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# A THEOREM ON COINCIDENCE POINT FOR MULTIVALUED MAPPINGS IN A CLASS OF PROBABILISTIC METRIC SPACES

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**Abstract.** A coincidence point theorem for multivalued mappings in probabilistic metric spaces is proved. The obtained result is a generalization of a fixed point theorem from [5].

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## 1. Introduction

T. L. Hicks proved in [6] a fixed point theorem for the so-called $C$ -contraction in probabilistic metric spaces $(S,\ \mathcal{F},\ T_M)$. A mapping $f : S \to S$ is a $C$-contraction if there is a $k \in (0,\ 1)$ such that for every $p,\ q \in S$ and $x > 0$

$$F_{p,q}(x) > 1 - x \Rightarrow F_{fp,fq}(kx) > 1 - kx.$$

V. Radu proved in [10] that $f : S \to S$ is a $C$-contraction on $S$, where $(S,\ \mathcal{F},\ T)$ is a complete Menger space with $T \geq T_L$, $T_L(a,\ b) = \max\ \{a + b - 1,\ 0\}$ $(a,\ b \in [0,\ 1])$, if and only if $f$ is a metric contraction on the metric space $(S,\ \beta)$,

$$\beta(p,\ q) = \inf\{h;\ F_{p,q}(h^+) > 1 - h\}.$$

If $\sup_{x<1} T(x,\ x) = 1$ V. Radu proved that a $C$-contraction $f : S \to S$, where $(S,\ \mathcal{F},\ T)$ complete Menger space, has a fixed point.

A generalization of the notion of a $C$-contraction for multivalued mappings is given by E. Pap, O. Hadžić and R. Mesiar in [9] where they proved a fixed point theorem for a $(\Psi,\ C)$- multivalued contraction in the Menger space.

Let $2_c^{f(S)}$ denote the family of all nonempty, closed subsets of $f(S)$.

In this paper we proved a theorem on coincidence point for singlevalued mapping $f : S \to S$ and multivalued mappings $A,\ B : S \to 2_c^{f(S)}$.

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## 2. Preliminaries

Let $\Delta^+$ be the set of all distribution functions $F$ such that $F(0) = 0$ ($F$ is a nondecreasing, left continuous mapping from $\mathbf{R}$ into $[0,1]$ such that $\sup_{x\in\mathbf{R}} F(x) = 1$).

The ordered pair $(S, \mathcal{F})$ is said to be **a probabilistic metric space** if $S$ is a nonempty set and $\mathcal{F} : S \times S \to \Delta^+$ ($\mathcal{F}(p,q)$ is written by $F_{p,q}$ for every $(p,q) \in S \times S$) satisfies the following conditions:

1. $F_{u,v}(x) = 1$ for every $x > 0 \Rightarrow u = v$ $(u, v \in S)$.
2. $F_{u,v} = F_{v,u}$ for every $u, v \in S$.
3. $F_{u,v}(x) = 1$ and $F_{v,w}(y) = 1 \Rightarrow F_{u,w}(x+y) = 1$ for $u, v, w \in S$ and $x, y \in \mathbf{R}^+$.

A **Menger space** (see [11]) is a triple $(S, \mathcal{F}, T)$, where $(S, \mathcal{F})$ is a probabilistic metric space, $T$ is a triangular norm (abbreviated t -norm) and the following inequality holds

$$F_{u,v}(x+y) \geq T(F_{u,w}(x), F_{w,v}(y)) \text{ for every } u, v, w \in S \text{ and every } x > 0, y > 0.$$

Recall that a mapping $T : [0,1] \times [0,1] \to [0,1]$ is called a **triangular norm** (a t-norm) if the following conditions are satisfied:
$T(a, 1) = a$ for every $a \in [0,1]$ ; $T(a,b) = T(b,a)$ for every $a, b \in [0,1]$;

$$a \geq b,\ c \geq d \Rightarrow \ T(a,c) \geq T(b,d) \ \ (a,b,c,d \in [0,1]);$$

$$T(a, T(b,c)) = T(T(a,b),c) \ \ (a,b,c \in [0,1]).$$

**Example 1.** The following are the four basic t-norms :

(i) The **minimum** t-norm, $T_M$, is defined by

$$T_M(x,y) = \min(x,y),$$

(ii) The **product** t-norm, $T_P$, is defined by

$$T_P(x,y) = x \cdot y,$$

(iii) The **Lukasiewicz** t-norm $T_L$ is defined by

$$T_L(x,y) = \max(x+y-1, 0),$$

(iv) The weakest t-norm, the **drastic product** $T_D$, is defined by

$$T_D(x,y) = \begin{cases} \min(x,y) & \text{if } \max(x,y) = 1, \\ 0 & \text{otherwise.} \end{cases}$$

□

Let $T$ be a t-norm and let for every $n \in \mathbf{N}$, $T_n : [0, 1] \to [0, 1]$ be defined by

$$T_1(x) = T(x, x),\ T_n(x) = T(T_{n-1}(x), x),\ n > 2,\ x \in [0, 1].$$

If the family of functions $\{T_n(x)\}_{n \in \mathbf{N}}\}$ is equicontinuous at the point $x = 1$ then $T$ norm t is said to be of H-type. A nontrivial example of a T-norm of H-type is given in [9].

As regards the pointwise ordering, we have the inequalities

$$T_D < T_L < T_P < T_M.$$

The $(\epsilon, \lambda)-$ topology in $S$ is introduced by the family of neighbourhoods $\mathcal{U} = \{U_v(\epsilon, \lambda)\}_{(v,\epsilon,\lambda) \in S \times \mathbf{R}_+ \times (0,1)}$, where

$$U_v(\epsilon, \lambda) = \{u;\ F_{u,v}(\epsilon) > 1 - \lambda\}.$$

If a t-norm $T$ is such that $\sup_{x<1} T(x, x) = 1$, then $S$ is in the $(\epsilon, \lambda)$-topology a metrizable topological space.

The t-norm $T$ is called **strict** if it is continuous and strictly monotone. The t-norm $T$ is called **nilpotent** if each $a \in (0, 1)$ is a nilpotent element of $T$, i.e., for every $a \in (0, 1)$ there exists $b \in (0, 1)$ such that $T(a, b) = 0$. A function $T : [0, 1]^2 \to [0, 1]$ is a **continuous Archimedean triangular norm** if for all $x \in (0, 1)$ we have $T(x, x) < x$. The class of continuous Archimedean t-norms consists of two disjoint classes: the class of strict t-norms and the class of nilpotent t-norms.

The following representation theorem holds, see [8].

**Theorem 1.** *A function $T : [0, 1]^2 \to [0, 1]$ is a continuous Archimedean triangular norm if and only if there exists a continuous, strictly increasing function $\theta : [0, 1] \to [0, 1]$ such that $\theta(1) = 1$ (so-called multiplicative generator), and such that for all $x, y \in [0, 1]$*

$$T(x, y) = \theta^{(-1)}(\theta(x)\theta(y)).$$

*Moreover, $T$ is a strict t-norm if and only if every continuous multiplicative generator $\theta$ of $T$ satisfies $\theta(0) = 0$.*

As it is known, a multiplicative generator $\theta$ is uniquely determined by $T$ up to a positive constant exponent. If $T$ is a t-norm with a multiplicative generator $\theta$ then the function $\xi : [0, 1] \to [0, 1]$ given by $\xi(x) = \theta(1 - x)$ is a multiplicative generator of the dual t-conorm $S$.

Each t-norm $T$ can be extended (by associativity) in a unique way to an $n$-ary operation taking for $(x_1, \ldots, x_n) \in [0, 1]^n$ the values

$$T_{i=1}^{0} x_i = 1,\ T_{i=1}^{n} x_i = T(T_{i=1}^{n-1} x_i, x_n).$$

Since each t-norm $T$ is weaker than $T_M$, we can extend $T$ to a countable infinitary operation taking for any sequence $(x_i)$ from $[0, 1]$ the values

$$T_{i=1}^{\infty} x_i = \lim_{n \to \infty} T_{i=1}^{n} x_i.$$

## 3. A coincidence point theorem

We shall prove a generalization of the fixed point theorem from [5].

**Definition 1.** *Let $(S, \mathcal{F}, T)$ be a Menger space, $\emptyset \neq M \subset S$, $f : M \to M$ and $A : M \to 2^M$ (the family of nonempty subsets of $M$). The mapping $A$ is $f$-strongly demicompact if for every sequence $\{x_n\}_{n\in\mathbf{N}}$ from $M$, such that $\lim\limits_{n\to\infty} F_{fx_n,y_n}(\epsilon) = 1$, for some sequence $\{y_n\}_{n\in\mathbf{N}}$, $y_n \in Ax_n$, $n \in \mathbf{N}$ and every $\epsilon > 0$, there exists a convergent subsequence $\{fx_{n_k}\}_{k\in\mathbf{N}}$.*

A mapping $A : M \to 2^M$ is weakly commuting with $f : M \to M$ if for every $x \in M$

$$f(Ax) \subset A(fx).$$

**Theorem 2.** *Let $(S, \mathcal{F}, T)$ be a complete Menger space, $\sup_{x<1} T(x, x) = 1$, $M$ a nonempty and closed subset of $S$, $f : M \to M$ a continuous mapping, $A, B : M \to 2_c^{f(M)}$ and $\psi : [0, \infty) \to [0, \infty)$, and there exists $s > 1$ such that the series $\sum_{n=1}^{\infty} \psi^n(s)$ is convergent. Suppose that the following implication holds for every $u, v \in M$ and every $\epsilon > 0$*

$$F_{fu,fv}(\epsilon) > 1-\epsilon \quad \Rightarrow \quad \begin{array}{l} \text{for every } p \in Au \text{ there exists } q \in Bv \\ \text{such that } F_{p,q}(\psi(\epsilon)) > 1-\psi(\epsilon) \text{ and} \\ \text{for every } p' \in Bv \text{ there exists } q' \in Au \\ \text{such that } F_{p',q'}(\psi(\epsilon)) > 1-\psi(\epsilon). \end{array}$$

*If $A$ and $B$ are weakly commuting with $f$ and (a) or (b) are satisfied, then there exists $x \in M$ such that $fx \in Ax \cap Bx$, where*

*(a) $A$ or $B$ are $f$-strongly demicompact.*

*(b) The t-norm $T$ satisfies the condition*

$$\lim_{n\to\infty} \mathrm{T}_{i=n}^{\infty}(1-\psi^i(s)) = 1 .$$

*Proof.* Let $x_0 \in M$ and $x_1 \in M$ be such that $fx_1 \in Ax_0$. Since $s > 1$, then $F_{fx_0,fx_1}(s) > 1-s$ and so there exists $x_2 \in M$ such that $F_{fx_1,fx_2}(\psi(s)) > 1-\psi(s)$ and $fx_2 \in Bx_1$. Continuing in this way we obtain a sequence $\{x_n\}_{n\in\mathbf{N}}$ in $M$ such that for every $n \in \mathbf{N}$.

(i) $fx_{2n+1} \in Ax_{2n}$, $fx_{2n+2} \in Bx_{2n+1}$

(ii) $F_{fx_n,fx_{n+1}}(\psi^n(s)) > 1-\psi^n(s)$.

Since $\lim\limits_{n\to+\infty} \psi^n(s) = 0$, from (ii) it is easy to prove that for every $\epsilon > 0$ and $\lambda \in (0, 1)$ there exists $n_1(\epsilon, \lambda) \in \mathbf{N}$ such that for every $n \geq n_1(\epsilon, \lambda)$, $F_{fx_n,fx_{n+1}}(\epsilon) > 1-\lambda$. That means that for every $\epsilon > 0$

$$\lim_{n\to\infty} F_{fx_n,fx_{n+1}}(\epsilon) = 1. \tag{1}$$

If we suppose that $A$ is $f$-strongly demicompact, using

$$\lim_{n\to\infty} F_{fx_{2n},fx_{2n+1}}(\epsilon) = 1 \text{ and } fx_{2n+1} \in Ax_{2n}\,(n \in \mathbf{N}),$$

we conclude that there exists a convergent subsequence $\{fx_{2n_k}\}_{k\in\mathbf{N}}$ of the sequence$\{fx_{2n}\}_{n\in\mathbf{N}}$.

We shall prove that if $T$ satisfies the condition (b), the sequence $\{fx_n\}_{n\in\mathbf{N}}$ is convergent.

Let $\epsilon > 0$ and $\lambda \in (0,\,1)$. Since the series $\sum_{n=1}^{\infty} \psi^n(s)$ is convergent, there exists $n'(\epsilon,\,s) \in \mathbf{N}$ such that $\sum_{n\geq n'(\epsilon,\,s)} \psi^n(s) < \epsilon$. Then, for every $n \geq n'(\epsilon,\,s)$ and for every $p \in \mathbf{N}$, we have that

$$F_{fx_{n+p+1},fx_n}(\epsilon) \geq \mathrm{T}_{i=n}^{n+p} F_{x_{i+1},x_i}(1-\psi^i(s)),$$

Since $\lim\limits_{n\to\infty} \psi^n(s) = 0$ implies that there exists $n_1(s) \in \mathbf{N}$ such that $\psi^n(s) < 1$ for every $n \geq n_1(s)$, the condition (ii) implies that

$$F_{fx_{n+p+1},fx_n}(\epsilon) \geq \mathrm{T}_{i=n}^{n+p}(1-\psi^i(s))$$

for every $n \geq \max(n_1(s),\, n'(\epsilon,\,s))$ and for every $p \in \mathbf{N}$.

It is obvious that for every $p \in \mathbf{N}$ and $n \geq \max(n_1(s),\, n'(\epsilon,\,s))$

$$\mathrm{T}_{i=n}^{n+p}(1-\psi^i(s)) \geq \mathrm{T}_{i=n}^{\infty}(1-\psi^i(s)),$$

since $T$ is monotone nondecreasing. Hence, for every $p \in \mathbf{N}$ and every $n \geq \max(n_1(s),\, n'(\epsilon,\,s))$

(2) $$F_{fx_{n+p+1},fx_n}(\epsilon) \geq \mathrm{T}_{i=n}^{\infty}(1-\psi^i(s)).$$

From (b) it follows that there exists $n''(s,\,\lambda) \in \mathbf{N}$ such that

(3) $$\mathrm{T}_{i=n}^{\infty}(1-\psi^i(s)) > 1-\lambda,$$

for every $n \geq n''(s,\,\lambda)$.

Hence, (2) and (3) implies that

$$F_{fx_{n+p+1},fx_n}(\epsilon) > 1-\lambda$$

for $n \geq n_0(\epsilon,\,\lambda) = \max(n_1(s),\, n'(\epsilon,\,s),\, n''(s,\,\lambda))$ and every $p \in \mathbf{N}$.

Since $S$ is complete and $M$ is closed we conclude that in both cases (a) and (b) there exists $x = \lim\limits_{k\to\infty} fx_{2n_k} \in M$. From (1) it follows that $x = \lim\limits_{k\to\infty} fx_{2n_k+1}$.

We shall prove that $fx \in Ax \cap Bx$. Since $Ax$ and $Bx$ are closed it remains to be proved that $fx \in \bar{Ax} \cap \bar{Bx}$ i.e. that for every $\epsilon > 0$ and $\lambda \in (0,\,1)$ there exists $q(\epsilon,\,\lambda) \in Ax$ and $r(\epsilon,\,\lambda) \in Bx$ such that

(4) $$q(\epsilon,\,\lambda) \in U_{fx}(\epsilon,\,\lambda), \qquad r(\epsilon,\,\lambda) \in U_{fx}(\epsilon,\,\lambda)$$

Since $\sup_{x<1} T(x, x) = 1$ it is easy to see that there exists $\delta(\lambda) \in (0, 1)$ such that

$$T(1-\delta(\lambda), T(1-\delta(\lambda), 1-\delta(\lambda))) > 1-\lambda.$$

From the continuity of the mapping $f$ and $x = \lim_{k\to\infty} fx_{2n_k}$ it follows that there exists $k_1 \in \mathbf{N}$ such that

$$F_{fx,ffx_{2n_k}}\left(\frac{\epsilon}{3}\right) > 1-\delta(\lambda), \quad \text{for every} \quad k \geq k_1. \tag{5}$$

From (1) implies that there exists $k_2 \in \mathbf{N}$ such that

$$F_{ffx_{2n_k}, ffx_{2n_k+1}}\left(\frac{\epsilon}{3}\right) > 1-\delta(\lambda), \quad \text{for every} \quad k \geq k_2. \tag{6}$$

Let $t_0 \in \mathbf{R}_+$ be such that $\psi(t_0) < \min\left(\frac{\epsilon}{3}, \delta(\lambda)\right)$ and $k_3 \in \mathbf{N}$ such that

$$F_{fx, ffx_{2n_k}}(t_0) > 1-t_0, \quad \text{for every} \quad k \geq k_3. \tag{7}$$

Since $fx_{2n_k+1} \in Ax_{2n_k}$ $(k \in \mathbf{N})$ and $A$ is weakly commuting with $f$ it follows that

$$ffx_{2n_k+1} \in f(Ax_{2n_k}) \subset Afx_{2n_k} \quad (k \in \mathbf{N}). \tag{8}$$

Using (7) and (8) and implication from the theorem, where $v = x$, $u = fx_{2n_k}$ and $p = ffx_{2n_k+1}$, we conclude that there exists a $r(\epsilon, \lambda) = q \in Bx$ such that for $k \geq k_3$

$$F_{ffx_{2n_k+1}, r(\epsilon,\lambda)}(\psi(t_0)) > 1-\psi(t_0)$$

which implies

$$\begin{aligned} F_{ffx_{2n_k+1}, r(\epsilon,\lambda)}\left(\tfrac{\epsilon}{3}\right) \geq\ & F_{ffx_{2n_k+1}, r(\epsilon,\lambda)}(\psi(t_0)) > \\ 21mm] >\ & 1-\psi(t_0) > 1-\delta(\lambda), \quad \text{for every} \quad k \geq k_3. \end{aligned} \tag{9}$$

Hence, for $k \geq \max(k_1, k_2, k_3)$ and using (5), (6) i (9) we obtained

$$\begin{aligned} F_{fx,r(\epsilon,\lambda)}(\epsilon) &\geq T\left(F_{fx,ffx_{2n_k}}\left(\frac{\epsilon}{3}\right), T\left(F_{ffx_{2n_k}, ffx_{2n_k+1}}\left(\frac{\epsilon}{3}\right), F_{ffx_{2n_k+1}, r(\epsilon,\lambda)}\left(\frac{\epsilon}{3}\right)\right)\right) \\ &> 1-\lambda. \end{aligned}$$

Hence, we have $r(\epsilon, \lambda) \in U_{fx}(\epsilon, \lambda)$. From (1) and

$$x = \lim_{k\to\infty} fx_{2n_k} = \lim_{k\to\infty} fx_{2n_k+1} = \lim_{k\to\infty} fx_{2n_k+2}$$

we obtain that there exists $k_1', k_2', k_3' \in \mathbf{N}$ such that

$$F_{fx,ffx_{2n_k+1}}\left(\frac{\epsilon}{3}\right) > 1-\delta(\lambda), \quad \text{for every} \quad k \geq k_1' \tag{10}$$

$$F_{ffx_{2n_k+1}, ffx_{2n_k+2}}\left(\frac{\epsilon}{3}\right) > 1-\delta(\lambda), \quad \text{for every} \quad k \geq k_2' \tag{11}$$

$$F_{fx,ffx_{2n_k+1}}(t_0) > 1-t_0, \quad \text{for every} \quad k \geq k_3'. \tag{12}$$

Since $fx_{2n_k+2} \in Bx_{2n_k+1}$ ( $k \in \mathbf{N}$) and $B$ weakly commutes with $f$ it follows that $ffx_{2n_k+2} \in f(Bx_{2n_k+1}) \subset B(fx_{2n_k+1})$. Hence, there exists $q(\epsilon, \lambda) \in Ax$, so that for every $k \geq k'_3$

$$F_{ffx_{2n_k+2},q(\epsilon,\lambda)}(\psi(t_0)) > 1 - \psi(t_0)$$

which implies for $k \geq \max(k'_1, k'_2, k'_3)$

$$\begin{aligned} F_{fx,q(\epsilon,\lambda)}(\epsilon) \geq & T(F_{fx,ffx_{2n_k+1}}(\frac{\epsilon}{3}), T(F_{ffx_{2n_k+1},ffx_{2n_k+2}}(\frac{\epsilon}{3}), F_{ffx_{2n_k+2},q(\epsilon,\lambda)}(\frac{\epsilon}{3}))) \\ > & 1 - \lambda. \end{aligned}$$

**Example 2.** Let $(M, d)$ be a separable metric space, $(\Omega, \Sigma, P)$ a probability space and $S$ the space of all classes of measurable mappings from $\Omega$ into $M$. Then $(S, \mathcal{F}, T_L)$ is a Menger space where

$$F_{X,Y}(u) = P(\{\omega;\ \omega \in \Omega,\ d(X(\omega), Y(\omega)) < u\}),\ u \in \mathbf{R}^+,\ X, Y \in S.$$

The Ky Fan metric in $S$ is

$$d(X, Y) = \sup\{u;\ F_{X,Y}(u) < 1 - u,\ \ u > 0\}$$

and the $(\epsilon, \lambda)$-topology and the topology induced by $d$ coincide.

Let $f : S \to S$ be a continuous mapping and $A, B : S \to 2_c^{f(S)}$ such that

$$D(AX, BY) \leq \psi(d(fX, fY)),\ \ X, Y \in S,$$

where $\psi$ is a strictly increasing mapping from $\mathbf{R}^+$ into $\mathbf{R}^+$ and $D$ is a Hausdorff metric induced by the metric $d$.

If $F_{fX,fY}(u) > 1 - u$ for some $u$, then $d(fX, fY) < u$ and since $\psi$ is stricly increasing we have that $\psi(d(fX, fY)) < \psi(u)$. Hence

$$\sup_{U \in AX} \inf_{V \in BY} d(U, V) < \psi(u),\quad \sup_{V \in BY} \inf_{U \in AX} d(U, V) < \psi(u),$$

which implies that for every $U \in AX$ there exists $V \in BY$ such that $d(U, V) < \psi(u)$ and that for every $V' \in BY$ there exists $U' \in AX$ such that $d(U', V') < \psi(u)$. So

$$F_{U,V}(\psi(u)) > 1 - \psi(u)\ \text{ i }\ F_{U',V'}(\psi(u)) > 1 - \psi(u).$$

**Example 3.** [5] Let $\{x_n\}_{n \in \mathbf{N}}$ be a sequence of numbers from $[0, 1]$ such that $\lim_{n \to \infty} x_n = 1$ and the t-norm $T$ is of $H$-type. Then

$$\lim_{n \to \infty} \mathrm{T}_{i=n}^{\infty} x_i = 1.$$

**Corollary 1.** *Let $(S, \mathcal{F}, T)$ be a complete Menger space, $M$ a nonempty and closed subset of $S$, $f : M \to M$ a continuous mapping, $A, B : M \to 2_c^{f(M)}$ and $\psi : [0, \infty) \to [0, \infty)$, so that there exists $s > 1$ such that the series $\sum_{n=1}^{\infty} \psi^n(s)$ is convergent. Suppose that the following implication holds for every $u, v \in M$ and every $\epsilon > 0$*

$$F_{fu, fv}(\epsilon) > 1 - \epsilon \quad \Rightarrow \quad \begin{array}{l} \text{for every } p \in Au \text{ there exists } q \in Bv \\ \text{such that } F_{p,q}(\psi(\epsilon)) > 1 - \psi(\epsilon) \text{ and} \\ \text{for every } p' \in Bv \text{ there exists } q' \in Au \\ \text{such that } F_{p',q'}(\psi(\epsilon)) > 1 - \psi(\epsilon). \end{array}$$

*If $A$ and $B$ are weakly commuting with $f$ and t-norm $T$ is of H-type then there exists $x \in M$ such that $fx \in Ax \cap Bx$.*

**Corollary 2.** *Let $(S, \mathcal{F}, T)$ be a complete Menger space, $M \in 2_{cl}^M$, $A$, $B$, $f$ and $\psi$ satisfy all condition of Theorem 2, and t-norm $T$ is strict with the multiplicative generator $\theta$. If $A$ or $B$ is a $f$ strongly demicompact or*

$$\lim_{n\to\infty} \prod_{i=n}^{\infty} \theta(1 - \psi^i(s)) = 1$$

*then there exists at least one element $x \in M$ such that $x \in Ax \cap Bx$.*

*Proof.* Since $\theta^{-1}$ is continuous and from Theorem 2 it follows that

$$\lim_{n\to\infty} T_{i=n}^{\infty}(1 - \psi^i(s)) = \theta^{-1}(\lim_{n\to\infty} \prod_{i=n}^{\infty} \theta(1 - \psi^i(s))) = 1$$

then there exists at least one element $x \in M$ such that $x \in Ax \cap Bx$. □

We shall give examples for some families of t-norms sequences $\{x_n\}_{n\in\mathbf{N}}$ from $(0, 1]$, which tend to 1 and for which $\lim_{n\to\infty} \mathrm{T}_{i=n}^{\infty} x_i = 1$ (see [5]).

**Example 4.** *[5]* Let $(T_\lambda^D)_{\lambda\in(0,\infty)}$ be the Dombi family of t-norms given by

$$T_\lambda^D(x, y) = \frac{1}{1 + ((\frac{1-x}{x})^\lambda + (\frac{1-y}{y})^\lambda)^{1/\lambda}}.$$

Then for every sequence $\{x_n\}_{n\in\mathbf{N}}$ of elements from $[0, 1)$ such that for some $\alpha > \max(1, \lambda)$, some $A > 0$ and some $n_0 \in \mathbf{N}$

$$1 - x_n \geq \frac{n^{\alpha/\lambda}}{A + n^{\alpha/\lambda}}, \qquad \text{for every } n \geq n_0$$

we have

$$\lim_{n\to\infty} (\mathrm{T}_\lambda^D)_{i=n}^{\infty}(1 - x_i) = 1, \quad \sum_{i=1}^{\infty} x_i < \infty.$$

**Example 5.** *[5]* Let $(T_\lambda^{SS})_{\lambda\in(-\infty,0)}$ be the Schweizer-Sklar family of t-norms given by

$$T_\lambda^{SS}(x, y) = (x^\lambda + y^\lambda - 1)^{1/\lambda}.$$

Then for every sequence $\{x_n\}_{n\in\mathbf{N}}$ of elements from $[0, 1)$ such that for some $\alpha > 1$, some $A > 0$ and some $n_0 \in \mathbf{N}$

$$1 - x_n \geq \left(1 + \frac{A}{n^\alpha}\right)^{1/\lambda}, \quad \text{for every } n \geq n_0,$$

we have

$$\lim_{n\to\infty} (T_\lambda^{SS})_{i=n}^\infty (1 - x_i) = 1, \ \sum_{i=1}^\infty x_i < \infty.$$

**Remark**: (i) t-norm $T_P$ is a member of the Schweizer-Sklar family for $\lambda = 0$. In this case $\theta_0^{SS}(x) = x$ and then for every sequence $\{x_n\}_{n\in\mathbf{N}}$ from $(0, 1]$ the condition $\prod_{i=1}^\infty x_i \neq 0$ implies that $\lim\limits_{n\to\infty} \prod_{i=n}^\infty x_i = 1$.

(ii) For t-norm $T_P$ it is well-known that $\lim\limits_{n\to\infty} \prod_{i=n}^\infty x_i = 1$ if and only if $\sum_{i=1}^\infty x_i < \infty$.

**Example 6.** *Let $(T_\lambda^{AA})_{\lambda\in(0,\infty)}$ be the Aczél-Alsina family of t-norms given by*

$$T_\lambda^{AA}(x, y) = e^{-(|\log x|^\lambda + |\log y|^\lambda)^{1/\lambda}}.$$

*Then for every sequence $\{x_n\}_{n\in\mathbf{N}}$ of elements from $[0, 1)$ such that for some $\alpha > \max(1, \lambda)$, some $A > 0$ and some $n_0 \in \mathbf{N}$*

$$x_n \leq 1 - e^{-An^{-\alpha/\lambda}}, \quad \text{for every } n \geq n_0$$

*we have*

$$\lim_{n\to\infty} (T_\lambda^{AA})_{i=n}^\infty (1 - x_i) = 1, \quad \sum_{i=1}^\infty x_i < \infty.$$

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