-scheme) as modifications of Hermite's scheme. The first one was introduced in [3], and the second in [16]. Both of them have the same coefficients $a_1(i)$, $a_0(i)$ and $a_2(i)$, as the above scheme. In the H-scheme the other coefficients are:

$$
b_1(i) = -a_1(i)(h_i^2 - h_{i+1}^2 + h_i h_{i+1})/12,
$$

$$
b_0(i) = -a_1(i)(h_i^2 + h_{i+1}^2 + 3h_i h_{i+1})/12,
$$

$$
b_2(i) = -a_2(i)(h_{i+1}^2 - h_i^2 + h_i h_{i+1})/12.
$$

The other coefficients in the V-scheme are:

$$
b_1(i) = \frac{2h_i - h_{i+1}}{6(h_i + h_{i+1})}, \qquad b_0(i) = \frac{5}{6}, \qquad b_2(i) = \frac{2h_{i+1} - h_i}{6(h_i + h_{i+1})}.
$$

The difference second-order scheme (KS-scheme) where the solution of (2) is sought in the form of a cubic spline, studied by Surla [12], have the following coefficients:

$$
a_1(i) = \frac{6}{h_i\,(h_i + h_{i+1})}, \qquad a_0(i) = -\frac{6}{h_i h_{i+1}}, \qquad a_2(i) = \frac{6}{h_{i+1}\,(h_i + h_{i+1})},
$$

$$
b_1(i) = -\frac{h_i}{h_i + h_{i+1}}, \qquad b_0 = -2, \qquad b_2(i) = -\frac{h_{i+1}}{h_i + h_{i+1}}.
$$

4. Numerical results

In order to eliminate the errors induced by nonlinear solvers we shall test first the methods derived when applying all schemes from the previous section to solve the problem (2) in its linear form, on all the described meshes. Our example is taken from [17]:

$$-\varepsilon u'' + (1 + x(1 - x))\,u = f(x), \qquad u(0) = u(1) = 0.$$

Its exact solution

$$u(x) = 1 - (1 - x)\exp\left(-\frac{x}{\sqrt{\varepsilon}}\right) - x\exp\left(-\frac{1-x}{\sqrt{\varepsilon}}\right),$$

will be used to find the errors

$$E_n = \|u_\varepsilon^h - u^h\|_\infty,$$

where u^h is the numerical solution on a mesh with n sub-intervals. We compare the errors induced by the same number of mesh points, but differently distributed in the interval [0,1] and the results are given in Tables 2 and 3. In Table 2 the meshes of Bakhvalov's type are used with $a = 2$ and $q = 0.4$, while the new mesh parameter δ is taken to be equal to $\sqrt{\varepsilon}$, while the parameters in Table 3 take the following values: $q = 0.4$, $a = 0.2$ and $\delta = 0.03$. Because of the large amount of results we present only the cases when $\varepsilon = 2^{-20}$ and $\varepsilon = 1$.

We also determined numerically the order of uniform convergence of the following nonlinear example, taken from [18]:

$$-\varepsilon u'' + \left(u^2 + u - 0.75\right)\left(u^2 + u - 3.75\right) = 0, \qquad u(0) = u(1) = 0.$$

The exact solution is unknown and, furthermore, the condition (3) is not satisfied, so the order is determined as

$$Ord_n^* = \frac{\ln E_n^* - \ln E_{2n}^*}{\ln 2}$$

where $E_n^* = \|u_\varepsilon^* - u\|_\infty$ and for each fixed ε, u_ε^* is the numerical solution to the problem with $n = 1024$. The nonlinear systems are solved using Newton's method, with the initial guess $u_0 = [0, 1.5, 1.5, \ldots, 1.5, 0]^T \in R^{n+1}$.

Finally, for some methods, we present in Table 4 the Ord^*, the average of all the Ord_n^*, $n = 16, 32, 64, 128, 256$, while ε takes the values of $2^{-8}, 2^{-10}, \ldots, 2^{-32}$. The parameters in the mesh generating functions were the same as used in testing the linear example when $\varepsilon = 2^{-20}$, except for $a = 1$. In the case when the methods included Shishkin's mesh the convergence rate was determined the same way as in [11], but the results turned out to be unsatisfactory, because of the choice of the transition point, where the condition (3) is needed, see (7). Different values of the parameter γ_0 were tested.

Table 2: Linear Example ($\varepsilon = 2^{-20}$)

	n	R mesh	V mesh	S mesh	B mesh	H mesh
C scheme	64	8.91(−4)	8.92(−4)	1.52(−2)	5.33(−4	7.38(−4)
	256	5.56(−5)	5.56(−5)	1.82(−3)	3.23(−5)	4.64(−5)
	1024	3.47(−6)	3.48(−6)	1.80(−4)	1.96(−6)	2.90(−6)
H scheme	64	5.22(−5)	1.20(−4)	8.67(−4)	2.90(−3)	1.91(−5)
	256	1.99(−7)	4.46(−7)	1.10(−5)	8.71(−5)	7.27(−8)
	1024	7.74(−10)	1.74(−9)	1.06(−7)	1.47(−6)	2.84(−10)
V scheme	64	1.67(−5)	2.90(−5)	8.67(−4)	2.16(−4)	6.14(−6)
	256	6.47(−8)	1.32(−7)	1.10(−5)	1.53(−5)	2.37(−8)
	1024	2.53(−10)	5.17(−10)	1.06(−7)	3.69(−7)	9.26(−11)
KS scheme	64	8.76(−4)	8.76(−4)	1.84(−2)	5.19(−4)	8.15(−4)
	256	5.47(−5)	5.47(−5)	1.86(−3)	3.32(−5)	5.10(−5)
	1024	3.42(−6)	3.42(−6)	1.80(−4)	1.97(−6)	3.18(−6)

Table 3: Linear Example ($\varepsilon = 1$)

	n	R mesh	S mesh	H mesh
C scheme	8	6.29(−4)	8.09(−4)	7.71(−4)
	32	4.10(−5)	5.07(−5)	4.83(−5)
	128	2.57(−6)	3.17(−6)	3.02(−6)
H scheme	8	5.46(−7)	9.13(−7)	7.73(−7)
	32	1.95(−9)	3.57(−9)	3.01(−9)
	128	7.83(−12)	2.23(−10)	1.18(−11)
V scheme	8	9.76(−6)	9.13(−7)	9.80(−7)
	32	5.80(−8)	3.57(−9)	5.10(−9)
	128	2.76(−10)	2.23(−10)	4.36(−11)
KS scheme	8	9.49(−4)	8.14(−4)	8.28(−4)
	32	5.96(−5)	5.07(−5)	5.16(−5)
	128	3.72(−6)	3.17(−6)	3.23(−6)

Table 4: Nonlinear Example ($\varepsilon = 2^{-20}$)

n	16	32	64	128	256	
KS scheme + H mesh	1.18	1.92	1.96	2.06	2.32	Odr^*
H scheme + V mesh	3.57	4.03	4.00	4.01	4.09	Odr^*
V scheme + R mesh	4.13	4.00	3.99	4.00	4.09	Odr^*

Table 5: Linear Example ($\varepsilon = 2^{-20}, a = 1, q = 0.4$)

		δ					
	n	2^{-4}	2^{-6}	2^{-9}	2^{-10}	2^{-11}	2^{-32}
KS scheme + H mesh	64	1.20(−1)	1.61(−2)	8.01(−4)	5.18(−4)	8.22(−4)	1.34(−3)
	256	9.97(−3)	9.20(−4)	5.00(−5)	3.23(−5)	5.16(−5)	8.39(−5)
	1024	6.03(−4)	5.72(−5)	3.13(−6)	2.02(−6)	3.22(−6)	5.25(−6)

		δ					
	n	2^{-4}	2^{-6}	2^{-7}	2^{-8}	2^{-9}	2^{-32}
H scheme + H mesh	64	2.62(−2)	6.78(−4)	8.09(−5)	1.17(−5)	4.49(−5)	3.33(−4)
	256	2.87(−4)	2.73(−6)	3.23(−7)	4.58(−8)	1.58(−7)	1.17(−6)
	1024	1.18(−6)	1.07(−8)	1.26(−9)	1.79(−10)	6.14(−10)	4.56(−9)

Conclusion. All numerical experiments confirm the theoretically obtained results of the convergence rate. Using different meshes, the same number of mesh points is differently distributed in the observed interval. Bakhavalov's type meshes have more points in the boundary layers, but numerical results suggest that "a large" number of mesh points in the boundary layers do not guarantee good results when a high accuracy is needed, see Table 2 (H-scheme, V-scheme). Table 5 shows how the number of mesh points in the boundary layers influences the accuracy of the method. Two schemes are used on the H-mesh, where by changing the parameter δ, we get different number mesh points in the boundary layers.

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