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JOVAN KARAMATA (1902-1967)

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Jovan Karamata was born in Zagreb on February 1st 1902. He started his studies in 1920 at the Faculty of Engineering, but in 1922, he transferred to the faculty of Philosophy to study mathematics. He graduated from this faculty in 1925 and was immediately appointed as a teaching assistant to Professor Mihailo Petrović. He obtained his doctoral degree in 1926. He was promoted to the position of assistant professor in 1930, became an associate professor in 1937 and professor at Belgrade Unviersity in 1950. He left Belgrade in 1951 when he was appointed a professor at the University in Geneva, where he stayed until the end of his life. He died on August 14th, 1967.

As a highly respected mathematician and lecturer he participated in the work of numerous congresses and was a visiting professor at many universities in Europe and America. He became a member of Yugoslav Academy of Science (in 1933), Czech Royal Society (1936) and Serbian Royal Academy (1939) as well as a fellow of Serbian Academy of Sciences (1948). Karamata was member of the Swiss, French and German mathematical societies, the French Association for the Development of Science, the permanent reviewer in referee journals and the editor-in-chief of the journal L'Ensignement Mathématique. He took an active part in the activities at the Belgrade University and in the work of the Serbian Academy of Science and its Institute of Mathematics, thus contributing a great deal to the world reputation Belgrade mathematics had in those days. He also thought mathematical analyse at new-open Faculty of philosophy in Novi Sad and left significant trace in development of mathematics at University of Novi Sad.

Jovan Karamata was a prolific writer. In his works, he paid great attention to form and style. He published 122 scientific papers, 15 monographs and textbooks as well as 7 professional-pedagogical papers. His most significant results are in the field of classical mathematical analysis, more precisely in Tauberian theorems and in the theory of summability in general. These results, as well as those related to slowly varying functions, Mercer's theorems, inequalities, trigonometrical integrals, Froullani's integrals etc., have frequently been quoted in various papers and monographs. The originality of the approach to the various subject in mathematics, as well as the simplicity and elegance in proofs of theorems, confirm in the best manner not only Karamata's exceptional mathematical talent, but also his broad mathematical interest.

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There are some results which represent real progress in science, results which are further developed and which stay forever remembered as lasting scientific value. We pointed out here three such results of Karamata. All three are from the late twenties and the early thirties of previous century and almost equally known today as in the time when were created.

The first one is his proof of Abel's inverse statement i.e. the new proof of Littlewood's theorem. In 1910 Littlewood[2] proved that if the series $\sum_{\nu=0}^{\infty} a_\nu$ is Abel summable and if the condition $\nu a_\nu = O(1)$ holds, then $\lim_{n\to\infty} s_n = \lim_{n\to\infty} \sum_{\nu=0}^{n} a_\nu = s$, i.e. the given series is convergent.

In spite of the efforts of many mathematicians (e.g. Landau, Hardy, R. Schmidt), the proof of Littlewood's theorem remained rather difficult and far from the apparent for twenty years. But in 1930 in the journal Mathematische Zeitschrift appeared Karamata's two-page paper entitled *Über die Hardy-Littlewoodschen Umkehrungen des Abelschen Stätigkeitssatzes* which brought new proof to Littlewood's theorem, and created a sensation in mathematical circles. Karamata became world-renowned immediately. The editorial board of the "Mathematische Zeitschrift", on the occasion of its sixtieth anniversary, quoted this Karamata's paper in its selection of 50 of the most important papers, out of many thousands published.

In this paper Karamata first proved the following key theorem:

If the series $\sum_{\nu=0}^{\infty} a_\nu$ is Abel summable, i.e. if

$$\lim_{x\to 1-0}(1-x)\sum_{\nu=0}^{\infty} s_\nu x^\nu = s$$

and if

$$s_\nu \geq -M, \quad M \geq 0$$

then

$$\lim_{x\to 1-0}(1-x)\sum_{\nu=0}^{\infty} s_\nu f(x^\nu) x^\nu = s\int_0^1 f(t)dt$$

for each function $f(t)$ which is bounded and Riemann integrable in the interval $(0,1)$. The idea of Karamata's proof consists of two simple steps. The first one is that this statement obviously applies for the power, i.e. when $f(t) = x^\alpha$, $\alpha = 0, 1, 2, \ldots$ consequently for every polynomial of an arbitrary degree. The second one is in the direct application of the well-known Weierstrass's theorem

[2] *The converse of Abel's theorem on power series*, Proc. London Math. Soc. (2), vol. 9, pp. 434-448.

on uniform approximation of continuous functions by polynomials upon the function $f(t)$ for $0 \le t \le 1$. Here x and $f(t)$ are especially chosen as

$$x = e^{-\frac{1}{n}} \quad \text{and} \quad f(t) = \begin{cases} 0 & \text{za} \quad 0 \le t < e^{-1}, \\ \frac{1}{t} & \text{za} \quad e^{-1} \le t \le 1. \end{cases}$$

Becouse of condition $\nu a_\nu = O(1)$, it directly follows that the series $\sum\limits_{\nu=0}^{\infty} a_\nu$ is summable in the sense of Cesàro. That, however, implies by a simple lemma of Hardy that the series converges.

Karamata's proof was characterized as surprisingly simple by Knopp or extremely elegant by Titchmarsh.[3] In this, to the math world invaluable proof, he devised a new method of approach enabling other applications and results to follow. It entered later in the well-known books of Titchmarsh, Knopp, Doetch, Widder, Hardy, Favard. Although the great American mathematician and creator of cybernetics, Norbert Wiener in 1932 gave a general theory on inverse theorems which contains Hardy-Littlewood's theorem, Karamata's method did not loose its significance and has been used in proofs of a number of new results.

Discovery and introduction of slowly varying and regularly varying functions is the second Karamata's famous result. He defined slowly varying function by

$$\frac{L(tx)}{L(x)} \to 1, \quad x \to \infty\,, \quad t \ge 0\,,$$

L being continuous and positive on a positive half-axis, and regularly varying function by

$$\frac{r(tx)}{r(x)} \to t^\rho, \quad x \to \infty\,, \quad t \ge 0\,,$$

r being continuous and positive on a positive half-axis and ρ is called the index of regularity. This two definitions imply that any regularly varying function has the form $r(x) = x^\rho L(x)$.

Starting from these two simple definitions, Karamata developed the whole new theory of slowly and regularly varying functions which has included the majority of the most important properties of these functions. Above definitions appeared in 1930 in the paper entitled *Sur une mode de croissance régulière des fonctions* in, at that time, a less known Romanian journal *Mathematica* (Cluj). The intent of this paper was to generalize the Tauberian conditions in some inverse theorems of the Tauberian type for the Laplace transform. But it was soon realized that those functions could be successfully applied to many branches of mathematical analysis, not only when the mere convergence but also

[3]"Previously known proofs of Littlewood's theorem were very complicated, in spite of the number of research devoted to it, till 1930 J. Karamata found a *surprisingly simple* proof" (K.Knopp, *Theory and Application of Infinite Series* [London: Blackie and Son Ltd., 1954], p. 501; "We shall give an *extremely elegant* proof which has recently been obtained by Karamata" (E.C. Titchmarsh, *The Theory of Functions* [Oxford University Press, 1939], p. 226.

when other additional information were needed. These are, besides Abelian and Tauberian theorems, Mercerian theorems, Fourier analysis, analytic numbers theory, complex analysis, differential equations etc.

Those results, however, were not noticed and evaluated in a suitable manner until 1966 when they appeared in Feller's well-known monograph *An Introduction to Probability Theory and its Applications* whose second volume contained the elements of Karamata's theory, but not always with the precise presuppositions and clear conditions. This book has shown the great potential of regular variation for probability theory and stochastic processes in general. Hence Karamata's theory has grown, beyond all his expectations into a great mathematical edifice whose significance is still paramount. Theory of regularly varying functions, in recent years, has been expanded to the functions of several variables.

For further development of the theory of regularly varying functions and its applications, in addition to Karamata himself, the great importance have his colleagues and pupils (eg. Avakumović, Aljančić, Bajšanski, Bojanić, Tomić, Marić, Adamović, Aranđelović), recognized as "Karamata's (Yugoslav) school of mathematics", as well as Bingham, Goldie, Teugels, Seneta, Geluk, de Haan and many others. Even today, Karamata is the most frequently cited Serbian mathematician.

Using both the concepts of regular varying functions and generalization of Abel's and Tauber's theorems in Laplace-Stiltjes transform, Karamata obtained the third one result of lasting value. This is the following result today known as Hardy-Littlewood-Karamata theorem, which extends the Hardy-Littlewood's result for the Laplace transform where $L(x) = 1$:[4]

Let $L(x)$ and $L\left(\frac{1}{x}\right)$ be slowly varying functions, and let $A(x)$ be a non-decreasing function on $[a, \infty)$ such that the function

$$f(x) = \int_0^\infty e^{-xt} d\{A(t)\}$$

converges for every $x > 0$. Then the following statements hold:

$$\text{If } f(x) \sim x^{-\rho} L\left(\frac{1}{x}\right) \text{ for } x \to +0,\ \rho \geq 0 \text{ then } A(x) \sim x^{\rho} \frac{L(x)}{\Gamma(\rho+1)} \text{ for } x \to \infty.$$

$$\text{If } f(x) \sim x^{-\rho} L(x) \quad \text{for } x \to \infty \quad \text{then} \quad A(x) \sim x^{\rho} \frac{L\left(\frac{1}{x}\right)}{\Gamma(\rho+1)} \quad \text{for} \quad x \to +0.$$

E. Seneta ([6], p. 59.) said that this theorem was one of the most famous and very widely useful theorems in probabilistic (among other) context, and N.H. Bingham marked it as one of the major results in the analysis of the previous century.

[4] *Neuer Beweis und Verallgemeinerung der Tauberschen Sätze welche die Laplasche und Stieltjesche Transformationen betreffen*, Journal für die reine und angewandte Mathematik, B. 164 H. 1, 1931, pp. 27-39.

It must be remembered, as it is offten emphasized, that with Mihailo Petrović Serbian mathematics enterd the world of mathematical science, but with Jovan Karamata it reached its culmination. Karamata's original results in many areas of mathematics earned him recognition in the world of mathematicians. For history of Serbian mathematics there are also significant Karamata's school of mathematics and the great number of his pupils who later became eminent mathematician. It seems that even today, 35 years of Karamata's death and 50 years of his departure to Geneva, his work and his pupils give specific character to development of Serbian top-level mathematics.

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