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ON STABILITY OF SOME DIFFERENCE SCHEMES FOR PARABOLIC DIFFUSION EQUATIONS

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Abstract. Two uniformly stable difference schemes for the singularly perturbed parabolic boundary value problem are derived. Numerical results indicate the uniform convergence.

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1. Introduction

On the domain $Q = D \times (0,T]$, $D = (0,1)$ is an open interval in $\mathbf{R^n}$, $T \in \mathbf{R}$, we consider a boundary value problem for the parabolic equation

$$(1)\quad \begin{cases} Lu(x,t) = -\varepsilon u_{xx}(x,t) + d(x,t)u(x,t) + r(x,t)u_t(x,t) & \\ \qquad\qquad = f(x,t), & (x,t) \in Q, \\ u(x,t) = \varphi(x,t), \quad (x,t) \in S, & S = \bar{Q} \setminus Q \end{cases}$$

where $\bar{Q}$ is adherence of the set Q.

Here the functions $d(x,t)$, $r(x,t)$, $f(x,t)$ and also the function $\varphi(x,t)$, are sufficiently smooth on the sets $\bar{Q}$ and S respectively. Morever

$$d(x,t) \geq d_0 > 0, \quad r(x,t) \geq r_0 > 0 \quad (x,t) \in \bar{Q},$$

$\varepsilon \in (0,1]$. The solution of the boundary value problem is a function $u = u(x,t)$, which satisfies the equation on Q and the boundary condition on S. When the parameter ε tends to zero, a parabolic boundary layer appears in the neighbourhood of the lateral boundary of the set Q. In the set $\bar{Q}$ we define the grid

$$(2)\qquad \bar{Q}_h = \bar{\omega}_1 \times \bar{\omega}_0,$$

where $\bar{\omega}_1$, is a grid, generally nonuniform, on the interval $[0,1]$ and $\bar{\omega}_0$ is a uniform grid on the inerval $[0,T]$. Suppose $h_i = x_{i+1} - x_i, x_i, x_{i+1} \in \bar{\omega}_1$. By $N+1$ and N_0+1 we denote the number of nodes in the grids $\bar{\omega}_1$ and $\bar{\omega}_0$ respectively.

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On the grid $\bar{Q}_h$ we define the following difference scheme for problem (1)

$$(3)\quad \begin{cases} \Lambda z(x,t) \equiv \{-\varepsilon\delta_{x\bar{x}} + d(x,t) + r(x,t)\delta_{\bar{t}}\} z(x,t) = f(x,t),\ (x,t) \in Q_h, \\ \\ z(x,t) = \varphi(x,t), \qquad (x,t) \in S_h = S \bigcap \bar{Q}_h, \end{cases}$$

where $Q_h = Q \bigcap \bar{Q}_h$, $S_h = S \bigcap \bar{Q}_h$, $\delta_{x\bar{x}}$, $\delta_{\bar{t}}$ are the second and the first (backward) difference derivatives respectively. After the elementary calculations we obtain

$$(4)\quad \begin{cases} \Lambda z \equiv \frac{2\varepsilon}{h_i(h_i+h_{i-1})} z_{i+1}^j + (\frac{-2\varepsilon}{h_i h_{i-1}} - d_i^j - \frac{r_i^j}{\tau}) z_i^j + \frac{2\varepsilon}{h_{i-1}(h_i+h_{i-1})} z_{i-1}^j \\ \\ = -\frac{r_i^j}{\tau} z_i^{j-1} - f_i^j, \qquad i = 1,..,N, \qquad j = 1,2,...,N_0 \\ \\ z(x,t) = \varphi(x,t), \qquad (x,t) \in S_h. \end{cases}$$

The second scheme we obtain via the cubic spline for fixed t and the first (backward) difference derivative for a fixed x,

$$(5)\quad \begin{cases} \Lambda_1 z \equiv (-\frac{1}{h_{i-1}} + \frac{h_{i-1} d_{i-1}^j}{6\varepsilon} + \frac{r_{i-1}^j h_{i-1}}{6\tau\varepsilon}) z_{i-1}^j \\ \\ +(\frac{1}{h_{i-1}} + \frac{1}{h_i} + \frac{h_{i-1} d_i^j}{3\varepsilon} + \frac{h_i d_i^j}{3\varepsilon} + \frac{r_i^j(h_i+h_{i-1})}{3\tau\varepsilon}) z_i^j \\ \\ +(-\frac{1}{h_i} + \frac{h_i d_{i+1}^j}{6\varepsilon} + \frac{r_{i+1}^j h_i}{6\tau\varepsilon}) z_{i+1}^j \\ \\ = \frac{r_{i-1}^j h_{i-1}}{6\varepsilon\tau} z_{i-1}^{j-1} + \frac{r_i^j(h_{i-1}+h_i)}{3\varepsilon\tau} z_i^{j-1} + \frac{r_{i+1}^j h_i}{6\varepsilon\tau} z_{i+1}^{j-1} \\ \\ = \frac{h_{i-1}}{6\varepsilon} f_{i-1}^j + \frac{h_{i-1}+h_i}{3\varepsilon} f_i^j + \frac{h_i}{6\varepsilon} f_{i+1}^j, \qquad i = 1,..,N, \qquad j = 1,2,...,N_0 \\ \\ z(x,t) = \varphi(x,t), \qquad (x,t) \in S_h \end{cases}$$

Throughout the paper the symbols M and M_i denote different constants independent of ε, N and N_0.

Using the maximum principle on the uniform mesh in [1] the convergence of the difference scheme (4) for a fixed value of the parameter ε,

$$(6)\qquad |u(x,t) - z(x,t)| \le M(\varepsilon^{-1}N^{-2} + N_0^{-1}), \qquad (x,t) \in \bar{Q}_h,$$

is proved. Using the same technique and uniform mesh it is easy to verify that the esimate (see [4])

$$(7)\qquad |u(x,t) - z(x,t)| \le M(\varepsilon^{-1/2}N^{-1} + N_0^{-1}), \qquad (x,t) \in \bar{Q}_h,$$

is valid. The convergences given in (6) and (7) are not uniform with respect to the small parameter. The schemes have neither uniform stability nor uniformly bounded truncation errors. In the next section we will consider schemes (4), (5) and (12) on the non-uniform mesh $\bar{Q}_h^*$ for the next special problem of the type (1),

$$(8)\qquad \begin{cases} L_1 u(x,t) \equiv -\varepsilon u_{xx}(x,t) + u_t(x,t) = 0, \ (x,t) \in Q \\ \\ u(x,t) = W(x,t), \quad (x,t) \in S = \bar{Q}/Q. \end{cases}$$

The special mesh provides the uniform stability, which will be shown. For that purpose we will specialize the mesh $\bar{Q}$ (as in [1]) by condensing it in the boundary layers,

$$(9)\qquad \bar{Q}_h^* = \bar{\omega}_1^* \mathrm{x} \bar{\omega}_0,$$

where $\bar{\omega}_1^*$ is a piecewise grid on $[0,1]$ with the step-size $h_1 = 4\sigma N^{-1}$ on the intervals $[0,\sigma]$, $[1-\sigma,1]$ and the step-size $h_2 = 2(1-2\sigma)N^{-1}$ on the interval $[\sigma, 1-\sigma]$. The value σ is chosen to satisfy the condition

$$\sigma = \min{(1/4, m\sqrt{\varepsilon}\ln N)},$$

where m is a fixed number.

2. The stability of the schemes

In this section we consider a boundary value problem for the diffusion equation (8). The difference scheme (4) for problem (8) is

$$(10)\qquad \begin{cases} \bar{\Lambda} z \equiv \frac{2\varepsilon}{h_i(h_i+h_{i-1})} z_{i+1}^j + [\frac{-2\varepsilon}{h_i h_{i-1}} - \frac{1}{\tau}] z_i^j + \frac{2\varepsilon}{h_{i-1}(h_i+h_{i-1})} z_{i-1}^j \\ \\ = -\frac{1}{\tau} z_i^{j-1}, \quad i = 1,..,N, \quad j = 1,2,...,N_0 \\ \\ z(x,t) = W(x,t), \quad (x,t) \in S_h \end{cases}$$

For the non-uniform mesh $\bar{Q}_h^*$ the convergence

$$|u(x,t) - z(x,t)| \le M(N^{-2} + N_0^{-1})$$

is achieved in [1].

The second scheme we obtain from scheme 5,

$$\begin{cases} \bar{\Lambda}_1 z \equiv (-\frac{1}{h_{i-1}} + \frac{h_{i-1}}{6\tau\varepsilon}) z_{i-1}^j + (\frac{1}{h_{i-1}} + \frac{1}{h_i} + \frac{h_i + h_{i-1}}{3\tau\varepsilon}) z_i^j + (-\frac{1}{h_i} + \frac{h_i}{6\tau\varepsilon}) z_{i+1}^j \\ \\ = \frac{h_{i-1}}{6\varepsilon\tau} z_{i-1}^{j-1} + \frac{h_{i-1}+h_i}{3\varepsilon\tau} z_i^{j-1} + \frac{h_i}{6\varepsilon\tau} z_{i+1}^{j-1}, \quad i = 1,..,N, \quad j = 1,2,...,N_0 \\ \\ z(x,t) = W(x,t), \quad (x,t) \in S_h \end{cases}$$

(11)

The third scheme is derived so that we use the cubic spline difference scheme along the x-axis, and Cranc-Nikolson scheme along the t-axis,

$$
\left\{
\begin{array}{l}
\bar{\Lambda}_2 z \equiv (-\frac{1}{2h_{i-1}} + \frac{h_{i-1}}{6\tau\varepsilon}) z_{i-1}^j + (\frac{1}{2h_{i-1}} + \frac{1}{2h_i} + \frac{h_i + h_{i-1}}{3\tau\varepsilon}) z_i^j + (-\frac{1}{2h_i} + \frac{h_i}{6\tau\varepsilon}) z_{i+1}^j \\
= (\frac{1}{2h_{i-1}} + \frac{h_{i-1}}{6\varepsilon\tau}) z_{i-1}^{j-1} + (-\frac{1}{2h_{i-1}} - \frac{1}{2h_i} + \frac{h_{i-1} + h_i}{3\varepsilon\tau}) z_i^{j-1} \\
+(\frac{1}{2h_i} + \frac{h_i}{6\varepsilon\tau}) z_{i+1}^{j-1}, \quad i = 1, .., N, \quad j = 1, 2, ..., N_0 \\
z(x,t) = W(x,t), \quad (x,t) \in S_h
\end{array}
\right.
$$

(12)

We will prove the following theorem

Teorema 2.1. *Let $N_0\tau = T$ and let the mesh $\bar{Q}_h^*$ be defined by (9). Let z_i^j be the solution of the scheme (11) and*

$$b^{max} = \max_{(x,t)\in S_h} W(x,t), \quad \text{and} \quad b^{min} = \min_{(x,t)\in S_h} W(x,t),$$

$\tau \geq \frac{8m^2 \ln^2 N}{3N^2} = g(m,N)$ and $\tau \geq N^{-2}\ln^2 N$. Then for $0 \leq j\tau \leq T$ the estimate

$$b^{min} - MT \leq z_i^j \leq b^{max} + MT \tag{13}$$

is valid for $i = 1, .., N$.

Proof. The scheme (11) can be written in the form

$$(Lz)_i^j = (Rz)_i^j,$$

where

$$(Lz)_i^j \equiv r_i^{j+} z_{i+1}^j + r_i^{jc} z_i^j + r_{i-1}^{j-} z_{i-1}^j,$$

and

$$(Rz)_i^j \equiv q_i^{j+} z_{i+1}^{j-1} + q_i^{jc} z_i^{j-1} + q_{i-1}^{j-} z_{i-1}^{j-1}.$$

Let $0 \leq i \leq i_0 - 1$. Then $h_1 = h_i = h_{i-1} = 4m\sqrt{\varepsilon}N^{-1}\ln N$ and we have

$$r_i^{j-} = -\frac{1}{h_{i-1}} + \frac{h_{i-1}}{6\tau\varepsilon} \leq 0,$$

if

$$\frac{1}{h_{i-1}} \geq \frac{h_{i-1}}{6\tau\varepsilon},$$

i.e.

$$h_{i-1} \leq \frac{6\tau\varepsilon}{h_{i-1}}.$$

Since $h_{i-1} = 4m\sqrt{\varepsilon}N^{-1}\ln N$, the last inequality reduces to

$$16m^2\varepsilon N^{-2}\ln^2 N \leq 6\tau\varepsilon,$$

or

$$\tau \geq \frac{8m^2\ln^2 N}{3N^2} = g(m,N),$$

which is the assumption of the theorem. Thus, $r_i^{j-} \leq 0$. It is also clear that

$$r_i^{j+} = -\frac{1}{h_i} + \frac{h_i}{6\tau\varepsilon} \leq 0.$$

Now we introduce the notation

$$z_k^j = \max_{0\leq i\leq i_0-1} z_i^j, \qquad z_p^j = \min_{0\leq i\leq i_0-1} z_i^j,$$

and

$$z_{max}^j = \max_{0\leq i\leq N} z_i^{j-1}, \qquad z_{min}^j = \min_{0\leq i\leq N} z_i^{j-1}.$$

Since $r_i^{j-} \leq 0$, $r_i^{j+} \leq 0$, $q_i^{j-} \geq 0$ i $q_i^{j+} \geq 0$, we have

$$(r_i^{j-} + r_i^{jc} + r_i^{j+})z_k^j \leq (Lz)_k^j = (Rz)_k^j \leq (q_i^{j-} + q_i^{jc} + q_i^{j+})z_{max}^{j-1}.$$

By using the fact

$$r_i^{j-} + r_i^{jc} + r_i^{j+} = q_i^{j-} + q_i^{jc} + q_i^{j+} = \frac{h_1}{\tau\varepsilon},$$

we obtain

$$z_k^j \leq z_{max}^{j-1}.$$

Similarly, we obtain that

$$z_{min}^{j-1} \leq z_p^j.$$

Further, let $i = i_0$. Then $r_{i_0}^{j-} \leq 0$ and if $r_{i_0}^{j+} \leq 0$, we have the previous case. Suppose that $r_{i_0}^{j+} > 0$. Then

$$(r_{i_0}^{j-} + r_{i_0}^{jc} + r_{i_0}^{j+})z_{i_0}^j - r_{i_0}^{j+}z_{i_0}^j + r_{i_0}^{j+}z_{i_0+1}^j \leq (Lz)_{i_0}^j =$$

$$= (Rz)_{i_0}^j \leq (q_{i_0}^{j-} + q_{i_0}^{jc} + q_{i_0}^{j+})z_{max}^{j-1},$$

which gives

$$\frac{h_{i_0} + h_{i_0-1}}{2\varepsilon\tau}z_{i_0}^j \leq \frac{h_{i_0} + h_{i_0-1}}{2\varepsilon\tau}z_{max}^{j-1} + r_{i_0}^{j+}(z_{i_0}^j - z_{i_0+1}^j),$$

i.e.

$$z_{i_0}^j \leq z_{max}^{j-1} + G,$$

where

$$G = \frac{2\varepsilon\tau}{h_{i_0} + h_{i_0-1}} z_{max}^{j-1} + (-\frac{1}{h_{i_0}} + \frac{h_{i_0}}{6\tau\varepsilon})(z_{i_0}^j - z_{i_0+1}^j).$$

From that we obtain

$$G = \frac{-6\varepsilon\tau + h_{i_0}^2}{3h_{i_0}(h_{i_0} + h_{i_0-1})}(z_{i_0}^j - z_{i_0+1}^j),$$

or

$$G \leq M_1 N^{-2} \ln^2 N.$$

The difference $z_{i_0}^j - z_{i_0+1}^j$ is estimated by using the results from [4]. Finally,

$$z_{i_0}^j \leq z_{max}^{j-1} + M_1 N^{-2} \ln^2 N.$$

In the same way we can show that

$$z_{min}^{j-1} - M_1 N^{-2} \ln^2 N \leq z_{i_0}^j.$$

Let $i_0 < i \leq N - i_0 - 1$. Then $h_2 = h_i = h_{i-1}$. The last case which we will analyze in detail is $r_i^{j+} \geq 0$ i $r_i^{j-} \geq 0$. The other cases reduce to the considered situations. This time we have

$$(r_k^{j-} + r_k^{jc} + r_k^{j+})z_k^j - r_k^{j-} z_{k-1}^j + r_k^{j+} z_{k+1}^j - r_k^{j-} z_k^j + r_k^{j+} z_k^j \leq (Lz)_k^j$$
$$= (Rz)_k^j \leq (q_k^{j-} + q_k^{jc} + q_k^{j+}) z_{max}^{j-1},$$

which gives

$$z_k^j \leq z_{max}^{j-1} + r_k^{j-}(z_k^j - z_{k-1}^j)\frac{\tau\varepsilon}{h_2} + r_k^{j+}(z_k^j - z_{k+1}^j)\frac{\tau\varepsilon}{h_2}.$$

From [4] it follows that

$$z_k^j \leq z_{max}^{j-1} + M_2 N^{-2}.$$

Similar analysis shows that

$$z_{min}^{j-1} - M_2 N^{-2} \leq z_{i_0}^j.$$

Further flow of the proof is derived according to the mathematical induction on time indexes j. Namely, for $j = 0$ i $M = \max(M_1, M_2)$ from the previous inequalites we obtain that

$$z_k^1 \leq z_{max}^0 + MN^{-2} \ln^2 N \leq b^{max} + MN^{-2} \ln^2 N \leq b^{max} + MT.$$

Let us suppose that the statement is valid for some j ($j\tau < T$). Then,

$$z_k^{j+1} \leq z_{max}^j + MN^{-2} \ln^2 N \leq b^{max} + jMN^{-2} \ln^2 N \leq b^{max} + \frac{T}{\tau} MN^{-2} \ln^2 N.$$

The conditions of the theorem give $\tau \geq N^{-2}\ln^2 N$ and accordingly that we have

$$z_k^{j+1} \leq b_{max} + MT.$$

In the same way we obtain the second inequality. The stability of the scheme(12) gives the following theorem.

Teorema 2.2. *Let $N_0\tau = T$, the mesh $\bar{Q}_h^*$ being defined by (9) and let z_i^j be the solution of the system (12). Let*

$$b^{max} = \max_{(x,t)\in S_h} W(x,t), \quad i \quad b^{min} = \min_{(x,t)\in S_h} W(x,t),$$

and $\tau \geq \frac{16m^2\ln^2 N}{3N^2} = g(m,N)$ i $\tau \geq N^{-2}\ln^2 N$. Then, for $0 \leq j\tau \leq T$ and $i = 0,...,N$ the estimate

$$b^{min} - MT \leq z_i^j \leq b^{max} + MT. \tag{14}$$

holds.

The proof is very similar to the proof of the previous theorem and we omit it.

3. Numerical example

Let us consider the boundary value problem

$$\begin{cases} Lu(x,t) \equiv -\varepsilon u_{xx}(x,t) + u_t(x,t) = 0, \ (x,t) \in Q \\ u(x,t) = W(x,t), \quad (x,t) \in S = \bar{Q}/Q. \end{cases} \tag{15}$$

where

$$W(x,t) = \mathrm{erfc}(\frac{x}{2\varepsilon\sqrt{t}})(\frac{x^2}{2\varepsilon^2} + t) - \frac{1}{\pi}\exp(-\frac{x^2}{4\varepsilon^2 t})\frac{x\sqrt{t}}{\varepsilon}, \ 0 < x < \infty, \ t \geq 0.$$

and

$$W(x,0) = 0, \quad 0 \leq x < \infty, \quad W(0,t) = t, \quad t \geq 0.$$

Using the solutions of the difference schemes (11) and (12) on the grids (2) and (9) for $m = 2$ we calculated the values

$$E_n = \max_{\bar{Q}_h} |u(x,t) - z(x,t)|,$$

for various values of $\varepsilon = 2^{-k}$, $N = N_0$ and $T = 1$.

k	N							
	8	16	32	64	128	256	512	
2	9.622(−3)	6.154(−3)	3.432(−3)	1.802(−3)	9.232(−4)	4.670(−4)	42.349(−4)	E_n
4	6.636(−2)	1.323(−2)	1.783(−3)	6.233(−4)	6.284(−4)	3.933(−4)	2.164(−4)	E_n
6	2.442(−1)	1.862(−1)	7.734(−2)	1.878(−2)	4.220(−3)	8.268(−4)	1.114(−4)	E_n
8	2.664(−1)	2.622(−1)	2.461(−1)	1.891(−1)	8.017(−2)	2.025(−2)	4.940(−3)	E_n
10	2.678(−1)	2.675(−1)	2.665(−1)	2.624(−1)	2.466(−1)	1.899(−1)	8.089(−2)	E_n
12	2.679(−1)	2.679(−1)	2.678(−1)	2.676(−1)	2.665(−1)	2.625(−1)	2.467(−1)	E_n

Table 1. The scheme (11) on the uniform mesh $\bar{Q}_h$.

k	N							
	8	16	32	64	128	256	512	
2	9.622(−3)	6.154(−3)	3.432(−3)	1.802(−3)	9.232(−4)	4.670(−4)	42.349(−4)	E_n
4	6.636(−2)	1.323(−2)	1.783(−3)	6.233(−4)	6.284(−4)	3.933(−4)	2.164(−4)	E_n
6	7.379(−2)	3.214(−2)	1.174(−2)	3.715(−3)	9.935(−4)	2.050(−4)	4.463(−5)	E_n
8	7.435(−2)	3.214(−2)	1.174(−2)	3.715(−3)	9.935(−4)	2.050(−4)	4.463(−5)	E_n
10	7.435(−2)	3.214(−2)	1.174(−2)	3.715(−3)	9.935(−4)	2.050(−4)	4.463(−5)	E_n
12	7.454(−2)	3.214(−2)	1.174(−2)	3.715(−3)	9.935(−4)	2.050(−4)	4.463(−5)	E_n

Table 2. The scheme (11) on the nonuniform mesh $\bar{Q}^*_h$.

k	N							
	8	16	32	64	128	256	512	
2	9.860(−3)	4.446(−3)	2.133(−3)	9.952(−4)	4.384(−4)	2.261(−4)	1.110(−4)	E_n
4	8.139(−2)	2.096(−2)	5.408(−3)	1.599(−3)	6.162(−4)	2.779(−4)	1.333(−4)	E_n
6	8.935(−2)	3.998(−2)	1.564(−2)	5.657(−3)	1.946(−3)	6.507(−4)	2.203(−4)	E_n
8	9.011(−2)	3.997(−2)	1.564(−2)	5.657(−3)	1.946(−3)	6.507(−4)	2.203(−4)	E_n
10	9.031(−2)	3.997(−2)	1.564(−2)	5.657(−3)	1.946(−3)	6.507(−4)	2.203(−4)	E_n
12	9.036(−2)	3.997(−2)	1.564(−2)	5.657(−3)	1.946(−3)	6.507(−4)	2.203(−4)	E_n

Table 3. The scheme (12) on the nonuniform mesh $\bar{Q}^*_h$.

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