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# SEMI-BROWDER ESSENTIAL SPECTRA OF QUASISIMILAR OPERATORS

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**Abstract.** If $T$ and $S$ are quasisimilar bounded operators on Banach spaces, we prove that each closed-and-open subset of the lower semi-Browder essential spectrum of $T$ intersects one special part of the upper semi-Browder essential spectra of $T$ and $S$.

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## 1. Introduction

Let $X$ and $Y$ be Banach spaces and $\mathcal{L}(X,Y)$ be the Banach space of all bounded operators from $X$ into $Y$. For $T \in \mathcal{L}(X,Y)$ we use the following notations: $\mathcal{N}(T)$ is the kernel and $\mathcal{R}(T)$ is the range of $T$. Also, $\alpha(T) = \dim \mathcal{N}(T)$ and $\beta(T) = \dim \mathcal{N}(T^*) = \dim X/\overline{\mathcal{R}(T)}$. Here $X^*$ denotes the dual space of $X$ and $T^* \in \mathcal{L}(X^*)$ is the adjoint operator of $T$. We use $\sigma(T)$ to denote the spectrum of $T$. Recall that the approximate point spectrum of $T$ is defined by

$$\sigma_a(T) = \{\lambda \in \mathbb{C} : \lambda - T \text{ is not one-to-one with closed range}\}$$

and the defect spectrum of $T$ is defined by

$$\sigma_d(T) = \{\lambda \in \mathbb{C} : \lambda - T \text{ is not onto}\}.$$

The next sets of semi-Fredholm operators are well-known: $\Phi_+(X) = \{T \in \mathcal{L}(X) : \mathcal{R}(T) \text{ is closed and } \alpha(T) < \infty\}$ and $\Phi_-(X) = \{T \in \mathcal{L}(X) : \mathcal{R}(T) \text{ is closed and } \beta(T) < \infty\}$. $\Phi_+(X)$ and $\Phi_-(X)$, respectively, form the multiplicative semigroups of upper and lower semi-Fredholm operators on $X$. The set of Fredholm operators is defined as $\Phi(X) = \Phi_+(X) \cap \Phi_-(X)$. For a semi-Fredholm operator $T$ the index is defined as $i(T) = \alpha(T) - \beta(T)$. The sets of upper and lower semi-Fredholm essential spectra of $T$, respectively, are defined as

$$\sigma_{le}(T) = \{\lambda \in \mathbb{C} : \lambda - T \notin \Phi_+(X)\} \text{ and } \sigma_{re}(T) = \{\lambda \in \mathbb{C} : \lambda - T \notin \Phi_-(X)\}.$$

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The Fredholm essential spectrum of $T$ is defined as

$$\sigma_e(T) = \{\lambda \in \mathbb{C} : \lambda - T \notin \Phi(X)\} = \sigma_{le}(T) \cup \sigma_{re}(T).$$

We shall consider the set of Weyl operators, which is defined as $\Phi_0(X) = \{T \in \Phi(X) : i(T) = 0\}$. Also, the Weyl essential spectrum of $T$ is defined by $\sigma_w(T) = \{\lambda \in \mathbb{C} : \lambda - T \notin \Phi_0(X)\}$.

Recall that $\mathrm{asc}(T)$ (respectively $\mathrm{des}(T)$), the ascent (respectively descent) of $T$, is the smallest non-negative integer $n$, such that $\mathcal{N}(T^n) = \mathcal{N}(T^{n+1})$ (respectively $\mathcal{R}(T^n) = \mathcal{R}(T^{n+1})$). If no such $n$ exists, then $\mathrm{asc}(T) = \infty$ (respectively $\mathrm{des}(T) = \infty$) [1]. It is well-known that if the ascent and the descent of $T$ are finite, then they are equal.

The set of all upper (respectively lower) semi-Browder operators on $X$ is considered (under various names) in [3], [5], [7], [8], [9], [10], [11] and defined as: $\mathcal{B}_+(X) = \{T \in \Phi_+(X) : \mathrm{asc}(T) < \infty\}$ ($\mathcal{B}_-(X) = \{T \in \Phi_-(X) : \mathrm{des}(T) < \infty\}$). The notion "semi-Browder operator" firstly appears in [5], and also in [11] and [7]. The set of Browder (Riesz-Schauder [1]) operators on $X$ is defined as $\mathcal{B}(X) = \mathcal{B}_+(X) \cap \mathcal{B}_-(X)$. The Browder essential approximate point spectrum of $T$ is defined as

$$\sigma_{ab}(T) = \bigcap_{\substack{AK=KA \\ K \in K(X)}} \sigma_a(T+K) = \{\lambda \in \mathbb{C} : \lambda - T \notin \mathcal{B}_+(X)\},$$

the Browder essential defect spectrum of $T$ is defined as

$$\sigma_{db}(T) = \bigcap_{\substack{AK=KA \\ K \in K(X)}} \sigma_d(T+K) = \{\lambda \in \mathbb{C} : \lambda - T \notin \mathcal{B}_-(X)\}$$

and the Browder essential spectrum of $T$ is defined as

$$\sigma_b(T) = \bigcap_{\substack{AK=KA \\ K \in K(X)}} \sigma(T+K) = \{\lambda \in \mathbb{C} : \lambda - T \notin \mathcal{B}(X)\} = \sigma_{ab}(T) \cup \sigma_{db}(T).$$

We are pointing to the paper [9], where Rakočević introduced the notion of the Browder essential approximate point spectrum (and by duality the Browder essential defect spectrum) of $T$. By the analogy of the upper and lower semi-Fredholm essential spectra, we shall say that $\sigma_{ab}(T)$ and $\sigma_{db}(T)$, respectively, are the upper and lower semi-Browder essential spectra of $T$. Semi-Browder essential spectra are also considered in [7].

Recall the main statement concerning the semi-Browder operators and semi-Browder essential spectra.

**Lemma 1.1.** (a) *$\mathcal{B}_+(X)$ and $\mathcal{B}_-(X)$ are open subsets of $\mathcal{L}(X)$ [8, sect. 4].*

(b) $\partial\sigma_b(T)\imath\partial\sigma_{ab}(T)$ *[9, Corollary 2.5 (ii)], and by duality* $\partial\sigma_b(T)\imath\partial\sigma_{db}(T)$.

(c) $\sigma_{ab}(T)$ *and* $\sigma_{db}(T)$ *are compact non-empty subsets of* $\mathbb{C}$ *(follows from* (a) *and* (b)*).*

We also mention the next important and useful result [1, p. 57], [13].

**Lemma 1.2.** (a) *If at least one of the quantities* $\alpha(T)$, $\alpha(T^*)$ *is finite, then* $\mathrm{asc}(T) < \infty$ *implies* $\alpha(T) \le \alpha(T^*)$*, and* $\mathrm{des}(T) < \infty$ *implies* $\alpha(T^*) \le \alpha(T)$.

(b) *If* $\alpha(T) = \alpha(T^*) < \infty$*, then* $\mathrm{asc}(T)$ *is finite if and only if* $\mathrm{des}(T)$ *is finite.*

For $T \in \mathcal{L}(X)$ the Goldberg spectrum is defined as $\sigma_g(T) = \{\lambda \in \mathbb{C} : \mathcal{R}(\lambda - T) \text{ is not closed}\}$ (see [4] and [12]). Note that this spectrum may be empty, and also it is not necessarily closed or open subset of $\mathbb{C}$.

Recall that operators $T \in \mathcal{L}(X)$ and $S \in \mathcal{L}(Y)$ are quasisimilar, if there exist quasiaffinities $A \in \mathcal{L}(X, Y)$ and $B \in \mathcal{L}(Y, X)$, such that $AT = SA$ and $TB = BS$. Recall that $A$ is a quasiaffinity if $A$ is one-to-one and $\mathcal{R}(A)$ is dense. We shall frequently use the following fact: if $T$ and $S$ are quasisimilar, then $\alpha(\lambda - T) = \alpha(\lambda - S)$ and $\alpha(\lambda - T)^* = \alpha(\lambda - S)^*$ for all $\lambda \in \mathbb{C}$.

It is well-known that quasisimilar Banach space operators can have different spectra and different essential spectra (see [6] and references cited there). It seems interesting to consider the connections between various parts of the spectra of quasisimilar operators. These problems for bounded operators on Banach spaces and various essential spectra are considered (for example) in [2] and [6]. Upper and lower semi-Fredholm essential spectra of quasisimilar operators are considered in [16]. Results concerning some special cases of operators on Hilbert spaces, such as seminormal and quasinormal operators, may be found in [14] and [15].

It is natural to investigate the connection between the semi-Browder essential spectra of quasisimilar operators.

Finally, we recall one important Herrero's result [6].

**Lemma 1.3.** *If* $T \in \mathcal{L}(X)$ *and* $S \in \mathcal{L}(Y)$ *are quasisimilar, then every component of* $\sigma_e(T)$ *intersects* $\sigma_e(S)$ *and viceversa.*

## 2. Results

We begin with results which involve the semi-Browder essential spectra and the Goldberg spectrum.

**Theorem 2.1.** *If* $T \in \mathcal{L}(X)$ *and* $S \in \mathcal{L}(Y)$ *are quasisimilar, then:*

*(a)* $\sigma_{ab}(T) \backslash \sigma_g(T) \imath \sigma_{ab}(S)$ *and* $\sigma_{ab}(S) \backslash \sigma_g(S) \imath \sigma_{ab}(T)$;

*(b)* $\sigma_{db}(T) \backslash \sigma_g(T) \imath \sigma_{db}(S)$ *and* $\sigma_{db}(S) \backslash \sigma_g(S) \imath \sigma_{db}(T)$.

*Proof.* To prove (a), let $\lambda \in \sigma_{ab}(T)\backslash\sigma_g(T)$ and $\lambda \notin \sigma_{ab}(S)$. Since there exist quasiaffinities $A \in \mathcal{L}(X,Y)$ and $B \in \mathcal{L}(Y,X)$, such that $AT = SA$ and $TB = BS$, it follows that $A(\lambda - T)^n = (\lambda - S)^n A$ for all positive integers $n$. Since $\operatorname{asc}(\lambda - S) = p < \infty$, it follows that

$$A\mathcal{N}^{\infty}(\lambda - T)\imath\mathcal{N}^{\infty}(\lambda - S) = \mathcal{N}(\lambda - S)^p,$$

where we take $\mathcal{N}^{\infty}(T) = \bigcup_n \mathcal{N}(T^n)$. Since $\alpha(\lambda - S)^p < \infty$, and $A$ is one-to-one, it follows that $\dim \mathcal{N}^{\infty}(\lambda - T) < \infty$, so $\alpha(\lambda - T) < \infty$ and $\operatorname{asc}(\lambda - T) < \infty$. This contradicts the assumption $\lambda \in \sigma_{ab}(T) \setminus \sigma_g(T)$.

The rest of the proof follows in the same way. □

Now, we get a simple corollary. In the proof of this corollary we use Lemma 1.3.

**Corollary 2.2.** *If $T \in \mathcal{L}(X)$ and $S \in \mathcal{L}(Y)$ are quasisimilar, then*

$$\sigma_b(T) \setminus \sigma_g(T)\imath\sigma_b(S),$$

*so every component of $\sigma_b(T)$ intersects $\sigma_b(S)$.*

Also, we can prove the following result concerning the Weyl essential spectrum.

**Corollary 2.3.** *If $T \in \mathcal{L}(X)$ and $S \in \mathcal{L}(Y)$ are quasisimilar, then*

$$\sigma_w(T) \setminus \sigma_g(T)\imath\sigma_w(S),$$

*so every component of $\sigma_w(T)$ intersects $\sigma_w(S)$.*

*Proof.* Suppose that $\lambda \in \sigma_w(T) \setminus \sigma_g(T)$. It follows that $\mathcal{R}(\lambda - T)$ is closed and one of the following two cases may occur: $\alpha(\lambda - T) \neq \alpha(\lambda - T)^*$, or $\alpha(\lambda - T) = \infty$ and $\alpha(\lambda - T)^* = \infty$. We conclude that $\lambda \in \sigma_w(S)$. □

We shall use the following notations:

$$\begin{aligned}
H_{\infty\infty}(T) &= \{\lambda \in \mathbb{C} : \alpha(\lambda - T) = \infty, \alpha(\lambda - T)^* = \infty\},\\
H_{\alpha<\beta}(T) &= \{\lambda \in \mathbb{C} : \alpha(\lambda - T) < \alpha(\lambda - T)^*\},\\
H_{\beta<\alpha}(T) &= \{\lambda \in \mathbb{C} : \alpha(\lambda - T)^* < \alpha(\lambda - T)\},\\
K_{\infty\infty}(T) &= \{\lambda \in \mathbb{C} : \operatorname{asc}(\lambda - T) = \infty,\ \operatorname{asc}(\lambda - T)^* = \infty\},\\
A_{\infty}(T) &= \{\lambda \in \mathbb{C} : \operatorname{asc}(\lambda - T) = \infty\}\\
D_{\infty}(T) &= \{\lambda \in \mathbb{C} : \operatorname{asc}(\lambda - T)^* = \infty\}.
\end{aligned}$$

Also, let $\sigma_E(T) = \sigma_{ab}(T) \setminus [H_{\infty\infty}(T) \cup K_{\infty\infty}(T)]^{\circ}$. Here $D^{\circ}$ denotes the interior of $D$.

We also need the following auxiliary result.

**Lemma 2.4.** *If $T \in \mathcal{L}(X)$ and $\alpha(T) < \infty$, then $\alpha(T^n) \le n \cdot \alpha(T) < \infty$ for all positive integers $n$.*

We shall give a more precise information about the semi-Browder essential spectra of quasisimilar operators. The main result follows.

**Theorem 2.5** *If $T \in \mathcal{L}(X)$ and $S \in \mathcal{L}(Y)$ are quasisimilar, then every closed-and-open subset of $\sigma_{db}(T)$ intersects the set $\sigma_E(T) \cap \sigma_E(S)$.*

*Proof.* Let $\tau$ be an arbitrary closed-and-open subset of $\sigma_{db}(T)$. We distinguish two cases.

Case I. Suppose that $\tau$ is not an open subset of $\sigma_b(T)$. It follows that there exist: $t \in \tau$ and a sequence $(t_n)_n \imath \sigma_b(T) \setminus \sigma_{db}(T)$, such that $\lim t_n = t$. We conclude that $t \in \partial(\sigma_b(T) \setminus \sigma_{db}(T))$.

For arbitrary $\lambda \in \sigma_b(T) \setminus \sigma_{db}(T)$ we know that $\mathcal{R}(\lambda - T)$ is closed, $\alpha(\lambda - T)^* < \infty$ and $\mathrm{des}(\lambda - T) < \infty$. Since $\mathcal{R}(\lambda - T)^n$ is closed for all non-negative integers $n$, it follows that $\mathrm{asc}(\lambda - T)^* < \infty$. We get $\lambda \notin H_{\infty\infty}(T) \cup K_{\infty\infty}(T)$. Also, $\alpha(\lambda - T) = \infty$, or $\mathrm{asc}(\lambda - T) = \infty$, so $\lambda \in \sigma_{ab}(T) \setminus [H_{\infty\infty}(T) \cup K_{\infty\infty}(T)]^\circ$.

On the other hand, for the same $\lambda \in \sigma_b(T) \setminus \sigma_{db}(T)$ we have: $\alpha(\lambda - S)^* = \alpha(\lambda - T)^* < \infty$, so $\lambda \notin H_{\infty\infty}(S)$. There exist quasiaffinities $A \in \mathcal{L}(X, Y)$ and $B \in \mathcal{L}(Y, X)$ such that $AT = SA$, $TB = BS$, so $A^*[(\lambda - S)^*]^n = [(\lambda - T)^*]^n A^*$ for all non-negative integers $n$. Using the idea from Theorem 2.1, it follows that $A^* \mathcal{N}[(\lambda - S)^*]^n \imath \mathcal{N}[(\lambda - T)^*]^n$ for all $n$, so

$$A^* \mathcal{N}^\infty(\lambda - S)^* \imath \mathcal{N}^\infty(\lambda - T)^* = \mathcal{N}[(\lambda - T)^*]^p,$$

where $p = \mathrm{asc}(\lambda - T)^* < \infty$. Since $(\lambda - T)^*$ is semi-Fredholm and $A^*$ is one-to-one, it follows that

$$\alpha(\lambda - S)^* \le \dim \mathcal{N}^\infty(\lambda - S)^* \le \alpha[(\lambda - T)^*]^p < \infty.$$

It follows that $\mathrm{asc}(\lambda - S)^* < \infty$, so $\lambda \notin K_{\infty\infty}(S)$. We need to prove that $\lambda \in \sigma_{ab}(S)$. Suppose that $\lambda \notin \sigma_{ab}(S)$, so $\mathcal{R}(\lambda - S)$ is closed, $\alpha(\lambda - S) = \alpha(\lambda - T) < \infty$ and $\mathrm{asc}(\lambda - S) < \infty$. Using the previous method we know that these assumptions lead to the fact $\mathrm{asc}(\lambda - T) < \infty$, which contradicts $\lambda \in \sigma_{ab}(T)$. We have just proved that $\lambda \in \sigma_{ab}(S) \setminus [H_{\infty\infty}(S) \cup K_{\infty\infty}(S)]^\circ$.

It follows that $\sigma_b(T) \setminus \sigma_{db}(T) \imath \sigma_E(T) \cap \sigma_E(S)$. Since $\sigma_E(T) \cap \sigma_E(S)$ is closed, we get $t \in \sigma_E(T) \cap \sigma_E(S)$.

Case II. Let $\tau$ be an open subset of $\sigma_b(T)$. Since $\sigma_b(T)$ and $\sigma_{db}(T)$ are closed subsets of $\mathbb{C}$ and $\tau$ is a closed-and-open subset of $\sigma_{db}(T)$, it follows that $\tau$ is a closed-and-open subset of $\sigma_b(T)$. By Corollary 2.2 it follows that $\tau \cap \sigma_b(S) \neq \emptyset$.

Suppose that $\tau \cap \sigma_E(T) \cap \sigma_E(S) = \emptyset$. It is easy to prove the following:

$$\begin{aligned} \tau \cap \sigma_b(S) \quad & \imath (\sigma_{db}(T) \cap \sigma_b(S)) \setminus (\sigma_E(T) \cap \sigma_E(S)) \\ & \imath (\sigma_{db}(T) \setminus \sigma_E(T)) \cup (\sigma_b(S) \setminus \sigma_E(S)). \end{aligned}$$

Notice that

$$\sigma_{db}(T)\setminus\sigma_E(T)=(\sigma_{db}(T)\setminus\sigma_{ab}(T))\cup(\sigma_{db}(T)\cap[H_{\infty\infty}(T)\cup K_{\infty\infty}(T)]^\circ).$$

We shall prove that $\sigma_{db}(T)\setminus\sigma_E(T)\imath D(T)$, where

$$D(T)=[H_{\alpha<\beta}(T)\cap D_\infty(T)]^\circ\cup[H_{\infty\infty}(T)\cup K_{\infty\infty}(T)]^\circ.$$

Let $\lambda\in\sigma_{db}(T)\setminus\sigma_{ab}(T)$. It follows that $\mathcal{R}(\lambda-T)$ is closed, $\alpha(\lambda-T)<\infty$ and $\mathrm{asc}(\lambda-T)<\infty$. By Lemma 1.2 it follows that $\alpha(\lambda-T)\leq\beta(\lambda-T)$. If we admit $\alpha(\lambda-T)=\beta(\lambda-T)<\infty$, then it follows $\mathrm{des}(\lambda-T)=\mathrm{asc}(\lambda-T)<\infty$ (Lemma 1.2), so $\lambda-T$ is a Browder operator, which contradicts the fact $\lambda\in\sigma_{db}(T)$. It follows that $\lambda\in H_{\alpha<\beta}(T)$. Since $\lambda-T\in\mathcal{B}_+(X)\imath\Phi_+(X)$ we get $\lambda\in H_{\alpha<\beta}(T)^\circ$, so

$$\varepsilon_1=\mathrm{dist}\{\lambda;\ \mathbb{C}\setminus H_{\alpha<\beta}(T)\}>0.$$

Let $\varphi_0(T)=\{\mu\in\mathbb{C}:\mu-T\in\Phi_0(X)\}$. It is well-known that $\varphi_0(T)$ is an open subset of $\mathbb{C}$. Since $\lambda\in\Phi_+(X)\setminus\Phi_0(X)$, it follows that

$$\varepsilon_2=\mathrm{dist}\{\lambda;\ \varphi_0(T)\}>0.$$

Notice that

$$\varepsilon_3=\mathrm{dist}\{\lambda;\ \sigma_{ab}(T)\}>0.$$

Let $\varepsilon=\min\{\varepsilon_1,\varepsilon_2,\varepsilon_3\}(>0)$. We claim that if $|\mu-\lambda|<\varepsilon$, then $\mathrm{des}(\mu-T)=\mathrm{asc}(\mu-T)^*=\infty$. On the contrary, suppose that $\mathrm{des}(\mu-T)<\infty$. Since $\mu-T\in\mathcal{B}_+(X)$, it follows that $\beta(\mu-T)=\alpha(\mu-T)$, which contradicts the fact $\mu\in H_{\alpha<\beta}(T)$. We have just proved that

$$\lambda\in[H_{\alpha<\beta}(T)\cap D_\infty(T)]^\circ.$$

Now it is obvious that

$$\sigma_{db}(T)\setminus\sigma_E(T)\imath D(T).$$

By the same way we can prove that $\sigma_b(S)\setminus\sigma_E(S)\imath D(S)$, so

$$\tau\cap\sigma_b(S)\imath D(T)\cap D(S).$$

We prove that $D(T)=D(S)$. Firstly we prove

$$[H_{\alpha<\beta}(T)\cap D_\infty(T)]^\circ=[H_{\alpha<\beta}(S)\cap D_\infty(S)]^\circ.$$

Let $\lambda\in[H_{\alpha<\beta}(T)\cap D_\infty(T)]^\circ$. There exists $\varepsilon>0$, such that for all complex numbers $\mu$, if $|\mu-\lambda|<\varepsilon$, then $\alpha(\mu-T)<\alpha(\mu-T)^*$ and $\mathrm{asc}(\mu-T)^*=\infty$. It follows that $\alpha(\mu-S)<\alpha(\mu-S)^*$. Notice that $\mathrm{asc}(\mu-S)^*<\infty$ would imply $\alpha(\mu-S)^*\leq\beta(\mu-S)^*=\alpha(\mu-S)$ (Lemma 1.2), so we get $\mathrm{asc}(\mu-S)^*=\infty$ for all $\mu$, $|\mu-\lambda|<\varepsilon$, and $\lambda\in[H_{\alpha<\beta}(S)\cap D_\infty(S)]^\circ$.

Now we prove $H_{\infty\infty}(T) \cup K_{\infty\infty}(T) = H_{\infty\infty}(S) \cup K_{\infty\infty}(S)$. Since $H_{\infty\infty}(T) = H_{\infty\infty}(S)$, it is enough to prove

$$K_{\infty\infty}(T) \setminus H_{\infty\infty}(T) = K_{\infty\infty}(S) \setminus H_{\infty\infty}(S).$$

In order to prove the last equality, let $\lambda \in K_{\infty\infty}(T) \setminus H_{\infty\infty}(T)$. Then $\mathrm{asc}(\lambda - T) = \infty$ and $\mathrm{asc}(\lambda - T)^* = \infty$. Let us assume that $\infty > \alpha(\lambda - T) = \alpha(\lambda - S)$.

Suppose that $\mathrm{asc}(\lambda - S) = p < \infty$. Since $AT = SA$ we conclude

$$A\mathcal{N}^{\infty}(\lambda - T)\imath\mathcal{N}^{\infty}(\lambda - S) = \mathcal{N}(\lambda - S)^p.$$

Also, $A$ is a quasiaffinity, so

$$\alpha(\lambda - T) \leq \dim \mathcal{N}^{\infty}(\lambda - T) \leq \alpha(\lambda - S)^p \leq p \cdot \alpha(\lambda - S) < \infty \text{ (Lemma 2.4).}$$

It follows that $\mathrm{asc}(\lambda - T) < \infty$, which contradicts $\lambda \in K_{\infty\infty}(T) \setminus H_{\infty\infty}(T)$. We get that $\mathrm{asc}(\lambda - S) = \infty$.

Suppose that $\mathrm{asc}(\lambda - S)^* < \infty$. By Lemma 1.2 it follows that $\alpha(\lambda - S)^* \leq \beta(\lambda - S)^* = \alpha(\lambda - S) < \infty$ and by the known method we conclude $\mathrm{asc}(\lambda - T)^* < \infty$, which contradicts $\mathrm{asc}(\lambda - T)^* = \infty$. It follows that $\mathrm{asc}(\lambda - S)^* = \infty$, also. We have just proved $D(T) = D(S) = D$.

Notice that $D$ is an open subset of $\mathbb{C}$. Also, $D\imath\sigma_{db}(T)^{\circ}$ and $D\imath\sigma_b(S)^{\circ}$. We can prove that $\tau \cap D$ is a closed-and-open subset of $\mathbb{C}$, which contradicts the fact $\emptyset \neq D \neq \mathbb{C}$. Since $D$ is an open subset of $\mathbb{C}$ and $\tau$ is a closed-and-open subset of $\sigma_{db}(T)$, we can conclude that $\tau \cap D$ is open in $\mathbb{C}$. Since $\sigma_b(S) \setminus D\imath\sigma_E(S)$, we conclude $\partial D\imath\sigma_E(S)$. In the same way we can prove $\partial D\imath\sigma_E(T) \cap \sigma_E(S)$. Finally, suppose that $(t_n)_n\imath\tau \cap D$ and $\lim t_n = t \in \tau$. We get

$$t \in \tau \cap (D \cap \partial D)\imath(\tau \cap D) \cup (\tau \cap \sigma_E(T) \cap \sigma_E(S)) = \tau \cap D,$$

so $\tau \cap D$ is closed in $\mathbb{C}$.

It follows that $\tau \cap \sigma_E(T) \cap \sigma_E(S) \neq \emptyset$. □

Now, it is a routine to prove the following result.

**Corollary 2.6.** *If $T \in \mathcal{L}(X)$ and $S \in \mathcal{L}(Y)$ are quasisimilar and $\Omega$ is a subset of $\mathbb{C}$ such that*

$$\sigma_{db}(T) \cap \Omega \neq \emptyset, \quad \textit{but} \quad \sigma_{db}(T) \cap \partial\Omega = \emptyset,$$

*then*

$$\Omega \cap \sigma_E(T) \cap \sigma_E(S) \neq \emptyset.$$

In the next theorem we shall prove one result concerning the Browder essential spectrum. We use the notation $\sigma_{adb}(T) = \sigma_{ab}(T) \cap \sigma_{db}(T)$.

**Theorem 2.7.** *If $T \in \mathcal{L}(X)$ and $S \in \mathcal{L}(Y)$ are quasisimilar and $\Omega$ is a subset of $\mathbb{C}$ such that*

$$\sigma_b(T) \cap \Omega \neq \emptyset \quad \textit{and} \quad \sigma_b(T) \cap \partial\Omega = \emptyset,$$

*then $\Omega \cap \sigma_G(T) \cap \sigma_G(S) \neq \emptyset$. Here we use $\sigma_G(T) = \sigma_{adb}(T) \setminus G(T)$ and*

$$G(T) = [H_{\alpha<\beta}(T) \cap D_\infty(T)]^\circ \cup [H_{\beta<\alpha}(T) \cap A_\infty(T)]^\circ.$$

*Proof.* It is easy to conclude $\partial\sigma_b(T) \cap \Omega \neq \emptyset$. By Lemma 1.1 it follows that $\partial\sigma_b(T)\imath\partial\sigma_{ab}(T)$ and $\partial\sigma_b(T)\imath\partial\sigma_{db}(T)$. So, if $\lambda \in \partial\sigma_b(T) \cap \Omega$, we conclude $\lambda \in \sigma_{adb}(T)$. It is easy to notice $G(T)\imath\sigma_b(T)^\circ$, so $\lambda \in \sigma_{adb}(T) \setminus G(T) = \sigma_G(T)$. Now, $\lambda$ may or may not belong to $\sigma_b(S)$ and we distinguish two cases.

Case I. Let $\lambda \in \sigma_b(S)$ and $\lambda \notin \sigma_G(T) \cap \sigma_G(S)$. Then

$$\lambda \in \sigma_b(S) \setminus \sigma_G(S) = [\sigma_b(S) \setminus \sigma_{adb}(S)] \cup [\sigma_b(S) \cap G(S)].$$

Notice that $\sigma_b(S) \cap G(S) = G(S)$. If $\lambda \in \sigma_b(S) \setminus \sigma_{adb}(S)$, we conclude that $\lambda - S \in \mathcal{B}_+(Y) \cup \mathcal{B}_-(Y)$ and $\mathcal{R}(\lambda - S)$ is closed. If $\lambda - S \in \mathcal{B}_+(Y)$, then $\alpha(\lambda - S) < \infty$ and $\mathrm{asc}(\lambda - S) < \infty$. It follows that $\alpha(\lambda - S) \leq \alpha(\lambda - S)^*$. If we assume $\alpha(\lambda - S) = \alpha(\lambda - S)^*$, then it follows $\mathrm{asc}(\lambda - S) = \mathrm{asc}(\lambda - S)^* < \infty$ and $\lambda \notin \sigma_b(S)$, which contradicts $\lambda \in \sigma_b(S)$. We get that $\lambda - S \in \mathcal{B}_+(Y)$ implies $\lambda \in [H_{\alpha<\beta}(S) \cap D_\infty(S)]^\circ$ (recall the corresponding part of the proof of Theorem 2.5). Also, $\lambda - S \in \mathcal{B}_-(Y)$ implies $\lambda \in [H_{\beta<\alpha}(S) \cap A_\infty(S)]^\circ$. Anyway, it follows that $\sigma_b(S) \setminus \sigma_{adb}(S)\imath G(S)$ and

$$\sigma_b(S) \setminus \sigma_G(S) = G(S).$$

Using the corresponding part of the proof of Theorem 2.4, we conclude that $G(S) = G(T)$, so $\lambda \in \sigma_b(T)^\circ$. The obtained fact contradicts $\lambda \in \partial\sigma_b(T)$, so it follows that $\lambda \in \Omega \cap \sigma_G(T) \cap \sigma_G(S)$.

Case II. Suppose that $\lambda \notin \sigma_b(S)$. In this case let $\tau$ denote the component of $\sigma_b(T)$ containing $\lambda$. By Corollary 2.2 it follows that there exists $\mu \in \tau \cap \sigma_b(S)$, so it follows that $\tau \cap \partial\sigma_b(S) \neq \emptyset$. Let $\nu \in \tau \cap \partial\sigma_b(S)$. As in Case I we conclude that $\nu \in \sigma_{adb}(S) \setminus G(S) = \sigma_G(S)$. If $\nu \notin \sigma_G(S) \cap \sigma_G(T)$, then

$$\nu \in \sigma_b(T) \setminus \sigma_G(T) = G(T) = G(S)\imath\sigma_b(S)^\circ,$$

(use the corresponding part of Case I), which contradicts $\nu \in \partial\sigma_b(S)$. We get $\nu \in \sigma_G(T) \cap \sigma_G(S)$. Finally, suppose that $\nu \notin \Omega$. Since $\lambda \in \tau \cap \Omega$, it follows that $\tau \cap \partial\Omega \neq \emptyset$, which contradicts $\sigma_b(T) \cap \partial\Omega = \emptyset$. Again, it follows that $\nu \in \Omega \cap \sigma_G(T) \cap \sigma_G(S)$. □

Using Theorem 2.7 it is not difficult to prove the following result.

**Corollary 2.8.** *If the conditions from Theorem 2.7 are satisfied, then $\Omega \cap \partial(\sigma_G(T) \cap \sigma_G(S)) \neq \emptyset$.*

Finally, notice that using the same principles as in Theorem 2.7 and Corollary 2.8, we can prove one more result concerning the Weyl essential spectrum. We use the notation $\sigma_{lre}(T) = \sigma_{le}(T) \cap \sigma_{re}(T)$.

**Theorem 2.9.** *If $T \in \mathcal{L}(X)$ and $S \in \mathcal{L}(Y)$ are quasisimilar operators and $\Omega$ is a subset of $\mathbb{C}$ such that*

$$\sigma_w(T) \cap \Omega \neq \emptyset \quad \textit{and} \quad \sigma_w(T) \cap \partial\Omega = \emptyset,$$

*then $\Omega \cap \partial(\sigma_F(T) \cap \sigma_F(S)) \neq \emptyset$, where $\sigma_F(T) = \sigma_{lre}(T) \setminus F(T)$ and*

$$F(T) = [H_{\alpha<\beta}(T) \cup H_{\beta<\alpha}(T) \cup H_{\infty\infty}(T)]^{\circ}.$$

**Remark 2.10** Z. Yan proved analogous results for the lower and upper semi-Fredholm essential spectra and for the Fredholm essential spectrum in [16].

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