Novi Sad J. Math.
Vol. 31, No. 2, 2001, 69-71

ON ULTRAMETRIC SPACE

Ljiljana Gajić[1]

Abstract. Using well-known result about ultrametric spaces (see [3]) the fixed point theorem for a class of generalized contractive mapping on ultrametric space is proved.

AMS Mathematics Subject Classification (1991): 47H10

Key words and phrases: ultrametric space, spherically complete, fixed point

1. Introduction

Let (X, d) be a metric space. If the metric d satisfies strong triangle inequality: for all $x, y, z \in X$

$$d(x, y) \leq \max\{d(x, z), d(z, y)\}$$

it is called **ultrametric** on X [2].

Pair (X, d) now is **ultrametric space.**

Remark. Let $X \neq \emptyset$, metric d being defined on X by

$$d(x, y) = \begin{cases} 0, & \text{if } x = y \\ 1, & \text{if } x \neq y, \end{cases}$$

so-called discrete metric is ultrametric.

Example For $a \in \mathbb{R}$ let $[a]$ be the entire part of a. By

$$d(x, y) = \inf\{2^{-n} : n \in \mathbb{Z},\ [2^n(x - e)] = [2^n(y - e)]\}$$

(here e is any irrational number) an ultrametric d on $\mathbb{Q}$ is defined which determines the usual topology on $\mathbb{Q}$.

2. Result

In [1] the authors proved a generalization of a result from [2] for multivalued contractive function. We are going to generalize the result from [2] for single valued generalized contraction.

[1] Institute of Mathematics, University of Novi Sad, 21000 Novi Sad, Yugoslavia

Theorem 1. *Let (X, d) be spherically complete ultrametric space. If $T : X \to X$ is a mapping such that for every $x, y \in X$, $x \neq y$,*

$$d(Tx, Ty) < \max\{d(x, Tx), d(x, y), d(y, Ty)\} \tag{1}$$

then T has a unique fixed point.

Proof. Let $B_a = B(a; d(a, Ta))$ denote the closed spheres centered at a with the radii $d(a, Ta)$, and let $\mathcal{A}$ be the collection of these spheres for all $a \in X$.

The relation

$$B_a \leq B_b \quad \text{iff} \quad B_b \subseteq B_a$$

is a partial order on $\mathcal{A}$.

Now, consider a totally ordered subfamily $\mathcal{A}_1$ of $\mathcal{A}$. Since (X, d) is spherically complete we have that

$$\bigcap_{B_a \in \mathcal{A}_1} B_a = B \neq \emptyset.$$

Let $b \in B$ and $B_a \in \mathcal{A}_1$. Let $x \in B_b$.

Then

$$\begin{aligned} d(x, b) \leq \quad & d(b, Tb) \leq \max\{d(b, a), d(a, Ta), d(Ta, Tb)\} \\ & = \max\{d(a, Ta), d(Ta, Tb)\}. \end{aligned} \tag{2}$$

For $d(Ta, Tb) \leq d(a, Ta)$ implies that

$$d(x, b) \leq d(a, Ta).$$

In opposite case, $d(Ta, Tb) > d(a, Ta)$, and from (2) follows that

$$\begin{aligned} d(x, b) \leq d(b, Tb) \quad & \leq d(Ta, Tb) < \max\{d(a, Ta), d(a, b), d(b, Tb)\} \\ & = \max\{d(a, Ta), d(b, Tb)\} \end{aligned}$$

Now for $d(b, Tb) \leq d(a, Ta)$ we have

$$d(x, b) \leq d(a, Ta).$$

The inequality $d(b, Tb) > d(a, Ta)$ implies that $d(b, Tb) < d(b, Tb)$ which is a contradiction.

So we have proved that for $x \in B_b$

$$d(x, b) \leq d(a, Ta). \tag{3}$$

Now we have that

$$d(x, a) \leq d(a, Ta).$$

So $x \in B_a$ and $B_b \subseteq B_a$ for any $B_a \in \mathcal{A}_1$. Thus B_b is the upper bound for the family $\mathcal{A}$. By Zorn's lemma $\mathcal{A}$ has a maximal element, say B_z, $z \in X$. We are going to prove that $z = Tz$.

Let us suppose the contrary, i.e. that $z \neq Tz$. Inequality (1) implies that

$$d(Tz, T(Tz)) < d(z, Tz).$$

Now if $y \in B_{Tz}$ then $d(y, Tz) \leq d(Tz, T(Tz)) < d(z, Tz)$ so

$$d(y, z) \leq \max\{d(y, Tz), d(Tz, z)\} = d(Tz, z).$$

This means that $y \in B_z$ and that $B_{Tz} \subseteq B_z$.

On the other hand $z \notin B_{Tz}$ since

$$d(z, Tz) > d(Tz, T(Tz))$$

so $B_{Tz} \underset{\neq}{\subseteq} B_z$. This is a contradiction with the maximality of B_z. Hence, we have that $z = Tz$.

Let u be a different fixed point. For $u \neq z$ we have that

$$d(z, u) = d(Tz, Tu) < \max\{d(Tz, z), d(z, u), d(u, Tu)\} = d(z, u)$$

which is a contradiction.

The proof is completed. □

Remark. Since in ultrametric space the inequality

$$d(Tx, Ty) \leq \max\{d(Tx, x), d(x, y), d(y, Ty)\},\ x, y \in X,$$

is always satisfied so we can suppose only that for $x \neq y$

$$d(Tx, Ty) \neq \max\{d(Tx, x), d(x, y), d(y, Ty)\}.$$

References

[1] Kubiaczyk, J., Mostafa Ali, N., A Multivalued Fixed Point Theorem in Non - Archimedean Vector Spaces, Novi Sad, Journ. of Math. Vol 26, No 2, (1996), 111-116.

[2] Petalas,C., Vidalis, F., A fxed point theorem in non Archimedean vector spaces, Proc. Amer. Math. Soc. 118 (1993), 819-821.

[3] van Roovij, A. C. M., Non Archimedean Functional Analysis, Marcel Dekker, New York, 1978.

Received by the editors April 10, 2001