Novi Sad J. Math.
Vol. 31, No. 1, 2001,133-140

SOME COMPARISONS OF DIFFERENCE SCHEMES ON MESHES OF SHISHKIN AND BAKHVALOV TYPE

Katarina Surla[1], Dragoslav Herceg[1], Helena Maličić[1]

Abstract. The spline difference schemes for a linear singularly perturbed boundary value problem on the meshes of Shishkin and Bakhvalov type are compared. An example of the scheme which produces better results on Shishkin type mesh than on Bakhvalov type mesh is given. This scheme is obtained using a combination of a cubic spline scheme and a central difference scheme. In [5], is shown that the scheme is almost fourth order accurate (uniformly in the perturbation parameter) on the mesh of Shishkin type. Here, a uniform second-order convergence for Bakhvalov type mesh is proved. Numerical results are presented in support of this result.

AMS Mathematics Subject Classification (1991): 65L10, 65H10, 65F10

Key words and phrases: singular perturbation, spline difference schemes, Shishkin type mesh, Bakhvalov type mesh

1. Introduction

We consider the problem

$$(1)\qquad \begin{aligned} -\varepsilon y'' + p(x)\, y = f(x), \quad 0 \le x \le 1 \\ y(0) = \alpha_0, \quad y(1) = \alpha_1, \end{aligned}$$

where $\varepsilon \in (0, \varepsilon_0)$, $\varepsilon_0 \ll 1$, is a small perturbation parameter, and we shall assume that

$$p, f \in C^k[0,1], \quad k \ge 2, \quad p(x) \ge p > 0.$$

It is known [7] that the problem (2) has a unique solution $y(x) \in C^{k+2}[0,1]$, which exhibits boundary layers near 0 and 1.

The difference schemes derived via polynomial approximations or spline collocations on uniform meshes do not converge uniformly with respect to the small parameter ε. In many papers, special non-uniform meshes are used in order to obtain uniform convergence. we shall use two types of non-uniform meshes: Shishkin and Bakhvalov type (we shall often call them S–type and B–type meshes). Many known results considering the applications of these meshes on difference schemes for singularly perturbed boundary value problem, show

[1]Institute of Mathematics, University of Novi Sad, Trg Dositeja Obradovića 4, 21000 Novi Sad, Yugoslavia, e-mail: {ksurla, hercegd, helena}@unsim.im.ns.ac.yu

better results for the B–type of mesh. Namely, if for one scheme the B–type of mesh produces an $O(n^{-p})$ rate of convergence, then for the same scheme the S–type gives $O(n^{-p}\ln^p n)$. Since S–type of mesh is simpler to use, this small loss of accuracy can be considered as irrelevant. In this paper we shall present a scheme which produces better results on a mesh of S–type than of B–type.

2. Construction of the scheme

In [6], a family of difference schemes using spline collocation with cubic spline is derived on the mesh Δ^n defined as

$$\Delta^n : 0 = x_0 < x_1 < \cdots < x_{n-1} < x_n = 1,$$

with $h_j = x_j - x_{j-1}$, $j = 1, 2, \ldots, n$. Using the notation $f_j = f(x_j)$, these schemes can be represented as

$$\text{(2)}\qquad \begin{cases} Ru_j = Qf_j, & j = 0, 1, \ldots, n, \\ u_0 = \alpha_0, & u_n = \alpha_1, \end{cases}$$

where

$$Ru_j = r_j^- u_{j-1} + r_j^c u_j + r_j^+ u_{j+1}$$

$$Qf_j = q_j^- f_{j-1} + q_j^c f_j + q_j^+ f_{j+1}$$

and $u = [u_0, u_1, \ldots, u_n]^\top$ is a numerical approximation to the solution y at the mesh points $\{x_0, x_1, \ldots, x_n\}$.

If in (2)

$$r_j^- = -\frac{2\varepsilon}{h_j(h_j + h_{j+1})}, \qquad r_j^+ = -\frac{2\varepsilon}{h_{j+1}(h_j + h_{j+1})}$$

$$\text{(3)}\qquad r_j^c = \frac{2\varepsilon}{h_j h_{j+1}} + p_j,$$

$$q_j^- = 0, \qquad q_j^c = 1, \qquad q_j^+ = 0,$$

the central difference scheme is obtained for which in [7] a second order of convergence is proved on the B–type of non-uniform mesh. When the same scheme is observed on S–type of mesh in [3], the rate of convergence $O(n^{-2}\ln^2 n)$ is obtained.

Now, if in (2) the coefficients are

$$r_j^- = -\left(1 - \frac{h_j^2 p_{j-1}}{6\varepsilon}\right)\frac{1}{h_j}, \qquad r_j^+ = -\left(1 - \frac{h_{j+1}^2 p_{j+1}}{6\varepsilon}\right)\frac{1}{h_{j+1}},$$

$$r_j^c = \left(1 + \frac{h_j^2 p_j}{3\varepsilon}\right)\frac{1}{h_j} + \left(1 + \frac{h_{j+1}^2 p_j}{3\varepsilon}\right)\frac{1}{h_{j+1}} \tag{4}$$

$$q_j^- = \frac{h_j}{6\varepsilon}, \qquad q_j^c = \frac{h_j}{3\varepsilon} + \frac{h_{j+1}}{3\varepsilon}, \qquad q_j^+ = \frac{h_{j+1}}{6\varepsilon},$$

the scheme which corresponds to the spline from $C^2[0,1]$ is obtained. This is analyzed both in [2] and [4], where the rates of convergence $O(n^{-2})$ and $O(n^{-2}\ln^2 n)$ are proved on the meshes of Bakhvalov and Shishkin type, respectively.

If each equation from (4) is multiplied by $\varepsilon/\hbar_j$ and added to the corresponding equation from (3), where

$$\hbar_j = \frac{h_j + h_{j+1}}{2}, \qquad j = 1, 2, \ldots, n-1,$$

the new scheme has the coefficients

$$r_j^- = \frac{\varepsilon}{\hbar_j}\left(-\frac{2}{h_j} + \frac{h_j p_{j-1}}{6\varepsilon}\right), \qquad r_j^+ = \frac{\varepsilon}{\hbar_j}\left(-\frac{2}{h_{j+1}} + \frac{h_{j+1} p_{j+1}}{6\varepsilon}\right),$$

$$r_j^c = \frac{\varepsilon}{\hbar_j}\left(\frac{2}{h_j} + \frac{2}{h_{j+1}} + \frac{5 h_j p_j}{6\varepsilon} + \frac{5 h_{j+1} p_j}{6\varepsilon}\right), \tag{5}$$

$$q_j^- = \frac{h_j}{6\hbar_j}, \qquad q_j^c = \frac{5}{3}, \qquad q_j^+ = \frac{h_{j+1}}{6\hbar_j}.$$

3. The mesh construction

The S–type mesh appropriate to the problem (2) is defined as follows. The interval $[0,1]$ is divided into three subintervals $[0,\sigma]$, $[\sigma, 1-\sigma]$, $[1-\sigma, 1]$, where the transition points are σ and $1-\sigma$ with

$$\sigma = \min\left\{\frac{1}{4}, \frac{m}{p_0}\sqrt{\varepsilon}\ln n\right\}, \qquad p_0 = \min\{p, 1\}. \tag{6}$$

The mesh is piecewise equidistant with the mesh spacing $h_1 = 4\sigma n^{-1}$ on $[0,\sigma]$ and $[1-\sigma, 1]$, and $h_2 = 2(1-2\sigma)n^{-1}$ on $[\sigma, 1-\sigma]$.

The Bakhvalov type of mesh that we shall use has the mesh points $x_i = \lambda(i/n)$, $i = 0, 1, \ldots, n$, where λ is a mesh generating function from [7] defined with

$$\lambda(t) = \begin{cases} \mu(t) := \dfrac{a\sqrt{\varepsilon}t}{q-t}, & 0 \le t \le \alpha, \\ \mu(\alpha) + \mu'(\alpha)(t-\alpha), & \alpha \le t \le 0.5, \\ 1 - \lambda(1-t), & 0.5 \le t \le 1, \end{cases} \tag{7}$$

and $q \in (0, 0.5)$, $a \in (0, q/\sqrt{\varepsilon})$. Here α is the abscissa of the contact point of the tangent line from $(0.5, 0.5)$ to $\mu(t)$:

$$\alpha = \frac{q - \sqrt{aq\sqrt{\varepsilon}\,(1 - 2q + 2a\sqrt{\varepsilon})}}{1 + 2a\sqrt{\varepsilon}}.$$

4. Convergence results

Let M and M_1 denote constants independent of ε and n. The scheme (5) applied on the Shishkin mesh (6) is analyzed in [5]. It is proved that for $y(x) \in C^6[0,1]$ and $m = 4$ the scheme is almost fourth order convergent, i.e. it holds

$$|y(x_j) - u_j| \leq M\left(\varepsilon n^{-1} + n^{-4}\ln^4 n\right), \qquad j = 0, 1, \dots, n.$$

Here we shall consider the scheme (5) on the Bakhvalov mesh (7) and show second order of convergence.

Theorem 4.1 *Let $y(x) \in C^4[0,1]$ and $u = [u_0, u_1, \dots, u_n]^{\mathrm{T}}$ be the numerical solution of the system (2) using the scheme (5) on the Bakhvalov type of mesh defined in (7). Then*

$$|y(x_j) - u_j| \leq Mn^{-2}, \qquad j = 0, 1, \dots, n. \tag{8}$$

Proof. For the central scheme (3) on the B–mesh (5) the error estimate (8) holds (see [7]). An identical estimate is obtained for the scheme (4) on the same mesh in [2]. It is evident that their linear combination, i.e. the scheme (5), cannot have rate of convergence less than two. Analyzing the leading member of the truncation error, we shall prove that the rate of convergence cannot be greater than two.

In [7] is shown that the solution $y(x) \in C^{k+2}[0,1]$ of the problem (2) has the property

$$\left|y^{(i)}(x)\right| \leq \begin{cases} M\left(1 + \varepsilon^{-\frac{i}{2}}\exp\left(-\frac{\beta x}{\sqrt{\varepsilon}}\right)\right), & 0 \leq x \leq 0.5, \\ M\left(1 + \varepsilon^{-\frac{i}{2}}\exp\left(-\frac{\beta(1-x)}{\sqrt{\varepsilon}}\right)\right), & 0.5 \leq x \leq 1, \end{cases} \tag{9}$$

where $x \in [0,1]$, $\beta = \sqrt{p}$ and $i = 0, 1, \dots, k$. Also, from [7], for the Bakhvalov type of mesh (7) it holds

$$h_{j+1} - h_j \leq \frac{Mn^{-2}}{\sqrt[4]{\varepsilon}}, \qquad \lambda(\alpha) \geq M\sqrt[4]{\varepsilon}. \tag{10}$$

Now, let $(j-1)/n < \alpha \leq j/n$ for some j. Then $x_j \in [0, 0.5]$. The truncation error $\tau_j[y]$ for the scheme (5) has the form

$$\tau_j[y] = \frac{\varepsilon}{\hbar_j} \left(\frac{h_j^2 - h_{j+1}^2}{6} y_j''' + \frac{h_j^4 - h_{j+1}^4}{90} y_j^V - \frac{1}{360}\left(h_j^5 y^{VI}(\zeta_j) + h_{j+1}^5 y^{VI}(\zeta_{j+1})\right) \right.$$

$$\left. - \frac{1}{144}\left(h_j^5 y^V(\xi_j) + h_{j+1}^5 y^V(\xi_{j+1})\right)\right),$$

with $\zeta_j, \xi_j \in (x_{j-1}, x_j)$, $\zeta_{j+1}, \xi_{j+1} \in (x_j, x_{j+1})$. If A_j is the leading member in $\tau_j[y]$, then, using that $\lambda(t)$ is an increasing function, from (9) and (10) we have

$$|A_j| = \varepsilon \left| \frac{h_j - h_{j+1}}{3} y_j''' \right| \leq M n^{-2} \left(\varepsilon^{\frac{3}{4}} + \varepsilon^{-\frac{3}{4}} \exp\left(-\frac{M_1 \beta}{\sqrt{\varepsilon}} \right) \right) \leq M n^{-2}.$$

Therefore, the scheme (5) on the Bakhvalov type of mesh defined in (7) has the second order of convergence and the proof is complete. ◇

Furthermore, for the Shishkin mesh we shall analyze the same member of the truncation error $\tau_{i_0}[y] = \Phi_{i_0}[y] + \Psi_{i_0}[y]$ at the transition point $\sigma = x_{i_0}$, where $\Phi_{i_0}[y]$ and $\Psi_{i_0}[y]$ are the truncation errors which come from the the schemes (3) and (4), respectively. In the integral form

$$\Phi_{i_0}[y] = \frac{\varepsilon}{2\hbar_{i_0}} \int_{x_{i_0}}^{x_{i_0-1}} (x_{i_0-1} - s)^2 \left(1 + \frac{exp(-\beta s/\sqrt{\varepsilon})}{\sqrt{\varepsilon^3}} \right) ds$$

$$- \frac{\varepsilon}{2\hbar_{i_0}} \int_{x_{i_0}}^{x_{i_0+1}} (x_{i_0+1} - s)^2 \left(1 + \frac{exp(-\beta s/\sqrt{\varepsilon})}{\sqrt{\varepsilon^3}} \right) ds.$$

Using the properties of mesh and the choice of parameter $m = 4$, we have

$$|\Phi_{i_0}[y]| \leq \frac{M\varepsilon}{\hbar_{i_0}} \int_{x_{i_0}}^{x_{i_0+1}} (x_{i_0+1} - s)^2 \left(1 + \frac{exp(-\beta s/\sqrt{\varepsilon})}{\sqrt{\varepsilon^3}} \right) ds \leq M\left(\varepsilon n^{-1} + n^{-5}\right).$$

We can conclude that for the scheme (5) a free choice of m on the Shishkin mesh can give an almost arbitrary degree of accuracy in respect to exponential parts of the error. Instead, the Bakhvalov mesh at the transition points lowers rate of convergence (on an equidistant mesh, this rate is four). Let us just mention that the stability of the scheme (5) follows from stability of the schemes (3) and (4).

5. Numerical results

In this section we present numerical results for the scheme (5) applied to the problem (2) on the non-uniform meshes defined in Section 3.

Let $u^*(x)$ be the cubic spline interpolant of the numerical aproximation u. We denote by E_n the maximum of $|y(x_j) - u^*(x_j)|$, $x_j \in \Delta_S^{4096} \cup \Delta_B^{4096}$, $j =$

$0, 1, \ldots, n$, where Δ_S^{4096} and Δ_B^{4096} denote the meshes of Shishkin and Bakhvalov type respectively. Assuming the order of convergence $M\left(n^{-1}\ln n\right)^r$ on the S–type of mesh we estimate the classical convergence rate r with

$$R_\varepsilon^n = \frac{\ln \tilde{E}_\varepsilon^n - \ln \tilde{E}_\varepsilon^{2n}}{\ln\left(\frac{2k}{k+1}\right)}, \qquad n = 2^k, \qquad k = 5, 6, \ldots, 11,$$

$$\tilde{E}_\varepsilon^n = \max_{0\le j\le n} \left|u_j^n - \tilde{u}_{2j}^{2n}\right|,$$

and $\tilde{u}^n \in \mathbf{R}^{n+1}$ is the solution on the Shishkin type of mesh with the parameter altered slightly to

$$\sigma = \min\left\{\frac{1}{4}, \frac{m}{p_0}\sqrt{\varepsilon}\ln\frac{n}{2}\right\}, \quad m = 4.$$

Assuming convergence of the order Mn^{-t} for some t on the B–type of mesh, we compute

$$T_\varepsilon^n = \frac{\ln E_\varepsilon^n - \ln E_\varepsilon^{2n}}{\ln 2}, \qquad n = 2^k, \qquad k = 5, 6, \ldots, 11,$$

as an approximation to t, where

$$E_\varepsilon^n = \max_{0\le j\le n} \left|u_j^n - u_{2j}^{2n}\right|.$$

Different values of $\varepsilon = 2^{-s}$ and n are considered on both meshes, and we used $q = 0.48$ and $a = 1$ for the mesh (7). For other values of q and a similar results were obtained.

We tested the following examples, [1]. The first example is

$$-\varepsilon y'' + (1 + x(1-x))\, y = f(x), \qquad y(0) = y(1) = 0,$$

$$y(x) = 1 - (1-x)\exp\left(-\frac{x}{\sqrt{\varepsilon}}\right) - x\exp\left(-\frac{1-x}{\sqrt{\varepsilon}}\right),$$

and the second example is

$$-\varepsilon y'' + y = -\cos^2(\pi x) - 2\varepsilon\pi^2\cos(2\pi x), \qquad y(0) = y(1) = 0,$$

$$y(x) = \frac{\exp\left(-\frac{1-x}{\sqrt{\varepsilon}}\right) + \exp\left(-\frac{x}{\sqrt{\varepsilon}}\right)}{1 + \exp\left(-\frac{1}{\sqrt{\varepsilon}}\right)} - \cos^2(\pi x).$$

The following tables show the maximum error E_n and the rate of convergence *Ord* on the S–type and B–type of meshes.

Table 1: Example 1 — S–mesh

s	n							
	32	64	128	256	512	1024	2048	
4	1.72(−5)	1.17(−6)	7.67(−8)	4.90(−9)	3.09(−10)	1.95(−11)	1.00(−11)	E_n
	5.43	5.14	4.95	4.42	*error*			Ord
6	1.73(−4)	1.28(−5)	8.74(−7)	5.71(−8)	3.65(−9)	2.30(−10)	1.45(−11)	E_n
	5.42	5.14	4.95	4.81	4.16			Ord
8	1.69(−3)	1.46(−4)	1.07(−5)	7.30(−7)	4.76(−8)	3.04(−9)	1.92(−10)	E_n
	5.41	5.14	4.95	4.82	4.70			Ord
10	1.35(−2)	1.54(−3)	1.32(−4)	9.70(−6)	6.58(−7)	4.29(−8)	2.74(−9)	E_n
	5.38	5.13	4.95	4.82	4.72			Ord
12	5.29(−2)	1.29(−2)	1.46(−3)	1.25(−4)	9.18(−6)	6.22(−7)	4.05(−8)	E_n
	3.84	5.09	4.94	4.82	4.72			Ord
16	5.16(−2)	1.40(−2)	2.68(−3)	3.93(−4)	4.76(−5)	5.06(−6)	4.93(−7)	E_n
	3.56	3.99	3.95	3.99	4.00			Ord
20	5.13(−2)	1.39(−2)	2.66(−3)	3.89(−4)	4.71(−5)	5.01(−6)	4.88(−7)	E_n
	3.57	3.99	3.95	3.99	4.00			Ord

Table 2: Example 1 — B–mesh

s	n							
	32	64	128	256	512	1024	2048	
4	2.65(−4)	6.56(−5)	1.63(−5)	4.08(−6)	1.02(−6)	2.55(−7)	6.36(−8)	E_n
	1.98	2.00	2.00	2.00	2.00			Ord
6	1.58(−3)	3.88(−4)	9.63(−5)	2.41(−5)	6.01(−6)	1.50(−6)	3.76(−7)	E_n
	1.99	1.99	2.00	2.00	2.00			Ord
8	3.36(−3)	8.62(−4)	2.11(−4)	5.26(−5)	1.31(−5)	3.28(−6)	8.20(−7)	E_n
	2.00	2.00	2.00	2.00	2.00			Ord
10	1.37(−2)	1.00(−3)	2.42(−4)	5.99(−5)	1.49(−5)	3.74(−6)	9.35(−7)	E_n
	2.01	2.00	2.00	2.00	2.00			Ord
12	2.91(−2)	1.45(−3)	2.46(−4)	6.10(−5)	1.52(−5)	3.80(−6)	9.50(−7)	E_n
	2.02	2.00	2.00	2.00	2.00			Ord
16	2.19(−1)	1.79(−3)	2.50(−4)	6.19(−5)	1.55(−5)	3.86(−6)	9.65(−7)	E_n
	2.02	2.00	2.00	2.00	2.00			Ord
20	4.85(−1)	1.82(−3)	2.51(−4)	6.22(−5)	1.55(−5)	3.88(−6)	9.69(−7)	E_n
	2.02	2.00	2.00	2.00	2.00			Ord

Table 3: Example 2 — S–mesh

s	n							
	32	64	128	256	512	1024	2048	
4	2.13(−5)	1.33(−6)	8.26(−8)	5.12(−9)	3.19(−10)	1.99(−11)	5.20(−12)	E_n
	5.43	5.15	4.95	4.78	3.26			Ord
6	8.84(−5)	6.74(−6)	4.66(−7)	3.06(−8)	1.96(−9)	1.24(−10)	7.81(−12)	E_n
	5.42	5.14	4.95	4.80	3.67			Ord
8	1.36(−3)	1.16(−4)	8.54(−6)	5.79(−7)	3.77(−8)	2.40(−9)	1.52(−10)	E_n
	5.41	5.14	4.95	4.82	4.66			Ord
10	1.24(−2)	1.39(−3)	1.18(−4)	8.66(−6)	5.86(−7)	3.81(−8)	2.43(−9)	E_n
	5.38	5.13	4.95	4.82	4.72			Ord
12	5.12(−2)	1.24(−2)	1.39(−3)	1.18(−4)	8.66(−6)	5.87(−7)	3.82(−8)	E_n
	3.87	5.10	4.94	4.82	4.00			Ord
16	5.12(−2)	1.39(−2)	2.65(−3)	3.88(−4)	4.69(−5)	4.99(−6)	4.86(−7)	E_n
	3.57	3.99	3.95	3.99	4.00			Ord
20	5.12(−2)	1.39(−2)	2.65(−3)	3.88(−4)	4.69(−5)	4.99(−6)	4.86(−7)	E_n
	3.57	3.99	3.95	3.99	4.00			Ord

Table 4: Example 2 — B–mesh

s	n							
	32	64	128	256	512	1024	2048	
4	3.61(−4)	9.22(−5)	2.30(−5)	5.74(−6)	1.43(−6)	3.59(−7)	8.97(−8)	E_n
	1.97	2.01	2.00	2.00	2.00			Ord
6	1.90(−3)	4.75(−4)	1.19(−4)	2.96(−5)	7.40(−6)	1.85(−6)	4.63(−7)	E_n
	2.00	2.00	2.00	2.00	2.00			Ord
8	3.95(−3)	1.04(−3)	2.56(−4)	6.36(−5)	1.59(−5)	3.97(−6)	9.93(−7)	E_n
	1.96	1.99	2.00	2.00	2.00			Ord
10	1.25(−2)	1.11(−3)	2.64(−4)	6.52(−5)	1.63(−5)	4.07(−6)	1.02(−6)	E_n
	2.01	2.00	2.00	2.00	2.00			Ord
12	2.89(−2)	1.33(−3)	2.52(−4)	6.25(−5)	1.56(−5)	3.90(−6)	9.74(−7)	E_n
	2.02	2.00	2.00	2.00	2.00			Ord
16	2.21(−1)	1.20(−2)	9.04(−4)	6.23(−5)	1.55(−5)	3.88(−6)	9.70(−7)	E_n
	2.02	2.00	2.00	2.00	2.00			Ord
20	4.21(−1)	6.28(−3)	4.20(−3)	2.91(−4)	1.86(−5)	3.88(−6)	9.71(−7)	E_n
	2.02	2.00	2.00	2.00	2.00			Ord

References

[1] Doolan E.P., Miller J.J.H., Schilders W.H.A., Uniform numerical methods for problems with initial and boundary layers, Dublin, Boole Press, 1980.

[2] Herceg D., Surla K., Rapajić S., On a mesh of Bakhvalov type, Novi Sad Journal of Mathematics, 28 (1998), 41–49.

[3] Sun G., Stynes M., A uniformly convergent method for a singularly perturbed semilinear reaction–diffusion problem with multiple solutions, Math. Comput., 65 (1996), 1085–1109.

[4] Surla K., On modelling of semilinear singularly perturbed reaction–diffusion problem, Nonlinear Analysis, Theory, Methods & Applications, 30 (1997), 61–66.

[5] Surla K., Vukoslavčević V., Uniform almost fourth order scheme for a singular perturbation problem, Novi Sad Journal of Mathematics (in print)

[6] Vukoslavčević V., Surla K., Rapajić S., Some uniformly convergent schemes on Shishkin mesh, Novi Sad Journal of Mathematics (in print)

[7] Vulanović R., Mesh construction for discretization of singularly perturbed boundary value problems, Ph.D. thesis, University of Novi Sad, 1986.