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ON COLLOCATION METHODS FOR SINGULAR PERTURBATION PROBLEMS OF CONVECTION-DIFFUSION TYPE

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Abstract. A collocation method using cubic spline for second-order linear singularly-perturbed two-point boundary value problem of convection-diffusion type is developed. The approximate solution is determined by collocation at the nodes of a piecewise equidistant mesh of Shishkin type. We improve the method by changing the location of the collocation points from the mesh points to the Gauss-Legendre points. Numerical example comparing the methods is presented.

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1. Introduction

We consider the numerical solution of the singularly-perturbed second-order linear boundary value problem

$$L_\varepsilon y(x) \equiv \varepsilon y''(x) + p(x)y'(x) = f(x), \tag{1.1}$$

$$y(0) = 0, \quad y(1) = 0, \tag{1.2}$$

where ε is a small positive parameter. Under the basic assumptions that the functions $p(x), f(x)$ are smooth enough on $I = [0,1]$ and

$$p(x) \geq p > 0 \quad \text{for all} \quad x \in I,$$

problem (1.1), (1.2) has a unique solution. The reduced problem obtained by setting $\varepsilon = 0$ in (1.1), (1.2) also has a unique solution $y_r(x)$. When $y_r(0) \neq 0$, $y(x)$ exhibits a boundary layer of exponential type at $x = 0$, when ε is near to zero. For $\varepsilon = 1$, the problem (1.1), (1.2) and the collocation methods with polynomial splines on a regular mesh are considered in [1]. The authors have shown that the maximal order of convergence is achieved by collocation

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at the Gauss-Legandre points. For the cubic spline forth order of convergence is achieved. The singularly perturbed problem (1.1), (1.2) with the collocation methods which use cubic and tension splines are considered in [2]. The rules for selecting the tension parameters and collocation points are developed and the convergence of the method outside the boundary layer region is proved.

It is known that most classical methods fail when ε is small related to the mesh width h used for the discretization of the operator L_ε. In most cases it is impractical to require a partition with subinterval length of order ε. Our aim in this paper is to show that collocation methods with piecewise polynomials can furnish accurate numerical approximations of (1.1), (1.2) when h/ε either is small or large, if the mesh used for the discretization is a piecewise equidistant mesh of Shishkin type.

In [3], the collocation with quadratic spline is considered when the collocation points are the midpoints of the piecewise equidistant mesh of Shishkin type.

In Section 2 we will describe the construction of a piecewise equidistant mesh of Shishkin type. In Section 3, we derive and analyze the collocation method with cubic splines. We introduce Gauss-Legendre points as collocation points in order to improve the accuracy of computed approximations. The numerical experiments presented in Section 4 indicate the method which converges ε−uniformly with the error of the order $\mathcal{O}(N^{-4}\ln^5 N)$.

2. The mesh

The Shishkin mesh appropriate to the problem (1.1), (1.2) is defined as follows. Given a even positive integer N, we divide the interval $[0,1]$ into two subintervalas

$$[0,\tau], \quad [\tau,1]$$

and use equidistant meshes with $N/2$ points in each of them. The transition point τ from the fine to the coarse mesh depends on ε and N, and is defined with

$$\tau = \min\left\{\frac{1}{2}, \frac{m\varepsilon \ln N}{p_0}\right\}, \quad p_0 = \min\{p,1\},$$

$$\text{(2.1)} \qquad \begin{aligned} \Delta^S = \{x_i \mid x_i &= ih, \quad i = 1,\dots,\frac{N}{2}, \quad h = \frac{2\tau}{N}; \\ x_i &= \tau + iH, \quad i = 1,\dots,\frac{N}{2}-1, \quad H = \frac{2(1-\tau)}{N}\}. \end{aligned}$$

3. Derivation of the scheme

For a given integer N, we denote by Δ^N an arbitrary mesh

$$\text{(3.1)} \qquad \Delta^N = \{x_i |\ 0 = x_0 < x_1 < \dots < x_i < \dots < x_N = 1\}$$

$$\text{with} \quad h_i = x_{i+1} - x_i \quad \text{for} \quad i = 0, 1, ... N-1.$$

We seek to determine an approximation to the solution with a cubic spline $u(x) \in C^1(I)$. Let $u_i(x)$ be a cubic polynomial which represents the spline $u(x)$ on the interval $[x_i, x_{i+1}]$:

$$u_i(x) = u_i + (x - x_i)u_i' + (x - x_i)^2 \frac{u_i''}{2} + (x - x_i)^3 \frac{u_i'''}{6}, \tag{3.2}$$

$$x \in [x_i, x_{i+1}] \quad i = 0, ..., N-1.$$

We specify $u(x)$ with collocation at $2N$ points, i.e. by enforcing

- $u(x)$ satisfies the problem (1.1) at the points

$$a_i = x_i + t_i h_i, \quad b_i = x_{i+1} - t_i h_i, \quad i = 0, ..., N-1,$$

$$\text{where} \qquad t_i = \frac{1 - 1/\sqrt{3}}{2}$$

and boundary conditions

- $u(x) \in C^1([0, 1])$.

The collocation points a_i and b_i are symmetrically disposed on each subinterval and for these choice of t_i we select the Gauss-Legendre points.

These conditions give the system of $4N$ unknowns with the same number of equations:

$$\begin{aligned}
1.\quad Lu_i(x)_{x=a_i} &= f(x)_{x=a_i}, \quad i = 0, ..., N-1 \\
2.\quad Lu_i(x)_{x=b_i} &= f(x)_{x=b_i}, \quad i = 0, ..., N-1 \\
3.\quad u_i(x)_{x=x_{i+1}} &= u_{i+1}(x)_{x=x_{i+1}}, \quad i = 1, ..., N-1 \\
4.\quad u_i'(x)_{x=x_{i+1}} &= u_{i+1}'(x)_{x=x_{i+1}}, \quad i = 1, ..., N-1
\end{aligned}$$

$$u_0 = 0, \quad u_N = 0.$$

Let us introduce the following notations:

$$q_i = a_i - x_i, \quad d_i = b_i - x_i,$$

$$p_i = p(a_i), \quad \tilde{p}_i = p(b_i), \qquad f_i = f(a_i), \quad \tilde{f}_i = f(b_i),$$

$$i = 0, ..., N-1.$$

Equations 1 - 4 on the interval $[x_i, x_{i+1}]$ have the form:

$$
(3.3)\qquad
\begin{array}{lrcl}
1. & \varepsilon(u_i'' + q_i u_i''') + p_i\left(u_i' + q_i u_i'' + \dfrac{q_i^2}{2}u_i'''\right) & = & f_i \\[2ex]
2. & \varepsilon(u_i'' + d_i u_i''') + \tilde{p}_i\left(u_i' + d_i u_i'' + \dfrac{d_i^2}{2}u_i'''\right) & = & \tilde{f}_i \\[2ex]
3. & u_i + h_i u_i' + \dfrac{h_i^2}{2}u_i'' + \dfrac{h_i^3}{6}u_i''' & = & u_{i+1} \\[2ex]
4. & u_i' + h_i u_i'' + \dfrac{h_i^2}{2}u_i''' & = & u_{i+1}'.
\end{array}
$$

Equations 1 and 2 in (3.3) can be written in the form:

$$
(3.4)\qquad
\begin{array}{lrcl}
1. & u_i'' & = & \gamma_i f_i + T_i u_i' - C_i R_i \tilde{f}_i \\[1ex]
2. & u_i''' & = & -\dfrac{1}{G_i}(K_i u_i' - \tilde{f}_i + M_i f_i),
\end{array}
$$

$$i = 0, ..., N-1$$

where

$$A_i = \varepsilon + \frac{p_i q_i}{2}, \quad B_i = \varepsilon + \frac{\tilde{p}_i d_i}{2}, \quad C_i = \frac{1}{2A_i - \varepsilon}, \quad M_i = C_i(2B_i - \varepsilon),$$

$$K_i = \tilde{p}_i - p_i M_i, \quad G_i = d_i B_i - q_i M_i A_i,$$

$$R_i = \frac{q_i A_i}{G_i}, \quad \gamma_i = C_i(1 + R_i M_i), \quad T_i = C_i(R_i K_i - p_i).$$

Substituting (3.4) into the third and fourth equation of the system (3.3) we get

$$(3.5)\qquad 3.\quad u_i' = \frac{1}{\alpha_i}(u_{i+1} - u_i - \theta_i f_i - \chi_i \tilde{f}_i),$$

$$(3.6)\qquad 4.\quad \beta_i u_i' + \eta_i f_i + \rho_i \tilde{f}_i = u_{i+1}',$$

$$i = 0, ..., N-1,$$

where

$$\alpha_i = h_i\left(1 + \frac{h_i}{2}T_i - \frac{h_i^2}{6G_i}K_i\right), \quad \theta_i = \frac{h_i^2}{2}\left(\gamma_i - \frac{h_i}{3G_i}M_i\right),$$

$$\chi_i = \frac{h_i^2}{2}\left(\frac{h_i}{3G_i} - C_i R_i\right),$$

$$\beta_i = 1 + h_i T_i - \frac{h_i^2}{2G_i} K_i, \quad \eta_i = h_i\left(\gamma_i - \frac{h_i}{2G_i} M_i\right), \quad \rho_i = h_i\left(\frac{h_i}{2G_i} - C_i R_i\right).$$

Substituting (3.5) into (3.6) we get the scheme:

$$(3.7) \qquad r_i^- u_{i-1} + r_i^c u_i + r_i^+ u_{i+1} = s_{i-1} f_{i-1} + \tilde{s}_{i-1} \tilde{f}_{i-1} + S_i f_i + \tilde{S}_i \tilde{f}_i,$$

$$i = 1, 2, ..., N-1, \quad u_0 = 0, \quad u_N = 0,$$

where

$$r_i^- = -\frac{\beta_i}{\alpha_i}, \qquad r_i^+ = -\frac{1}{\alpha_i}, \qquad r_i^c = -(r_i^- + r_i^+),$$

$$s_i = \frac{\beta_i \theta_i}{\alpha_i} - \eta_i, \qquad \tilde{s}_i = \frac{\beta_i \chi_i}{\alpha_i} - \rho_i, \qquad S_i = -\frac{\theta_i}{\alpha_i}, \qquad \tilde{S}_i = -\frac{\chi_i}{\alpha_i}.$$

For $t_i = 0, \ i = 0, ..., N-1$, the Gauss-Legendre points are reduced to the mesh points and the method (3.7) is reduced to

$$(3.8) \qquad r_i^- u_{i-1} + r_i^c u_i + r_i^+ u_{i+1} = q_i^- f_{i-1} + q_i^c f_i + q_i^+ f_{i+1},$$

$$i = 1, 2, ..., N-1, \quad u_0 = 0, \quad u_N = 0,$$

where

$$r_i^- = -\frac{c_i}{h_i \mu_i}, \qquad r_i^+ = -\frac{1}{h_{i+1}\mu_{i+1}}, \qquad r_i^c = -(r_i^- + r_i^+),$$

$$q_i^- = \frac{h_i c_i}{6\varepsilon\mu_i}\left(1 + \frac{1}{a_i}\right) - \frac{h_i}{2\varepsilon a_i}, \qquad q_i^+ = \frac{h_{i+1}}{6\varepsilon a_{i+1}\mu_{i+1}},$$

$$q_i^c = \frac{h_i c_i}{6\varepsilon a_i \mu_i} - \frac{h_{i+1}}{6\varepsilon\mu_{i+1}}\left(1 + \frac{1}{a_{i+1}}\right) - \frac{h_i}{2\varepsilon a_i},$$

with

$$a_i = 1 + \frac{h_i p_i}{2\varepsilon}, \qquad b_i = 1 - \frac{h_i p_{i-1}}{2\varepsilon}, \qquad \mu_i = b_i - \frac{h_i}{6\varepsilon a_i}(p_i - p_{i-1}(2a_i - 1)).$$

The spline $u(x)$ obtained by (3.8) belongs to $C^2(I)$.

4 Numerical results

In this section we give some numerical experiments for the schemes (3.7) and (3.8) applied to the problem (1.1), (1.2).

Example. Consider the problem

$$\varepsilon y''(x) + (1 + x(1-x))y'(x) = f(x,\varepsilon)$$

$$y(0) = y(1) = 0.$$

Its exact solution is

$$y(x) = \frac{1 - e^{-\frac{x}{\varepsilon}}}{1 - e^{-\frac{1}{\varepsilon}}} - \cos(\frac{\pi}{2}(1-x)).$$

For this example the constant p_0 at the transition point of the Shishkin mesh has the value $p_0 = 1$ and $m = 4$.

Let $\mathbf{u}_{\mathbf{G}}^{\mathbf{n}} = (u_{G,0}, \ldots, u_{G,n})^T$ be the solution to (3.7) on the Shishkin mesh, and let $\mathbf{u}^{\mathbf{n}} = (u_0, \ldots, u_n)^T$ be the solution to (3.8) on the Shishkin mesh.

We use a double-mesh method to compute the experimental rates of convergence. For each $N = 2^{-k}$, $k = 5, 6, \ldots, 10$ and $\varepsilon = 2^{-l}$, $l = 1, 2, \ldots, 15$ we shall report

$$E_N = \max_{0 \le i \le N} |y(x_i) - u_i|,$$

and

$$E_{G,N} = \max_{0 \le i \le N} |y(x_i) - u_{G,i}|,$$

in Tables 1 and 2 below.

Table 1. Errors E_N for the scheme (3.8)

l	N					
	32	64	128	256	512	1024
1	1.073(-4)	2.681(-5)	6.701(-6)	1.675(-6)	4.188(-7)	1.047(-7)
2	4.215(-4)	1.052(-4)	2.631(-5)	6.576(-6)	1.644(-6)	4.110(-7)
3	1.734(-3)	4.318(-4)	1.080(-4)	2.700(-5)	6.750(-6)	1.687(-6)
4	7.272(-3)	1.782(-3)	4.435(-4)	1.109(-4)	2.771(-5)	6.927(-6)
5	2.386(-2)	7.510(-3)	1.840(-4)	4.577(-4)	1.143(-4)	2.857(-5)
6	2.437(-2)	8.301(-3)	2.768(-3)	8.986(-4)	2.841(-4)	8.763(-5)
7	2.466(-2)	8.408(-3)	2.800(-3)	9.100(-4)	2.876(-4)	8.870(-5)
8	2.482(-2)	8.466(-3)	2.818(-3)	9.163(-4)	2.895(-4)	8.928(-5)
9	2.492(-2)	8.496(-3)	2.827(-3)	9.195(-4)	2.905(-4)	8.959(-5)
10	2.499(-2)	8.513(-3)	2.832(-3)	9.211(-4)	2.910(-4)	8.974(-5)
11	2.508(-2)	8.527(-3)	2.834(-3)	9.220(-4)	2.912(-4)	8.982(-5)
12	2.522(-2)	8.544(-3)	2.836(-3)	9.224(-4)	2.913(-4)	8.986(-5)
13	2.550(-2)	8.577(-3)	2.838(-3)	9.226(-4)	2.914(-4)	8.988(-5)
14	6.513(-2)	8.640(-3)	2.842(-3)	9.229(-4)	2.914(-4)	8.989(-5)
15	2.643(-1)	8.767(-3)	2.851(-3)	9.234(-4)	2.915(-4)	8.989(-5)

Table 2. Errors E_N for the scheme (3.7)

l	N					
	32	64	128	256	512	1024
1	3.103(-9)	1.940(-10)	1.213(-11)	7.522(-13)	5.564(-14)	1.136(-14)
2	1.250(-7)	7.807(-9)	4.879(-10)	3.049(-11)	1.878(-12)	7.080(-14)
3	2.138(-6)	1.332(-7)	8.318(-9)	5.200(-10)	3.250(-11)	2.159(-12)
4	3.354(-5)	2.070(-6)	1.290(-7)	8.057(-9)	5.035(-10)	3.145(-11)
5	3.037(-4)	3.294(-5)	2.035(-6)	1.268(-7)	7.918(-9)	4.948(-10)
6	3.007(-4)	3.817(-5)	4.360(-6)	4.644(-7)	4.647(-8)	4.425(-9)
7	2.992(-4)	3.802(-5)	4.342(-6)	4.627(-7)	4.630(-8)	4.409(-9)
8	2.985(-4)	3.795(-5)	4.334(-6)	4.619(-7)	4.621(-8)	4.406(-9)
9	2.981(-4)	3.791(-5)	4.329(-6)	4.615(-7)	4.617(-8)	4.397(-9)
10	2.980(-4)	3.790(-5)	4.327(-6)	4.613(-7)	4.615(-8)	4.395(-9)
11	2.978(-4)	3.789(-5)	4.326(-6)	4.612(-7)	4.614(-8)	4.394(-9)
12	2.977(-4)	3.788(-5)	4.326(-6)	4.612(-7)	4.614(-8)	4.393(-9)
13	2.975(-4)	3.788(-5)	4.326(-6)	4.612(-7)	4.613(-8)	4.393(-9)
14	2.975(-4)	3.787(-5)	4.325(-6)	4.612(-7)	4.613(-8)	4.393(-9)
15	2.974(-4)	3.786(-5)	4.325(-6)	4.612(-7)	4.613(-8)	4.393(-9)

Table 3. Estimated convergence orders P_N for the scheme (3.8)

l	N				
	32	64	128	256	512
1	2.00	2.00	2.00	2.00	2.00
2	2.00	2.00	2.00	2.00	2.00
3	2.00	2.00	2.00	2.00	2.00
4	2.03	2.01	2.00	2.00	2.00
5	1.67	2.03	2.01	2.00	2.00
6	1.55	1.58	1.62	1.66	1.70
7	1.55	1.59	1.62	1.66	1.70
8	1.55	1.59	1.62	1.66	1.70
9	1.55	1.59	1.62	1.66	1.70
10	1.55	1.59	1.62	1.66	1.70
11	1.56	1.59	1.62	1.66	1.70
12	1.56	1.59	1.62	1.66	1.70
13	1.57	1.59	1.62	1.66	1.70
14	2.91	1.60	1.62	1.66	1.70
15	4.91	1.62	1.63	1.66	1.70

Table 4. Estimated convergence orders P_N for the scheme (3.7)

l	N				
	32	64	128	256	512
1	4.00	4.00	4.01	3.76	2.29
2	4.00	4.00	4.00	4.02	4.73
3	4.00	4.00	4.00	4.00	3.91
4	4.02	4.00	4.00	4.00	4.00
5	3.20	4.02	4.00	4.00	4.00
6	2.98	3.13	3.23	3.32	3.39
7	2.98	3.13	3.23	3.32	3.39
8	2.97	3.13	3.23	3.32	3.39
9	2.97	3.13	3.23	3.32	3.39
10	2.97	3.13	3.23	3.32	3.39
11	2.97	3.13	3.23	3.32	3.39
12	2.97	3.13	3.23	3.32	3.39
13	2.97	3.13	3.23	3.32	3.39
14	2.97	3.13	3.23	3.32	3.39
15	2.97	3.13	3.23	3.32	3.39

Assuming convergence of the order MN^{-p}, for some p, for each fixed ε we compute E_N (and $E_{G,N}$) for two consecutive values of k. Because of

$$\frac{E_N}{E_{2N}} \approx \frac{(N^{-k})^p}{(N^{-2k})^p} = 2^{-p},$$

in Table 3 we estimate the convergence order p for each fixed ε from

$$P_N = \frac{\ln E_N - \ln E_{2N}}{\ln 2}, \text{ for } N = 2^k \text{ and } k = 4, 5, \ldots 10,$$

and in Table 4 from

$$P_{G,N} = \frac{\ln E_{G,N} - \ln E_{G,2N}}{\ln 2}, \text{ for } N = 2^k \text{ and } k = 4, 5, \ldots 10.$$

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