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EXTENSION OF NULL-ADDITIVE SET FUNCTIONS ON ALGEBRA OF SUBSETS

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Abstract. In this note an extension-type theorem for null-additive fuzzy measures is proved.

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1. Introduction

We shall prove a theorem on extension of null-additive set function from a ring of subsets of a given set to the algebra of subsets generated by this ring.

Let $\mathcal{R}$ be a ring of subsets of the given set X.

Definition 1. *A set function $m, m : \mathcal{R} \to [0, \infty]$ with $m(\emptyset) = 0$ is called null-additive, if we have*

$$m(A \cup B) = m(A)$$

whenever $A, B \in \mathcal{R}, A \cap B = \emptyset$ and $m(B) = 0$.

For properties of null-additive set functions see E. Pap [20], [21], H. Suzuki [24] and Z. Wang [26],[27].

Example 1. Let $\perp$ be a $t-$conorm, i.e. a binary operation on $[0, 1]$ such that it is associative commutative and monotone with a neutral element 0. A set function $m : \mathcal{R} \to [0, 1]$ is called $\perp-$decomposable measure if $m(\emptyset) = 0$ and

$$m(A \cup B) = m(A) \perp m(B)$$

whenever $A, B \in \mathcal{R}$ and $A \cap B = \emptyset$ ([15], [28]).

m is monotone null-additive set function.

Example 2 ($\oplus$ - decomposable measure [9],[11],[19]). The operation $\oplus$ (pseudo-addition) is a function $\oplus : [0, \infty] \times [0, \infty] \to [0, \infty]$ which is commutative, nondecreasing, associative, continuous and has a zero element 0. For example, $x \oplus y = (x^p + y^p)^{1/p}$ for a fixed $p > 0$.

A set function $m : \mathcal{R} \to [0, \infty]$ is a $\oplus$ -decomposable measure if there hold $m(\emptyset) = 0$ and

$$m(A \cup B) = m(A) \oplus m(B)$$

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whenever $A, B \in \mathcal{R}$ and $A \cap B = \emptyset$. It is obvious that m is null-additive set function. There are further generalizations for operations defined on $[a, b] \subset [-\infty, +\infty]$.

Example 3. (k-triangular set functions, [7], [12], [13]). A set function $m : \mathcal{R} \to [0, \infty)$ is said to be $k-$triangular for $k \geq 1$ if $m(\emptyset) = 0$ and

$$m(A) - km(B) \leq m(A \cup B) \leq m(A) + km(B)$$

whenever $A, B \in \mathcal{R},\ \ A \cap B = \emptyset$. It is obvious that $k-$ triangular set function is always null - additive, although it may not be monotone. Special 1 - triangular set functions are submeasures.

2. Extension

We shall need in the proof the following known result (see [22], 1.1.9 (4))

Theorem 1. *Let $\mathcal{R}$ be a ring on X. Let*

$$\mathcal{R}_1 = \{A : A \subset X, A' \in \mathcal{R}\}.$$

Then $\mathcal{A} = \mathcal{R} \cup \mathcal{R}_1$ is the smallest algebra on X containing $\mathcal{R}$.

Theorem 2. *Let $\mathcal{R}$ be a ring of subsets of a set X such that $X \notin \mathcal{R}$. Let m be a null-additive monotone set function on $\mathcal{R}$. Let $\mathcal{A}$ be the algebra on X generated by $\mathcal{R}$. Then there exists a null-additive set function on $\mathcal{A}$ possibly taking the value infinity which is an extension of m from $\mathcal{R}$ to $\mathcal{A}$.*

Proof. We take

$$\mathcal{R}_1 = \{A : A \subset X, A' \in \mathcal{R}\}.$$

By Theorem 1 we have $\mathcal{A} = \mathcal{R} \cup \mathcal{R}_1$. Let $d = \sup_{R \in \mathcal{R}} m(R)$.

First case: $d = \infty$. We define $\overline{m}$ on $\mathcal{A}$ by the following

$$\overline{m}(A) = m(A) \quad \text{if} \quad A \in \mathcal{R} \ \text{ and}$$

$$\overline{m}(A) = \infty \quad \text{if} \ \ A \in \mathcal{R}_1.$$

By the definition $\overline{m}$ is an extension of m from $\mathcal{R}$ to $\mathcal{A}$. We shall prove that $\overline{m}$ is null-additive. Let $A, B \in \mathcal{A}$ and $A \cap B = \emptyset$.

Case (1a): $A, B \in \mathcal{R}$. If $\overline{m}(B) = 0$, then also $m(B) = 0$ and we have

$$\overline{m}(A \cup B) = m(A \cup B) = m(A) = \overline{m}(A).$$

Case (1b): $A \in \mathcal{R}, B \in \mathcal{R}_1$. We have always $\overline{m} = \infty$, i.e., $\overline{m}(B) \neq 0$. So case (b) does not influences the null-additivity.

Case (1c): $A \in \mathcal{R}_1, B \in \mathcal{R}$. We shall prove that $A \cup B \in \mathcal{R}_1$. Namely, we have

$$(A \cup B)' = A' \cap B' = A' \setminus B \in \mathcal{R},$$

since $A' \in \mathcal{R}, B \in \mathcal{R}$ and $\mathcal{R}$ is a ring. Now we have for $\overline{m}(B) = m(B) = 0$

$$\overline{m}(A \cup B) = \overline{m}(A).$$

Case (1d): $A, B \in \mathcal{R}_1$. We have

$$X = A' \cup B' \in \mathcal{R},$$

what is impossible, since $X \notin \mathcal{R}$. So this case does not arise.

Second case: $d < \infty$. We define $\overline{m}$ on $\mathcal{A}$ by the following equalities

$$\overline{m}(A) = m(A) \text{ if } A \in \mathcal{R} \text{ and}$$

$$\overline{m}(A) = d - m(A') \text{ if } A \in \mathcal{R}_1.$$

Let $A, B \in \mathcal{A}$ and $A \cap B = \emptyset$.

Case (2a): $A, B \in \mathcal{R}$. Then we have for $\overline{m}(B) = m(B) = 0$

$$\overline{m}(A \cup B) = m(A \cup B) = m(A) = \overline{m}(A).$$

Case (2b): $A \in \mathcal{R}, B \in \mathcal{R}_1$. Then in the same way as in the case (1b) we obtain $A \cup B \in \mathcal{R}_1$. Now we have for $\overline{m}(B) = 0$ that $d = m(B')$ and so

$$\overline{m}(A \cup B) = d - m((A \cup B)') = d - m(A' \cap B') = d - m(A') = \overline{m}(A).$$

Namely, since

$$m(B') = \sup_{R \in \mathcal{R}} m(R,)$$

we have $A' \subset B'$.

Case (2c): $A \in \mathcal{R}_1,\ B \in \mathcal{R}$. We have $A \cup B \in \mathcal{R}_1$. So for $\overline{m}(B) = 0$ it is also $m(B) = 0$ and so

$$\overline{m}(A \cup B) = d - m((A \cup B)') = d - m(A' \cap B') = d - m(A' \setminus B) = d - m(A') = \overline{m}(A),$$

since $m(A' \setminus B) = m(A')$ for $m(B) = 0$.

Case (2d): $A, B \in \mathcal{R}_1$. Then $X = A' \cup B' \in \mathcal{R}$, what is impossible by the supposition $X \notin \mathcal{R}$. So, this case does not arise.

This completes the proof of the theorem. □

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