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SAFE CONVERGENCE OF CHEBYCHEV-LIKE METHOD

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Abstract. Initial conditions for a safe and fast convergence of the Chebyshev-like iterative method for the simultaneous determination of all simple zeros of a polynomial are considered. It is proved that, under the stated conditions, this method converges cubically. The proposed convergence conditions are computationaly verifiable since they depend only on initial approximations and the degree of a given polynomial, which is of practical importance.

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1. Introduction

The state of initial conditions that provide both safe and fast convergence of an implemented iterative algorithm has a very important role in solving nonlinear equations. Actually, these conditions reduce to the choice of initial approximations to the desired solutions of a given equation. First and fundamental result in this domain is due to Smale who developed in [14] the so-called "point estimation theory" for Newton's method. This approach treats convergence conditions and the domain of convergence in solving the equation $f(z) = 0$ using only the information of f at the initial point $\mathbf{z}^{(0)}$. Later, Smale's result has been improved and extended in [2], [3], [5], [16], [17]. A special attention has been paid to iterative methods for the simultaneous determination of all zeros of algebraic polynomials, see [4], [6]–[13], [17]–[19].

Let P be a monic polynomial with the simple zeros $\zeta_1, \ldots, \zeta_n$. In this paper we will use the Newton and Weierstrass correction given respectively by

$$N_i^{(m)} = \frac{P(z_i^{(m)})}{P'(z_i^{(m)})} \quad \text{and} \quad W_i^{(m)} = \frac{P(z_i^{(m)})}{\prod\limits_{\substack{j=1 \\ j\neq i}}^{n} (z_i^{(m)} - z_j^{(m)})} \quad (i \in I_n;\ m = 0, 1, \ldots),$$

where $I_n := \{1, \ldots, n\}$ is the index set and $m = 0, 1, \ldots$ is the iteration index. Following Smale [14], initial approximations $z_1^{(0)}, \ldots, z_n^{(0)}$ which enable the safe

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convergence of a simultaneous method for finding polynomial zeros will be called *approximate zeros.*

Most of initial conditions considered in the literature for simultaneous iterative methods are not of practical importance since they depend on unattainable data (for instance, on desired zeros). Studying suitable forms of initial conditions it turned out that two different approaches presented in [18] and [7] led in a natural way to an initial condition of the form

$$w^{(0)} < c_n \cdot d^{(0)}, \tag{1}$$

where

$$w^{(0)} = \max_{1 \le i \le n} |W_i^{(0)}|, \quad d^{(0)} = \min_{j \ne i} |z_i^{(0)} - z_j^{(0)}|,$$

and c_n is the so-called inequality factor, for short i-factor, depending on the polynomial degree n. For more details see the recent references [6]–[13], [17]–[19]. During the last years a special attention has been directed to the increase of the i-factor c_n which multiplies the minimal distance $d^{(0)}$.

In this paper we study the initial conditions for a safe convergence of a simultaneous method of Chebyshev's type. The Chebyshev iterative method for solving a nonlinear equation $f(x) = 0$ is given by the formula

$$\hat{z} = z - \frac{f(z)}{f'(z)}\left(1 + \frac{f(z)}{f'(z)} \cdot \frac{f''(z)}{2f'(z)}\right), \tag{2}$$

where z is an approximation to the zero of f, and $\hat{z}$ is the improved approximation. The above formula is sometimes called Euler's formula or Schröder's method of the third order (see, e.g. [15]). We will consider a special case when f is a monic polynomial P of the nth degree, that is

$$P(z) = z^n + a_1 z^{n-1} + \ldots + a_{n-1} z + a_n.$$

Let $\zeta_1, \ldots, \zeta_n$ be the zeros of P and let $z_1, \ldots, z_n$ be some approximations to these zeros. If initial approximations $z_1, \ldots, z_n$ are reasonably close to the zeros $\zeta_1, \ldots, \zeta_n$, then it is easy to show that the following approximation is valid:

$$\frac{P''(z_i)}{2P'(z_i)} \approx \sum_{\substack{j=1 \\ j \ne i}}^{n} \frac{1}{z_i - z_j}. \tag{3}$$

Substituting (3) in Chebyshev's formula (2), we get the simultaneous method of the Chebyshev type for the determination of all zeros of P

$$z_i^{(m+1)} = z_i^{(m)} - \frac{P(z_i^{(m)})}{P'(z_i^{(m)})}\left[1 + \frac{P(z_i^{(m)})}{P'(z_i^{(m)})} \sum_{j \ne i} \frac{1}{z_i^{(m)} - z_j^{(m)}}\right], \quad (i \in I_n;\ m = 0, 1, \ldots). \tag{4}$$

The aim of this paper is to state practically applicable initial conditions of the form (1) which enable a safe convergence of the Chebyshev-like method. We establish initial conditions depending only on the vector $\mathbf{z}^{(0)} = (z_1^{(0)}, \dots, z_n^{(0)})$ of starting approximations and the values of P in the components of $\mathbf{z}^{(0)}$. For simplicity, in our analysis we will sometimes omit the iteration index m and new entries in the subsequent $(m+1)$-st iteration will be additionally stressed by the symbol $\widehat{\ }$ (*hat*). For example, instead of $z_i^{(m)}, z_i^{(m+1)}, W_i^{(m)}, W_i^{(m+1)}, d^{(m)}, d^{(m+1)}, N_i^{(m)}, N_i^{(m+1)}$, etc., we will write $z_i, \hat{z}_i, W_i, \widehat{W}_i, d, \hat{d}, N_i, \hat{N}_i$. According to this we denote

$$w = \max_i |W_i|, \quad \hat{w} = \max_i |\widehat{W}_i|.$$

2. Some necessary lemmas

In the sequel, the convergence speed of the method (4) will be investigated according to the point estimation theory. We find that the i-factor c_n appearing in (1) in our concrete case is $c_n = 1/5n$, that is, we will perform the convergence analysis of the Chebyshev-like method (4) under the condition

$$w < \frac{d}{5n}. \tag{5}$$

For the sake of simplicity, let us introduce the following denotations:

$$S_i = \sum_{j \neq i} \frac{W_j}{z_i - z_j}, \quad \hat{S}_i = \sum_{j \neq i} \frac{W_j}{\hat{z}_i - z_j}, \quad \Sigma_i = \sum_{j \neq i} \frac{1}{z_i - z_j}, \quad \delta_i = \sum_{j \neq i} \frac{1}{z_i - \zeta_j}.$$

In [10, Sec. 1.2] the following assertion is proved.

Lemma 1. *Let the condition*

$$w < \frac{d}{c_n},$$

be satisfied. Then each disk $|z - z_i| < |W_i|/(1 - nc_n)$ $(i \in I_n)$ contains one and only one zero of P.

According to this, in our concrete case $(c_n = 1/5n)$ we have

Corollary 1. *If the inequality* (5) *is valid, then*

$$|z_i - \zeta_i| \le \frac{5|W_i|}{4} \quad (i \in I_n).$$

The following assertion gives the relation between the latter and former Weierstrass' corrections $\widehat{W}_i$ and W_i.

Lemma 2. *Let $z_1, \dots, z_n$ be the approximations to the zeros $\zeta_1, \dots, \zeta_n$ of P and let $\hat{z}_1, \dots, \hat{z}_n$ be the new approximations obtained by the method (4). Then the following is valid*

$$\widehat{W}_i = -(\hat{z}_i - z_i)\prod_{j\neq i}\left(\frac{\hat{z}_i - z_j}{\hat{z}_i - \hat{z}_j}\right)\left[(\hat{z}_i - z_i)\sum_{j\neq i}\frac{W_j}{(\hat{z}_i - z_j)(z_i - z_j)} + \frac{W_i^2 {\Sigma_i}^2}{2W_i\Sigma_i + S_i + 1}\right].$$
(6)

Proof. Using the Carstensen relation [1]

$$\frac{P'(z_i)}{P(z_i)} = \Sigma_i + \frac{1}{W_i}(S_i + 1)$$

we get from the iterative formula (4) the following relation

$$\frac{W_i}{\hat{z}_i - z_i} = -\frac{W_i\left(\dfrac{P'(z_i)}{P(z_i)}\right)^2}{\dfrac{P'(z_i)}{P(z_i)} + \Sigma_i} = -\frac{W_i\left(\Sigma_i + \dfrac{1}{W_i}(S_i + 1)\right)^2}{\Sigma_i + \dfrac{1}{W_i}(S_i + 1) + \Sigma_i} = -\frac{(W_i\Sigma_i + S_i + 1)^2}{2W_i\Sigma_i + S_i + 1}.$$

On the basis of this we obtain

$$\begin{aligned}\sum_{j=1}^{n}\frac{W_j}{z_i - z_j} + 1 &= \frac{W_i}{z_i - z_i} + \sum_{j\neq i}\frac{W_j}{\hat{z}_i - z_j} + 1 = -\frac{(W_i\Sigma_i + S_i + 1)^2}{2W_i\Sigma_i + S_i + 1} + \hat{S}_i + 1\\ &= \hat{S}_i - S_i - \frac{W_i^2{\Sigma_i}^2}{2W_i\Sigma_i + S_i + 1}.\end{aligned}$$

Since

$$\hat{S}_i - S_i = \sum_{j\neq i}\frac{W_j}{\hat{z}_i - z_j} - \sum_{j\neq i}\frac{W_j}{z_i - z_j} = -(\hat{z}_i - z_i)\sum_{j\neq i}\frac{W_j}{(\hat{z}_i - z_j)(z_i - z_j)},$$

it follows

$$\sum_{j=1}^{n}\frac{W_j}{z_i - z_j} + 1 = -(\hat{z}_i - z_i)\sum_{j\neq i}\frac{W_j}{(\hat{z}_i - z_j)(z_i - z_j)} - \frac{W_i^2{\Sigma_i}^2}{2W_i\Sigma_i + S_i + 1}. \tag{7}$$

From Lagrange's interpolating formula for a polynomial P with the nodes $z_1, \dots, z_n$, it is possible to express P in terms of W_j's:

$$P(t) = \left(\sum_{j=1}^{n}\frac{W_j}{t - z_j} + 1\right)\prod_{j=1}^{n}(t - z_j).$$

Putting $t = \hat{z}_i$ in the above formula and using (7), we obtain

$$P(\hat{z}_i) = \Big(\sum_{j=1}^{n} \frac{W_j}{\hat{z}_i - z_j} + 1\Big) \prod_{j=1}^{n} (\hat{z}_i - z_j)$$

$$= -(\hat{z}_i - z_i)\left[(\hat{z}_i - z_i)\sum_{j\neq i} \frac{W_j}{(\hat{z}_i - z_j)(z_i - z_j)} + \frac{W_i^2 \Sigma_i^{\ 2}}{2W_i\Sigma_i + S_i + 1}\right] \prod_{j\neq i} (\hat{z}_i - z_j).$$

After dividing the last relation with $\prod_{j\neq i}(\hat{z}_i - \hat{z}_j)$, we get the formula (6). □

Several inequalities and bounds, which play a key role in the convergence analysis of the Chebyshev-like method (4), are given in the following lemma.

Lemma 3. *Let $z_1, \ldots, z_n$ be the approximations obtained by the method* (4) *and let $\varepsilon_i = z_i - \zeta_i$, $\hat{\varepsilon}_i = \hat{z}_i - \zeta_i$. If the inequality* (5) *holds, then for $n \geq 3$ we have*

(i) $d < \dfrac{9n}{9n-8}\hat{d} \leq \dfrac{27}{19}\hat{d}$;

(ii) $\hat{w} < 0.35w$;

(iii) $\hat{w} < \dfrac{\hat{d}}{5n}$;

(iv) $|\hat{\varepsilon}_i| \leq \dfrac{16}{9d^2}|\varepsilon_i|^2\Big[(n-1)^2|\varepsilon_i| + \dfrac{4n-1}{4n}\sum_{j\neq i}|\varepsilon_j|\Big]$.

Proof. On the basis of (5) and Corollary 1 we have

$$|\varepsilon_i| = |z_i - \zeta_i| \leq \frac{5}{4}w < \frac{d}{4n} \qquad (i \in I_n), \tag{8}$$

wherefrom there follows

$$|z_i - \zeta_j| \geq |z_i - z_j| - |z_j - \zeta_j| > d - \frac{d}{4n} = \frac{4n-1}{4n}d. \tag{9}$$

Using the identity

$$\frac{P'(z_i)}{P(z_i)} = \sum_{j=1}^{n} \frac{1}{z_i - \zeta_j} = \frac{1}{z_i - \zeta_i} + \sum_{j\neq i} \frac{1}{z_i - \zeta_j} = \frac{1}{\mu_i} + \delta_i,$$

from (4) we get

$$\hat{\varepsilon}_i = \hat{z}_i - \zeta_i = z_i - \zeta_i - \Big(\frac{P'(z_i)}{P(z_i)}\Big)^{-1}\Big(1 + \Big(\frac{P'(z_i)}{P(z_i)}\Big)^{-1}\Sigma_i\Big)$$

$$= \varepsilon_i - \frac{1}{1/\varepsilon_i + \delta_i}\Big(1 + \frac{\Sigma_i}{1/\varepsilon_i + \delta_i}\Big) = \frac{\varepsilon_i^2}{(1+\varepsilon_i\delta_i)^2}\left(\delta_i - \Sigma_i + \varepsilon_i\delta_i^2\right)$$

$$= \varepsilon_i^3\Big(\frac{\delta_i}{1+\varepsilon_i\delta_i}\Big)^2 - \frac{\varepsilon_i^2}{(1+\varepsilon_i\delta_i)^2}\sum_{j\neq i}\frac{\varepsilon_j}{(z_i - \hat{z}_j)(z_i - z_j)}, \tag{10}$$

since

$$\delta_i - \Sigma_i = \sum_{j\neq i} \frac{1}{z_i - \zeta_j} - \sum_{j\neq i} \frac{1}{z_i - z_j} = -\sum_{j\neq i} \frac{\varepsilon_j}{(z_i - \zeta_j)(z_i - z_j)}.$$

Moreover, from (4) we have

$$\begin{aligned} |\hat{z}_i - z_i| &= \left| \left(\frac{P'(z_i)}{P(z_i)}\right)^{-1} \left(1 + \left(\frac{P'(z_i)}{P(z_i)}\right)^{-1} \Sigma_i\right)\right| \\ &= \left|\frac{\varepsilon_i}{1+\varepsilon_i\delta_i}\left(1 + \frac{\varepsilon_i \Sigma_i}{1+\varepsilon_i\delta_i}\right)\right|. \end{aligned} \tag{11}$$

Using the estimations (8) and (9), and the inequalities (5) and $|z_i - z_j| \geq d$, we estimate

$$|\delta_i| \leq \left|\sum_{j\neq i} \frac{1}{z_i - \zeta_j}\right| \leq \frac{4n(n-1)}{d(4n-1)}, \tag{12}$$

$$|\Sigma_i| \leq \left|\sum_{j\neq i} \frac{1}{z_i - z_j}\right| \leq \frac{n-1}{d}, \tag{13}$$

$$|S_i| = \left|\sum_{j\neq i} \frac{W_j}{z_i - z_j}\right| \leq \frac{w(n-1)}{d} < \frac{n-1}{5n}, \tag{14}$$

$$\left|\frac{1}{1+\varepsilon_i\delta_i}\right| \leq \frac{1}{1 - |\varepsilon_i||\delta_i|} \leq \frac{1}{1 - \dfrac{d}{4n}\cdot\dfrac{4n(n-1)}{d(4n-1)}} = \frac{4n-1}{3n}, \tag{15}$$

$$\left|\frac{\delta_i}{1+\varepsilon_i\delta_i}\right| \leq \frac{4n(n-1)}{d(4n-1)} \cdot \frac{4n-1}{3n} = \frac{4(n-1)}{3d}. \tag{16}$$

On the basis of these estimations and using (8) we obtain from (11)

$$\begin{aligned} |\hat{z}_i - z_i| &\leq |\varepsilon_i|\left|\frac{1}{1+\varepsilon_i\delta_i}\right|\left(1 + \frac{|\varepsilon_i||\Sigma_i|}{|1+\varepsilon_i\delta_i|}\right) \\ &\leq |\varepsilon_i|\frac{4n-1}{3n}\left[1 + \frac{\dfrac{d}{4n}\cdot\dfrac{n-1}{d}\cdot(4n-1)}{3n}\right] \\ &\leq \frac{5}{4}|W_i|\frac{4n-1}{3n}\left(1 + \frac{(n-1)(4n-1)}{12n^2}\right). \end{aligned}$$

From this it follows

$$|\hat{z}_i - z_i| \leq \frac{20}{9}|W_i| \tag{17}$$

and

$$|\hat{z}_i - z_i| \leq \frac{20}{9}|W_i| < \frac{4}{9n} d. \tag{18}$$

Knowing that $|z_i - z_j| \geq d$, on the basis of (18) we have

$$(19)\qquad |\hat{z}_i - z_j| \geq |z_i - z_j| - |\hat{z}_i - z_i| > d - \frac{4}{9n}d = \frac{9n-4}{9n}d$$

and

$$(20)\qquad |\hat{z}_i - \hat{z}_j| \geq |z_i - z_j| - |\hat{z}_i - z_i| - |\hat{z}_j - z_j| > d - 2\cdot\frac{4}{9n}d = \frac{9n-8}{9n}d$$

Considering that $|\hat{z}_i - \hat{z}_j| \geq \hat{d}$, from the relation (20) we obtain

$$(21)\qquad \hat{d} > \frac{9n-8}{9n}d, \quad \text{that is,} \quad d < \frac{9n}{9n-8}\hat{d}.$$

Hence

$$(22)\qquad \frac{d}{\hat{d}} < \frac{9n}{9n-8} \leq \frac{27}{19} \quad \text{for} \quad n \geq 3,$$

which proves (i) of the lemma.

Using (6) and the bounds (18), (19), (20), (13), and (14) we estimate the quantities which appear in (6):

$$\left|\prod_{j\neq i}\frac{\hat{z}_i - z_j}{\hat{z}_i - \hat{z}_j}\right| \leq \prod_{j\neq i}\left(1+\frac{|\hat{z}_j - z_j|}{|\hat{z}_i - \hat{z}_j|}\right) \leq \prod_{j\neq i}\left(1+\frac{\frac{4}{9n}d}{\frac{9n-8}{9n}d}\right) = \left(1+\frac{4}{9n-8}\right)^{n-1} < e^{4/9} \approx 1.5596,$$

$$|\hat{z}_i - z_i|\left|\sum_{j\neq i}\frac{W_j}{(\hat{z}_i - z_j)(z_i - z_j)}\right| \leq \frac{4d}{9n}\,\frac{(n-1)\frac{1}{5n}d}{\frac{9n-4}{9n}d\cdot d} = \frac{4(n-1)}{5n(9n-4)},$$

$$\begin{aligned}\frac{1}{|1+S_i+2W_i\Sigma_i|} &\leq \frac{1}{1-|S_i|-2|W_i||\Sigma_i|} \leq \frac{1}{1-\dfrac{n-1}{d}w - 2w\dfrac{n-1}{d}} \\ &< \frac{1}{1-\dfrac{n-1}{5n}-\dfrac{2(n-1)}{5n}} = \frac{5n}{2n+3},\end{aligned}$$

$$|W_i^2\Sigma_i{}^2| \leq |W_i|^2|\Sigma_i|^2 \leq \left(\frac{d}{5n}\right)^2\left(\frac{n-1}{d}\right)^2 = \left(\frac{n-1}{5n}\right)^2.$$

According to the above bounds and (17), we get from (6)

$$\begin{aligned}|\widehat{W}_i| &\leq |\hat{z}_i - z_i|\left|\prod_{j\neq i}\frac{\hat{z}_i - z_j}{\hat{z}_i - \hat{z}_j}\right|\left[|\hat{z}_i - z_i|\left|\sum_{j\neq i}\frac{W_j}{(\hat{z}_i - z_j)(z_i - z_j)}\right| + \frac{|W_i|^2|\Sigma_i|^2}{|1+S_i+2W_i\Sigma_i|}\right] \\ &\leq \frac{20}{9}|W_i|e^{4/9}\left[\frac{4(n-1)}{5n(9n-4)} + \frac{(n-1)^2}{5n(2n+3)}\right],\end{aligned}$$

wherefrom it follows

$$|\widehat{W}_i| < 0.35|W_i|. \tag{23}$$

This proves (ii).

With the help of (23), (5) and (21) we estimate

$$\hat{w} < 0.35w < \frac{0.35}{5n}d < 0.35\frac{9n}{9n-8}\cdot\frac{\hat{d}}{5n},$$

which gives

$$\hat{w} < \frac{\hat{d}}{5n}. \tag{24}$$

With this (iii) is proved.

Using the bounds (9), (12), (15), and (16), as well as the estimate

$$\Big|\sum_{j\neq i}\frac{\varepsilon_j}{(z_i-\zeta_j)(z_i-z_j)}\Big| \le \sum_{j\neq i}\frac{|\varepsilon_j|}{|z_i-\zeta_j||z_i-z_j|} \le \frac{4n}{(4n-1)d^2}\sum_{j\neq i}|\varepsilon_j|,$$

from (7) we get

$$\begin{aligned}|\hat{\varepsilon}_i| &\le |\varepsilon_i|^3\Big|\frac{\delta_i}{1+\varepsilon_i\delta_i}\Big|^2 + \frac{|\varepsilon_i|^2}{|1+\varepsilon_i\delta_i|^2}\Big|\sum_{j\neq i}\frac{\varepsilon_j}{(z_i-\zeta_j)(z_i-z_j)}\Big|\\ &\le |\varepsilon_i|^3\Big(\frac{4(n-1)}{3d}\Big)^2 + |\varepsilon_i|^2\Big(\frac{4n-1}{3n}\Big)^2\frac{4n}{(4n-1)d^2}\sum_{j\neq i}|\varepsilon_j|,\end{aligned}$$

that is

$$|\hat{\varepsilon}_i| \le \frac{16}{9d^2}|\varepsilon_i|^2\Big[(n-1)^2|\varepsilon_i| + \frac{4n-1}{4n}\sum_{j\neq i}|\varepsilon_j|\Big]. \tag{25}$$

This proves (iv) of the lemma. □

3. Convergence theorem

In this section we give the convergence theorem for the Chebyshev-like method (4) which involves only computationally verifiable quantities, more precisely, initial approximations to the zeros and the degree n of a given polynomial.

Theorem 1. *Under the initial condition*

$$w^{(0)} < \frac{d^{(0)}}{5n} \tag{26}$$

the Chebyshev method (4) *is convergent with the order of convergence three.*

Proof. The convergence analysis is based on the estimate procedure of the error $\varepsilon_i^{(m)} = z_i^{(m)} - \zeta_i$. We perform the proof by induction applying the same argumentation used for the inequalities (i)–(iv) of Lemma 3.

First, we observe that the initial condition (26) coincides with (5) which means that all estimates given in Lemma 3 hold for the index $m = 1$. In fact, this is the part of the proof with respect to $m = 1$. Besides, the inequality (iii) reduces again to the condition of the form (5) and, for this reason, the assertions (i)–(iv) of Lemma 4 are valid for the subsequent index, and so on. Therefore, all estimates and bounds for the index m are derived in the same way as for $m = 0$. Actually, the implication

$$w^{(m)} < \frac{d^{(m)}}{5n} \quad \Rightarrow \quad w^{(m+1)} < \frac{d^{(m+1)}}{5n},$$

plays an essential role in our convergence analysis because it involves the initial condition (26) that provides the validity of all inequalities given in Lemma 3 for all $m = 0, 1, \ldots$. In particular, regarding (22) and (25) we have

$$\frac{d^{(m)}}{d^{(m+1)}} < \frac{27}{19}, \tag{27}$$

and

$$|\varepsilon_i^{(m+1)}| \le \frac{16}{9[d^{(m)}]^2}|\varepsilon_i^{(m)}|^2\Big[(n-1)^2|\varepsilon_i^{(m)}| + \frac{4n-1}{4n}\sum_{j\neq i}|\varepsilon_j^{(m)}|\Big]. \tag{28}$$

Let us substitute

$$t_i^{(m)} = \frac{2(n-1)}{d^{(m)}}|\varepsilon_i^{(m)}|. \tag{29}$$

Then (28) becomes

$$t_i^{(m+1)} \le \frac{4}{9}\frac{d^{(m)}}{d^{(m+1)}}[t_i^{(m)}]^2\Big[t_i^{(m)} + \frac{4n-1}{4n(n-1)^2}\sum_{j\neq i}t_j^{(m)}\Big]. \tag{30}$$

According to (29) and the estimate (8) for $m = 0$ one obtains

$$t_i^{(0)} \le \frac{12}{19}[t_i^{(m)}]^2\Big[t_i^{(m)} + \frac{4n-1}{4n(n-1)^2}\sum_{j\neq i}t_j^{(m)}\Big]. \tag{31}$$

Let $t^{(0)} = \max t_i^{(0)}$. Then from (30) it follows

$$t_i^{(1)} < \frac{12}{19}\Big(1 + \frac{4n-1}{4n(n-1)}\Big)\big[t^{(0)}\big]^3 \le \frac{35}{38}[t^{(0)}]^3 < \Big(\frac{1}{2}\Big)^3$$

By the successive application of the inequality (30) for $m = 1, 2, \ldots$ we find

$$t_i^{(m)} < \frac{35}{38}\Big(\frac{1}{2}\Big)^{3^m},$$

which means that $\{t_i^{(m)}\}$ $(i = 1, \ldots, n)$ are monotonically decreasing sequences which converge to the zero with the order of convergence 3. Taking into account the relation (29) the same is valid for the sequences $\{|\varepsilon_i^{(m)}|\}$, that is $\varepsilon_i^{(m)} \to 0$. Hence it follows that $z_i^{(m)} \to \zeta_i$ with the order of convergence 3. □

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