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THE FEFERMAN-VAUGHT THEOREM FOR REDUCED IDEAL-PRODUCTS

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Abstract. The Feferman-Vaught theorem for first order logic is generalized for reduced ideal-products of topological structures. Roughly, the theorem connects satisfaction of a topological formula in a reduced ideal-product of topological structures with the satisfaction of the adjoint Boolean formula in the corresponding Boolean algebra.

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1. Preliminaries

Throughout the paper $\{(\mathcal{S}_i, \mathcal{O}_i) \mid i \in I\}$ will be a family of topological structures of a given first-order language $\mathcal{L}$ (in the sense of [5]). By Λ and Ψ we will denote an ideal and a filter on the index set I respectively. $\pi_j : \prod_{i\in I} S_i \longrightarrow S_j$, $j \in I$, will be the canonical projections.

$\mathcal{O}^\Lambda$ is the topology on $\prod_{i\in I} S_i$ with the base $\mathcal{B}^\Lambda$ which consists of sets $\bigcap_{i\in L} \pi_i^{-1}(O_i)$, where $L \in \Lambda$ and $O_i \in \mathcal{O}_i$, for all $i \in L$. The topological structure $(\prod_{i\in I} \mathcal{S}_i, \mathcal{O}^\Lambda)$ will be denoted by $\prod^\Lambda \mathcal{S}_i$. The equivalence relation $\sim$ on $\prod_{i\in I} S_i$ defined by: $f \sim g$ iff $\{i \in I \mid f_i = g_i\} \in \Psi$ determines the quotient space (structure) $\prod^\Lambda \mathcal{S}_i/\sim$ which will be called the reduced ideal-product (shortly r.i. product or just r.i.p.) of the family $\{(\mathcal{S}_i, \mathcal{O}_i) \mid i \in I\}$. Such a r.i.p. will be denoted by $\prod_\Psi^\Lambda \mathcal{S}_i$ ([7]).

The natural mapping $q : \prod_{i\in I} S_i \longrightarrow \prod_{i\in I} S_i/\sim$ is given by $q(f) = [f]$, where $[f]$ is the equivalence class of f. Since q is an open mapping, $\mathcal{B}_\Psi^\Lambda = \{q(B) \mid B \in \mathcal{B}^\Lambda\}$ is a base for the topology $\mathcal{O}_\Psi^\Lambda$ on $\prod_\Psi^\Lambda S_i$.

It is proved in [6] that the r.i.p. preserves separation axioms T_0, T_1, T_2, T_3 and $T_{3\frac{1}{2}}$ if and only if the following condition holds:

$$\forall A \in \Psi\, \forall B \notin \Psi\, \exists L \in \Lambda (L \subseteq A \setminus B \text{ and } L^c \notin \Psi) \qquad (\Lambda\Psi).$$

Reduced ideal-products which satisfy the above condition were investigated in [6], [7] and [8]. Special $(\Lambda\Psi)$-r.i. products are: the Tychonov product (for

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$\Lambda = [I]^{<\omega}$ and $\Psi = \{I\}$), the full box product (for $\Lambda = P(I)$ and $\Psi = \{I\}$), the ultraproduct (for $\Lambda = P(I)$ and Ψ an arbitrary ultrafilter on I) and the Knight's box product (for $\Lambda = [I]^{<\kappa}$ and $\Psi = \{A \subseteq I \mid A^c \in [I]^{<\mu}\}$, where κ and μ are cardinals satisfying $|I| \geq \kappa > \mu \geq \omega$).

By $\equiv$ we will denote the well-known congruence relation on the Boolean algebra $P(I)$ given by:

$$A \equiv B \text{ iff for some } F \in \Psi, \;\; A \cap F = B \cap F.$$

The equivalence class containing the element $A \in P(I)$ will be denoted by $[A]$.

The structure $\langle P(I)/\equiv, +, \cdot, ', \mathbf{0}, \mathbf{1}\rangle$, where $[A] + [B] = [A \cup B]$, $[A] \cdot [B] = [A \cap B]$, $[A]' = [A^c]$, $\mathbf{0} = [\emptyset]$ and $\mathbf{1} = [I]$, is a Boolean algebra. If Λ is an ideal on I (more precisely on $P(I)$), then $\Lambda/\equiv = \{[L] \mid L \in \Lambda\}$ is an ideal on $P(I)/\equiv$. The structure $\mathbf{B} \stackrel{\text{def}}{=} \langle P(I)/\equiv, +, \cdot, ', \mathbf{0}, \mathbf{1}, \Lambda/\equiv\rangle$ is a Boolean algebra with distinguished ideal. Let $\mathcal{L}_{\mathbf{B}} = \{+, \cdot, ', 0, 1, \lambda\}$ be the associated first-order language. The theory we consider, $T_{\mathbf{B}}$, includes all axioms of the theory of Boolean algebras and the additional axioms:

$$\lambda(0); \qquad \lambda(x) \wedge \lambda(y) \Longrightarrow \lambda(x+y); \qquad \lambda(x) \wedge y \leq x \Longrightarrow \lambda(y).$$

2. The Feferman-Vaught-type theorem for r.i.p.

The classical theorem of S. Feferman and R. Vaught can be found, for instance, in [4] and [3]. In [5] M. Flum and J. Ziegler gave a topological version of this theorem concerning Tychonov products. L. Bertossi considered in [2] the "D-product" of a family of topological spaces and formulated the Feferman-Vaught-type theorem for such structures.

Here we extend the above results for any r.i.p. of a family of topological structures.

Let φ be an $\mathcal{L}_t$-formula ([5]) and let Y be a set variable. The formula φ^Y is obtained from φ substituting each free occurence of Y of the form $t \in Y$ by $t = t$.

Lemma 2.1. *If φ is an $\mathcal{L}_t$-formula and $(\mathcal{S}, \mathcal{O})$ a topological structure, then for each valuation v in $(\mathcal{S}, \mathcal{O})$ it holds:*

$$(\mathcal{S}, \mathcal{O}) \models (\varphi \Longleftrightarrow \varphi^Y)[v(Y/S)],$$

where $v(Y/S)$ is the valuation obtained from v substituting the value of Y by S.

Proof. If Y is not free in φ, then $\varphi \equiv \varphi^Y$ and the proof is completed. Otherwise, we proceed by induction on the complexity of the formula φ. We omit the case when φ is atomic and the induction steps when φ is of the form $\psi \wedge \theta$, $\neg\psi$ and $\exists x\, \psi$. So let $\varphi \equiv \exists X(t \in X \wedge \psi)$ and let $(\mathcal{S}, \mathcal{O})$ and v be arbitrary while $(\mathcal{S}, \mathcal{O}) \models \varphi[v(Y/S)]$. Then there is $U \in \mathcal{O}$ such that

(a) $(\mathcal{S}, \mathcal{O}) \models (t \in X)[v(Y/S)(X/U)]$ and (b) $(\mathcal{S}, \mathcal{O}) \models \psi[v(Y/S)(X/U)]$.

Y is free in φ, so $X \neq Y$ and $v(Y/S)(X/U) = v(X/U)(Y/S)$. By the induction hypothesis we have $(\mathcal{S},\mathcal{O}) \models (\psi \Longleftrightarrow \psi^Y)[v(X/U)(Y/S)]$ and by (b) $(\mathcal{S},\mathcal{O}) \models \psi^Y[v(Y/S)(X/U)]$. Now, by (a), $(\mathcal{S},\mathcal{O}) \models (t \in X \wedge \psi^Y)[v(X/U)(Y/S)]$ for some $U \in \mathcal{O}$, i.e. $(\mathcal{S},\mathcal{O}) \models \exists X(t \in X \wedge \psi^Y)[v(Y/S)]$, which gives $(\mathcal{S},\mathcal{O}) \models (\varphi \Longrightarrow \varphi^Y)[v(Y/S)]$. The proof of the converse implication is similar.□

Theorem 2.2. *(The Feferman-Vaught-type theorem for r.i. products). For each $\mathcal{L}_t$-formula $\varphi(x^1,\dots,x^p,X^1,\dots,X^q)$ there is a sequence of formulas $(\sigma;\psi_1,\dots,\psi_m)$ satisfying:*

(A) for all $j \in \{1,\dots,m\}$, ψ_j is an $\mathcal{L}_t$-formula and the free variables of ψ_j are among the free variables of φ. Moreover, if X is a set variable which is positive (negative) in φ, then X is positive (negative) in ψ_j;

(B) $\sigma(y_1,\dots,y_m)$ is a formula of the language $\mathcal{L}_{\mathbf{B}}$ which is monotonic, that is:

$$T_{\mathbf{B}} \vdash y_1 \leq t_1 \wedge \ldots \wedge y_m \leq t_m \wedge \sigma(y_1,\dots,y_m) \Longrightarrow \sigma(t_1,\dots,t_m);$$

(C) for each nonempty set I, any ideal Λ and any filter Ψ on I, for each family $\{(\mathcal{S}_i,\mathcal{O}_i) \mid i \in I\}$ of topological structures, each $f^1,\dots,f^p \in \prod S_i$ and each $U^1,\dots,U^q \in \mathcal{B}^\Lambda$ there holds:

$$\prod\nolimits_{\Psi}^{\Lambda} \mathcal{S}_i \models \varphi[[f^1],\dots,[f^p],q(U^1),\dots,q(U^q)] \quad \textit{iff} \quad \mathbf{B} \models \sigma[[I_{\psi_1}],\dots,[I_{\psi_m}]],$$

where $I_{\psi_j} = \{i \in I \mid \mathcal{S}_i \models \psi_j[f_i^1,\dots,f_i^p,U_i^1,\dots,U_i^q]\}$ for $j \in \{1,\dots,m\}$ and $\mathbf{B}$ is the above defined model.

We say that φ is determined by the sequence $(\sigma;\psi_1,\dots,\psi_m)$.

Proof. Our proof follows the proof of the Feferman-Vaught theorem for first-order logic. Like in [5], the sequences $x^1,\dots,x^p$; $X^1,\dots,X^q$; $f^1,\dots,f^p$; $U^1,\dots,U^q$; $[f^1],\dots,[f^p]$; $q(U^1),\dots,q(U^q)$; $f_i^1,\dots,f_i^p$; $U_i^1,\dots,U_i^q$ and $[I_{\psi_1}],\dots,[I_{\psi_m}]$ will be denoted respectively by $\overline{x}$, $\overline{X}$, $\overline{f}$, $\overline{U}$, $\overline{[f]}$, $\overline{q(U)}$, $\overline{f_i}$, $\overline{U_i}$ and $\overline{[I_\psi]}$. By $Fv(\varphi)$ we will denote the set of free variables of the formula φ and $Fv^+(\varphi)$ $(Fv^-(\varphi))$ will be the set of the second order variables which are positive (negative) in φ. The r.i.p. $\prod_{\Psi}^{\Lambda} \mathcal{S}_i$ will be denoted shortly by $\mathcal{S}$.

The proof is based on induction on the complexity of the formula φ; the basic logical connectives will be $\neg$ and $\wedge$ and the basic quantifier will be $\exists$. We omit the (easy) case when φ is atomic and the induction steps when φ is of the form: $\neg\phi$, $\phi \wedge \theta$ and $\exists x \phi$.

Let $\varphi \equiv \exists Y(t(x^1,\dots,x^p) \in Y \wedge \phi(x^1,\dots,x^p,X^1,\dots,X^q,Y^-))$. By the induction hypothesis there is a sequence $(\tau;\theta_1,\dots,\theta_m)$ determining ϕ. Let $l = 2^m$ and let $s_1 = \{1\}$, $s_2 = \{2\},\dots,s_m = \{m\}$, $s_{m+1},\dots,s_l$ be a list of all subsets of $\{1,2,\dots,m\}$. We will prove that the sequence $(\sigma;\psi_1,\dots,\psi_l,\eta_1,\dots,\eta_m)$, where:

$$\sigma \equiv \exists z_1,\dots,z_l(\bigwedge_{k=1}^{l} z_k \leq y_k \wedge \bigwedge_{s_i \cup s_j = s_k} z_i \cdot z_j = z_k \wedge \tau(z_1,\dots,z_m) \wedge \bigwedge_{k=1}^{m} \lambda(z_k \setminus v_k));$$

$$\psi_k \equiv \exists Y(t \in Y \wedge \bigwedge_{j\in s_k} \theta_j),\ k = 1,\dots,l;$$

$$\eta_j \equiv \theta_j^Y,\ \ j = 1,\dots,m,$$

determines φ. θ_j^Y is obtained from θ_j as in the preceding lemma. By convention, the empty conjunction is a true sentence (in "our" notation T).

(A) By the induction hypothesis, θ_j, $j = 1,\dots,m$, are $\mathcal{L}_t$-formulas and Y is negative in θ_j, so ψ_k and η_j are $\mathcal{L}_t$-formulas. Since $Fv(\theta_j) \subseteq Fv(\phi)$, we have $Fv(\psi_k) \subseteq Fv(\varphi)$ and $Fv(\eta_j) \subseteq Fv(\varphi)$. Also, $Fv^+(\varphi) = Fv^+(\phi) \subseteq \bigcap_{j=1}^m Fv^+(\theta_j)\setminus\{Y\} \subseteq \bigcap_{k=1}^l Fv^+(\psi_k) \cap \bigcap_{j=1}^m Fv^+(\eta_j)$ and the analogous result holds for $Fv^-(\varphi)$.

(B) Since τ is a formula of the language $\mathcal{L}_{\mathbf{B}}$ so is σ. Suposse $y_1 \le t_1,\dots,y_l \le t_l$, $v_1 \le w_1,\dots,v_m \le w_m$ and $\sigma(y_1,\dots,y_l,v_1,\dots,v_m)$. Then there exist $z_1,\dots,z_l$ satisfying $z_1 \le y_1,\dots,z_l \le y_l$, thus $z_1 \le t_1,\dots,z_l \le t_l$. Also, if $s_i \cup s_j = s_k$, then $z_i \cdot z_j = z_k$ and it holds $\tau(z_1,\dots,z_m)$. Finally, $v_k \le w_k$ implies $z_k \setminus w_k \le z_k \setminus v_k$ and from $\lambda(z_k \setminus v_k)$ it follows $\lambda(z_k \setminus w_k)$, for each $k \in \{1,\dots,m\}$. Hence $\sigma(t_1,\dots,t_l,w_1,\dots,w_m)$.

(C) Let I, Λ, Ψ, $\{(\mathcal{S}_i,\mathcal{O}_i) \mid i \in I\}$, $f^1,\dots,f^p \in \prod S_i$, $U^1,\dots,U^q \in \mathcal{B}^\Lambda$ be arbitrary. We will prove that

$$\mathcal{S} \models \exists Y(t \in Y \wedge \phi(Y))[\overline{[f]},\overline{q(U)}] \text{ iff } \mathbf{B} \models \sigma[[I_{\psi_1}],\dots,[I_{\eta_m}]].$$

($\Longrightarrow$) Let $V = \prod V_i \in \mathcal{B}^\Lambda$ be such that $t[[f^1],\dots,[f^p]] = [t[\overline{f}]] \in q(V)$ and $\mathcal{S} \models \phi[\overline{[f]},\overline{q(U)},q(V)]$. Let

$$Z^k = \{i \in I \mid \mathcal{S}_i \models \bigwedge_{j\in s_k} \theta_j[\overline{f_i},\overline{U_i},V_i]\},\ \ k = 1,\dots,l.$$

Because of $[t[\overline{f}]] \in q(V)$ we have $F = \{i \in I \mid t[\overline{f_i}] \in V_i\} \in \Psi$. Further, for $i \in Z^k \cap F$ it holds: $\mathcal{S}_i \models t[\overline{f_i}] \in V_i \wedge \bigwedge_{j\in s_k} \theta_j[\overline{f_i},\overline{U_i},V_i]$, whence $\mathcal{S}_i \models \psi_k[\overline{f_i},\overline{U_i}]$, i.e. $i \in I_{\psi_k}$. Therefore $Z^k \cap F \subseteq I_{\psi_k}$ and

$$[Z^k] \le [I_{\psi_k}],\ \ k = 1,\dots,l.$$

If $s_i \cup s_j = s_k$ then: $i \in Z^i \cap Z^j$ iff $\mathcal{S}_i \models \bigwedge_{j\in s_k} \theta_j(\overline{f_i},\overline{U_i},V_i)$ iff $i \in Z^k$. Thus $Z^i \cap Z^j = Z^k$, consequently:

$$s_i \cup s_j = s_k \implies [Z^i][Z^j] = [Z^k].$$

For $k \in \{1,\dots,m\}$, $s_k = \{k\}$, whence $Z^k = I_{\theta_k}$. Since the sequence $(\tau;\theta_1,\dots,\theta_m)$ determines the formula ϕ and $\mathcal{S} \models \phi[\overline{[f]},\overline{q(U)},q(V)]$, we have $\mathbf{B} \models \tau[[I_{\theta_1}],\dots,[I_{\theta_m}]]$, that is

$$\mathbf{B} \models \tau[[Z^1],\dots,[Z^m]].$$

Finally, let $k \in \{1,\dots,m\}$ and $j \in Z^k \setminus I_{\eta_k} = I_{\theta_k} \setminus I_{\theta_k^Y}$. Then $\mathcal{S}_j \models \theta_k[\overline{f_j}, \overline{U_j}, V_j]$ and $\mathcal{S}_j \not\models \theta_k^Y[\overline{f_j}, \overline{U_j}, V_j]$. $V = \prod V_i \in \mathcal{B}^\Lambda$, so, for some $L \in \Lambda$, $V = \bigcap_{i \in L} \pi^{-1}(O_i)$. Let us suppose that $j \notin L$. Then $V_j = S_j$, so it holds:

$$\mathcal{S}_j \models \theta_k[\overline{f_j}, \overline{U_j}, S_j] \qquad \text{and} \qquad \mathcal{S}_j \not\models \theta_k^Y[\overline{f_j}, \overline{U_j}, S_j],$$

which is, according to the previous lemma, impossible. Therefore $j \in L$ and $Z^k \setminus I_{\eta_k} \subseteq L \in \Lambda$, which implies $Z^k \setminus I_{\eta_k} \in \Lambda$. But then $[Z^k] \setminus [I_{\eta_k}] \in \Lambda/\equiv$, i.e. $\lambda([Z^k] \setminus [I_{\eta_k}])$, $k = 1,\dots,m$, which proves

$$\mathbf{B} \models \sigma[[I_{\psi_1}],\dots,[I_{\psi_l}],[I_{\eta_1}],\dots,[I_{\eta_m}]].$$

($\Longleftarrow$) Let $\mathbf{B} \models \sigma[[I_{\psi_1}],\dots,[I_{\psi_l}],[I_{\eta_1}],\dots,[I_{\eta_m}]]$. Then there exist $Z^1,\dots,Z^l \subseteq I$, such that the following conditions are satisfied:

(*a*) $[Z^k] \leq [I_{\psi_k}]$, $k = 1,\dots,l$;
(*b*) $s_i \cup s_j = s_k \implies [Z^i][Z^j] = [Z^k]$;
(*c*) $\mathbf{B} \models \tau([Z^1],\dots,[Z^m])$;
(*d*) $[Z^k] \setminus [I_{\eta_k}] \in \Lambda/\equiv$, $k = 1,\dots,m$.

Hence, for some sets $F^k, F^{ijk}, G^k \in \Psi$ it holds:

(*a*1) $Z^k \cap F^k \subseteq I_{\psi_k}$, $k = 1,\dots,l$;
(*b*1) $s_i \cup s_j = s_k \implies Z^i \cap Z^j \cap F^{ijk} = Z^k \cap F^{ijk}$;
(*d*1) $Z^k \setminus I_{\eta_k} \cap G^k \in \Lambda$, $k = 1,\dots,m$.

Let $F = \bigcap_{k=1}^l F^k \cap \bigcap_{s_i \cup s_j = s_k} F^{ijk} \cap \bigcap_{k=1}^m G^k$. Then $F \in \Psi$ and it holds:

(*a*2) $Z^k \cap F \subseteq I_{\psi_k}$, $k = 1,\dots,l$;
(*b*2) $s_i \cup s_j = s_k \implies Z^i \cap Z^j \cap F = Z^k \cap F$;
(*d*2) $Z^k \setminus I_{\eta_k} \cap F \in \Lambda$, $k = 1,\dots,m$.

From (*b*2) it follows (by a simple inductive argument):

(*b*3) $s_{j_1} \cup s_{j_2} \cup \ldots \cup s_{j_r} = s_h \implies Z^{j_1} \cap Z^{j_2} \cap \ldots \cap Z^{j_r} \cap F = Z_h \cap F$.

For $i \in \bigcup_{k=1}^m (Z^k \setminus I_{\eta_k} \cap F) = L$ let

$$s(i) \stackrel{\text{def}}{=} \{j \in \{1,\dots,m\} \mid i \in Z^j\}.$$

Then, for some $h \in \{1,\dots,l\}$, it holds:

$$s(i) = s_h = \{j_1,\dots,j_r\} = s_{j_1} \cup \ldots \cup s_{j_r},$$

whence, because of (*b*3) and (*a*2):

$$i \in \bigcap_{j \in s(i)} Z^j \cap F = Z^{j_1} \cap \ldots \cap Z^{j_r} \cap F = Z^h \cap F \subseteq I_{\psi_h},$$

and also $\mathcal{S}_i \models \psi_h[\overline{f_i}, \overline{U_i}]$. Thus we can choose $V_i \in \mathcal{B}_i$ so that the following holds:

$$t[\overline{f_i}] \in V_i \qquad \text{and} \qquad \mathcal{S}_i \models \bigwedge_{j \in s(i)} \theta_j[\overline{f_i}, \overline{U_i}, V_i].$$

For $i \notin L$ we define $V_i = S_i$. L is a finite union of elements of Λ, so $L \in \Lambda$ and $V = \prod V_i \in \mathcal{B}^{\Lambda}$. Let us prove: $\mathcal{S} \models \phi[\overline{[f]}, \overline{q(U)}, q(V)]$. Let

$$I_{\theta_k} = \{i \in I \mid \mathcal{S}_i \models \theta_k[\overline{f_i}, \overline{U_i}, V_i]\}, \;\; k = 1, \dots, m,$$

$j_0 \in \{1, \dots, m\}$ and $i \in Z^{j_0} \cap F$. We discuss the possible cases:

(*i*) $i \in L$. Now $\mathcal{S}_i \models \bigwedge_{j \in s(i)} \theta_j[\overline{f_i}, \overline{U_i}, V_i]$ and since $j_0 \in s(i)$ it holds: $\mathcal{S}_i \models \theta_{j_0}[\overline{f_i}, \overline{U_i}, V_i]$. Hence $i \in I_{\theta_{j_0}}$.

(*ii*) $i \notin L$. By our agreement $V_i = S_i$. Also $i \notin (Z^{j_0} \cap F) \setminus I_{\eta_{j_0}}$, thus $i \in I_{\eta_{j_0}} = I_{\theta^Y_{j_0}}$. Let us assume that $i \notin I_{\theta_{j_0}}$. Then:

$$\mathcal{S}_i \models \theta^Y_{j_0}[\overline{f_i}, \overline{U_i}, S_i] \qquad \text{and} \qquad \mathcal{S}_i \not\models \theta_{j_0}[\overline{f_i}, \overline{U_i}, S_i],$$

which is impossible because of the previous lemma. So, again $i \in I_{\theta_{j_0}}$.

From (*i*) and (*ii*) it follows $Z^{j_0} \cap F \subseteq I_{\theta_{j_0}}$, accordingly:

$$[Z^j] \leq [I_{\theta_j}], \;\; j = 1, \dots, m.$$

The condition (*c*) and the monotonicity of τ imply $\mathbf{B} \models \tau[[I_{\theta_1}], \dots, [I_{\theta_m}]]$, whence by induction hypothesis

$$\mathcal{S} \models \phi[\overline{[f]}, \overline{q(U)}, q(V)].$$

Further, $t[\overline{f}] \in V$, thus $[t[\overline{f}]] \in q(V) \in \mathcal{B}^{\Lambda}_{\Psi}$. We conclude: $\mathcal{S} \models [t[\overline{f}]] \in q(V) \wedge \phi[\overline{[f]}, \overline{q(U)}, q(V)]$, that is

$$\mathcal{S} \models \exists Y (t \in Y \wedge \phi(Y)[\overline{[f]}, \overline{q(U)}].\Box$$

Remark. Let us just recall that $\mathcal{L}_t$-formulas are invariant, in other words, for any $\mathcal{L}_t$-formula χ it holds in general:

$$(\mathcal{S}, \mathcal{O}) \models \chi \qquad \text{iff} \qquad (\mathcal{S}, \mathcal{B}) \models \chi,$$

where $\mathcal{B}$ is any base for the topology $\mathcal{O}$. In particular, in our case we have:

$$(\prod\nolimits^{\Lambda}_{\Psi} \mathcal{S}_i, \mathcal{O}^{\Lambda}_{\Psi}) \models \varphi \quad \text{iff} \quad (\prod\nolimits^{\Lambda}_{\Psi} \mathcal{S}_i, \mathcal{B}^{\Lambda}_{\Psi}) \models \varphi \quad \text{iff} \quad \mathbf{B} \models \sigma[[I_{\psi_1}], \dots, [I_{\psi_m}]].$$

It is seen from the proof that if φ is a sentence, then the formulas $\psi_1, \dots, \psi_m$ are sentences as well. Hence the following holds.

Corollary 2.3. *For any $\mathcal{L}_t$-sentence φ there exists a sequence of formulas $(\sigma; \psi_1, \dots, \psi_m)$ such that it holds:*

(A) for each $j \in \{1, \dots, m\}$, ψ_j is an $\mathcal{L}_t$-sentence;

(B) σ is a monotonic formula of the language $\mathcal{L}_{\mathbf{B}}$;

(C) for any set I, any ideal Λ, any filter Ψ on I and any family of topological structures $\{(\mathcal{S}_i, \mathcal{O}_i) \mid i \in I\}$ it holds:

$$(\prod\nolimits^{\Lambda}_{\Psi} \mathcal{S}_i, \mathcal{O}^{\Lambda}_{\Psi}) \models \varphi \qquad \textit{iff} \qquad \mathbf{B} \models \sigma[[I_{\psi_1}], \dots, [I_{\psi_m}]],$$

where $I_{\psi_j} = \{i \in I \mid (\mathcal{S}_i, \mathcal{O}_i) \models \psi_j\}$.$\Box$

Corollary 2.4. *Reduced ideal-product of topological spaces preserves $\mathcal{L}_t$-equivalence.* $\square$

Theorem 2.2 enables us to give a short proof of the Los theorem for topological structures.

Theorem 2.5. *Let $\varphi(x^1, \ldots, x^p, X^1, \ldots, X^q)$ be an $\mathcal{L}_t$-formula. If $\{(\mathcal{S}_i, \mathcal{O}_i) \mid i \in I\}$ is a family of topological structures and $\mathcal{U}$ an arbitrary ultrafilter on I, then for aech $f^1, \ldots, f^p \in \prod S_i$ and each $U^1, \ldots, U^q \in \mathcal{B}^{P(I)}$ it holds*

$$(1) \qquad \prod\nolimits_{\mathcal{U}}^{P(I)} \mathcal{S}_i \models \varphi[\overline{[f]}, \overline{q(U)}] \qquad \textit{iff} \qquad \{i \in I \mid \mathcal{S}_i \models \varphi[\overline{f_i}, \overline{U_i}]\} \in \mathcal{U}$$

Clearly, $\prod_{\mathcal{U}}^{P(I)} \mathcal{S}_i$ is the ultraproduct of the given family.

Proof. The condition (1) holds iff the formula φ is determined by the sequence $(y_1 = \mathbf{1}; \varphi)$ (that is iff: $\prod_{\mathcal{U}}^{P(I)} \mathcal{S}_i \models \varphi[\overline{[f]}, \overline{q(U)}] \iff \mathbf{B} \models (y = \mathbf{1})[[I_\varphi]]$). Also, since $\mathcal{U}$ is an ultrafilter, $\mathbf{B}$ is the two-element Boolean algebra (i.e. $P(I)/\equiv = \{\mathbf{0}, \mathbf{1}\}$). The theorem is proved by the usual induction. We consider the only nontrivial case:

$$\varphi \equiv \exists Y (t(x^1, \ldots, x^p) \in Y \wedge \phi(x^1, \ldots, x^p, X^1, \ldots, X^q, Y^-)).$$

By the induction hypothesis, the formula ϕ is determined by the sequence $(y_1 = \mathbf{1};\, \phi)$. Following the proof of Theorem 2.2, the sequence $(\sigma; \psi_1, \psi_2, \eta_1)$, where

$$\sigma(y_1, y_2, v_1) \equiv \exists z_1, z_2 (z_1 \le y_1 \,\wedge\, z_2 \le y_2 \,\wedge\, z_1 \cdot z_2 = z_1 \,\wedge\, z_1 = \mathbf{1} \,\wedge\, \lambda(z_1 \setminus v_1));$$

$$\psi_1 \equiv \exists Y (t \in Y \wedge \psi), \qquad \psi_2 \equiv \exists Y (t \in Y \,\wedge\, T);$$

$$\eta_1 \equiv \phi^Y,$$

determines φ. Evidently:

$$T_{\mathbf{B}} \vdash \sigma(y_1, y_2, v_1) \iff y_1 = \mathbf{1} \,\wedge\, y_2 = \mathbf{1} \,\wedge\, \lambda(\mathbf{1} \setminus v_1).$$

Also, $\psi_1 \equiv \varphi$ and ψ_2 is a true sentence; thus $[I_{\psi_1}] = [I_\varphi]$ and $[I_{\psi_2}] = [I] = \mathbf{1}$. Now

$$\prod\nolimits_{\mathcal{U}}^{P(I)} \mathcal{S}_i \models \varphi[\overline{[f]}, \overline{q(U)}] \quad \text{iff} \quad \mathbf{B} \models (y_1 = \mathbf{1} \,\wedge\, y_2 = \mathbf{1} \,\wedge\, \lambda(v_1'))[[I_\varphi], \mathbf{1}, [I_{\eta_1}]]$$

$$\text{iff} \quad [I_\varphi] = \mathbf{1} \,\wedge\, [I_{\eta_1}]^c \in \Lambda/\equiv \quad \text{iff} \quad [I_\varphi] = \mathbf{1},$$

since $\Lambda/\equiv\, = P(I)/\equiv$. So the sequence $(y = \mathbf{1};\, \varphi)$ determines φ. $\square$

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