Novi Sad J. Math.
Vol. 33, No. 1, 2003, 1–14

SOME RESULTS IN DISCRETE MATHEMATICS AND DIGITAL GEOMETRY BY PROFESSOR DRAGAN ACKETA

Dedicated to the memory of Dr. Dragan Acketa (1953-2000)
Professor of Mathematics at the University of Novi Sad

Sanja Lozić[1], Vojislav Mudrinski[2], Joviša Žunić[3]

Abstract. This article is dedicated to the memory of Professor Dragan Acketa. As a scientist, with a huge research potential and wide research interests, he made essential contributions in many areas of mathematics and computer science. Some of these are matroid and greedoid theory, combinatorial optimization, design theory, theory of algorithms, linear programming, pattern recognition and computer graphics. This is an overview of Professor Acketa's work on the problems related to matroid theory, design theory and digital geometry.

AMS Mathematics Subject Classification (2000): 05B30, 05A20, 52B40

Key words and phrases: matroid, greedoid theory, combinatorial optimization, computer graphics

1. Results in Matroid Theory

Matroid theory is the area of combinatorics to which the late Professor Acketa dedicated a great deal of his research activity. Dragan Acketa wrote his doctoral thesis on matroid theory and published about one hundred papers on the subject, many of which are frequently cited.

Matroids were first defined in 1935 by Hassler Whitney [43], who combined the linear dependence of columns of a matrix with graph theory. Matroids are an abstract generalization of matrices and graphs. This means that all graphs are matroids, but not all matroids can be represented by graphs. Similarly, every matrix with the entries belonging to a field is a matroid, but not every matroid is representable by a matrix whose entries belong to some field.

To better present the diverse topics in matroid theory that were the focus of Professor Acketa's research, some basic matroid concepts ([40], [42]) should be reviewed.

A *matroid* M is a finite set E (called the *ground set*) and a collection $\mathcal{I}$ of subsets of E (called *independent sets*) such that (I1)-(I3) are satisfied:

[1]Institute of Mathematics, University of Novi Sad
[2]Faculty of Technology, University of Novi Sad
[3]Faculty of Engineering, University of Novi Sad

(I1) $\emptyset \in \mathcal{I}$.
(I2) If $X \in \mathcal{I}$ and $Y \in X$ then $Y \in \mathcal{I}$.
(I3) If U, V are members of $\mathcal{I}$ with $|U| = |V| + 1$ there exists $x \in U \setminus V$ such that $V \cup x \in \mathcal{I}$.

A *circuit* of a matroid is a minimal dependent set.

The *rank* of a set $X \subseteq E$, denoted $r(X)$, is the size of the largest independent set contained in X. The rank of a matroid is the rank of the set E.

A set $X \subseteq E$ is called a *flat* (also known as a *subspace* or a *closed set*) of a matroid if $r(X \cup x) = r(X) + 1$ for all $x \in E \setminus X$.

We can associate a *geometric lattice* with the flats of a matroid. The rank of a matroid coincides with the height of the lattice, and the rank of a flat is its distance from the minimal element of the lattice. The minimal element of the lattice contains loops if a matroid has them, otherwise it is the empty set. Flats of rank 1 are called *atoms*. *Hyperplanes* are flats that are covered with the maximal element of the lattice which is the ground set.

The *essential flats* of a matroid, defined in [28], are cyclic flats (union of circuits), different from the ground set, the existence of which cannot be 'predicted' by using the family of all flats of lower rank.

A matroid is called *simple* if its atoms are one-element sets and called *semisimple* if it has parallel elements, i.e. at least one atom has more than one element. A *binary* (*ternary*) matroid is a matroid which can be represented by a matrix with entries from $GF(2)$ or $GF(3)$, respectively.

A matroid on the ground set E of rank r is a *paving* matroid if it has at least two hyperplanes, each hyperplane has cardinality at least $r - 1$ and every $r - 1$ element subset of E is contained in a unique hyperplane.

Professor Acketa's first paper [1] concerning matroids was published in 1978, in which he gave the formula for enumerating non-isomorphic matroids of rank 2 on a ground set of n elements:

$$m_2(n) = \sum_{p=2}^{n} \sum_{k=2}^{p} g(k)$$

where $g(k)$ is the number of partitions of the integer k such that all summands of the partition exceed 1. This is the only known closed form expression for general matroid enumeration, although it was rediscovered by Mark Dukes [31] in 2000 in a more elegant fashion. Professor Acketa himself congratulated Dukes on this result.

Professor Acketa was fascinated with making catalogues of semisimple matroids on less then 9 elements, for example in [3], [8] and [9]. He based his work on the Blackburn, Crapo and Higgs's catalogue of simple matroids [25] and followed the order of matroids listed there, but used the essential flats (together with their ranks) to describe the simple matroids. He felt that, especially for his purposes, essential flats describe simple matroids in a more economical way than hyperplanes – as explained in [2].

He developed his own methods for construction of semisimple matroids, without computer aid. One of his methods ('shortcuts' as he called them) is the use of positional partitions described in [6].

Positional partitions are equivalence relations among the atoms of a geometric lattice, 'positional' meaning that these partitions are determined by the position of atoms with respect to the other flats. There are two kinds of positional partitions: *weak* and *strong*. Weak positional partitions coincide with the orbits of the automorphism group of a matroid. On the other hand, two atoms x, y are in the same class of strong positional partition (i.e. are strongly related) if each essential flat F satisfies the following: if $|F \cap \{x, y\}| = 1$ then there exists essential flat $(F \setminus \{x, y\}) \cup (\{x, y\} \setminus F)$. He showed that two strongly related elements must be in the same class of a weak positional partition, the converse being not true.

His methods enabled him to add new elements (in parallel) to the atoms of a simple matroid in order to obtain all non-isomorphic semisimple matroids. In his doctoral thesis [10] he gave a list of all of 2198 non-isomorphic matroids (including matroids with loops) and analyzed their properties to be binary, graphic, paving, transversal, ...

The following table (given in [7]) lists some of the numerical values he was able to obtain:

n	0	1	2	3	4	5	6	7	8
$m(n)$	1	2	4	8	17	38	98	306	1724
$B(n)$	1	2	4	8	16	32	68	148	342

where $m(n)$ is the number of non-isomorphic semisimple matroids on n elements, and $B(n)$ the number of binary matroids on n elements.

Professor Acketa also focused on paving matroids. In [4] and [5] he constructed, again by hand, all 322 rank 4 paving matroids on 8 elements. For that purpose he used three auxiliary classes of graphs and some properties of Steiner system $S(3, 4, 8)$. He associated a denoted graph to each paving matroid in a unique way so the non-isomorphism of such construction was obvious.

He generalized the property of binary and paving matroids and proved the following theorems in [14]:

Theorem 1.1. *A matroid on an n-set is binary paving matroid if and only if it belongs to at least one of the following three classes:*

(a) matroids of rank 0, 1, or n,

(b) loopless rank 2 matroids with at most three hyperplanes,

(c) matroids $M(G_n^n)$, $M(G_n^{n-1})$, $M(A_4)$, $M(K_{3,2})$, F_7, F_7^*, D_8,

where G_n^k denotes any unicyclic graph on n edges, the unique cycle of which contains exactly k edges, graph A_4 is obtained by deleting one edge from K_4, F_7 and D_8 are the matroids on 7 (respectively 8) elements, the hyperplanes of which are the blocks of the Steiner system $S(2, 3, 7)$(respectively $S(3, 4, 8)$), and F_7^* is the dual of F_7.

Theorem 1.2. *The only 3-connected binary paving matroids are* $U_{0,0}$, $U_{0,1}$, $U_{1,1}$, $U_{1,2}$, $U_{1,3}$, $U_{2,3}$, $M(K_4)$, F_7, F_7^* *and* $AG(3,2)$.

In 1987, he produced a catalogue of non-isomorphic matroids on 9 elements [13], which contained 6705 semisimple matroids and matroids with loops.

Professor Dragan Acketa dedicated two papers to graphic matroids: [11] and [12]. He made the catalogue of non-isomorphic graphic representations of graphic matroids up to 9 elements. He was motivated by Whitney's theorem: If graph G is loopless and 3-connected, then there exists a unique graphic representation (without isolated vertices) of the associated matroid. We can imagine how lengthy and difficult Professor Acketa's work on the catalogue was by looking at one example: matroid C_6L_3 which consists of 9 elements, a circuit of length 6 and 3 loops. It can be represented by 16 different graphs, of which 7 are connected. In this manner Professor Acketa listed all 59646 non-isomorphic graphic representations by at least two mutually independent methods.

Enumeration of special classes of matroids continues to this day. A formula for the enumeration of binary and ternary matroids was given by Marcel Wild in 1994 in [44]. The formula is based, among other things, on the use of Burnsides Lemma for counting the number of orbits under the action of an automorphism group. In his paper, Wild cited Acketa's results on binary matroids, the only ones that were known up to that point.

2. Design Theory

Research in *design theory* belongs mostly to the combinatorial theory, thus the results are most often published in journals like *Ars Combinatoria*, Winipeg, Canada, (e.g. [19], [20]), and *Discrete Mathematics*, Elsevier (e.g. [17]). The design theory is applied in several fields of technical science such as: Optics, Telecommunications and Coding. Let us also mention that there is an International Seminar "Combinatorics", which is directed by the well–known mathematician D. Jungnickel. In the majority of our papers on the subject, the following three approaches were used:

1. Special or combinatorial permutation groups.
2. Basic Kramer-Mesner's algorithm [35] or improved algorithm [20].
3. Programming.

2.1. Some historic facts (written mainly for non–specialists in this field of mathematics)

Evariste Galois discovered the finite field at the beginning of the nineteenth century. He also introduced finite groups and algebras, and gave first deep results on them. In the twentieth century, this theory advanced strongly. The state of the art of that time was presented by B. Hupper and N. Blackburn in a three–volume books: *Endliche Gruppen I*, [32] (1967), *Finite Groups III*, [32] published by Springer Verlag in 1982. Of course, B. Hupper and N. Blackburn, in these books also presented their own numerous important results.

In parallel with this theory, the branches of finite geometries and matroids are also developed. The design theory is an independent branch, closely connected to them.

Designs arise as a generalization of configuration and other similar points of projective geometry. A $t-(v,k,\lambda)$ design is a collection $\mathcal{B}$ of k-subsets (called blocks) of a v-elements set Δ of points, which satisfies the property that each t-elements[4] subset of Δ is in exactly λ blocks. It is also required that blocks are not repeated.

In the last few decades, the main task of this theory was the construction of a new design with proving its existence. There are several approaches to such problem. In our first papers, the permutation groups $PSL(2,q)$, were used for the discovery of the designs.

For convenience of the reader, we will shortly repeat some facts about the groups mentioned above. Let $\Omega = \{GF(q), \infty\}$ be a projective line over the finite field $GF(q)$, and let $PGL(2,q)$ be the group of all invertible linear fractional mappings of Ω onto itself. Then $PGL(2,q)$ is the 3-transitive group of order $(q+1)q(q-1)$.

The group $PSL(2,q)$ consists of linear fractional mappings of the form, $x \longrightarrow \dfrac{ax+b}{cx+d}$, $a,b,c,d \in GF(q)$ where $ad-bc$ is a square in $GF(q)$.

$PSL(2,q)$ is of order $(q+1)q(q-1)k^{-1}$, where $k = gcd(q-1,2)$ and 2-transitively operates on $q+1$ points of the projective $P(1,q)$. Except the neutral element, its elements have at most two fixed points.

In [19], using PSL(2,37) we have:

Theorem 2.1. *There exist* $4-(38,5,\lambda)$ *designs with* $PSL(2,37)$ *as an automorphism group and with each* λ *in the set* $\{6,10,12,16\}$.

2.2. The Kramer-Mesner method

Group M acts on Δ. $\binom{\Delta}{s}$ is the set of all subsets of Δ that have cardinality s. Group M acts naturally on $\binom{\Delta}{s}$.

$\overline{T}_i$, $1 \le i \le m$, are orbits[5] of M on $\binom{\Delta}{t}$;

$\overline{K}_j$, $1 \le j \le m$, are orbits of M on $\binom{\Delta}{k}$.

---

[4]Let t-subsets denote a subset of t elements.

[5]Let G be a set and A be a group operating on G. Classes of equivalences based on this operation are called orbits [20].

If there is only one orbit of M on $\binom{\Delta}{s}$, then M is s-homogeneous on Δ.

Lemma 2.1. [24] *Given* $i \in \{1, \ldots, m\}$, $j \in \{1, \ldots, n\}$, *the number of sets* $K_j \in \overline{K}_j$ *containing a set* $T_i \in \overline{T}_i$ *is the same for all* T_i *sets.*

This invariant for the orbit pair $\overline{T}_i, \overline{K}_j$ will be denoted as λ_{ij}. The $m \times n$ matrix $[\lambda_{ij}]$ will be denoted as $\Lambda(M, t, k)$.

1. Let us delete some columns from $\Lambda(M, t, k)$ in such manner that the remaining matrix ΛD has the sum of entries equal to λ in each row. We then obtain a $t - (|\Delta|, k, \lambda)$ design.
2. Blocks of the above design are all k-subsets of Δ which belong o some of the k-orbits determined by the matrix $D\Lambda$.
3. Group M is an automorphism group of $t - (|\Delta|, k, \lambda)$.

If W is a subgroup of M, then W is also an automorphism group of $t - (|\Delta|, k, \lambda)$. The subgroup W induces a partition of M-orbits. Consequently, the matrix $\Lambda(M, t, k)$ has generally more rows and columns than $\Lambda(W, t, k)$.

It is of interest to note that four new 4-designs can be easily recognized from the table of incidence: 7×15 matrix $\Lambda(PGL(2, 37), 4, 5)$. The columns associated to the design $4 - (38, 5, 16)$ are marked.

	↓	↓				↓	↓					↓	↓	
8	8	8	2	8	0	0	0	0	0	0	0	0	0	0
4	0	4	0	0	4	8	4	4	2	4	0	0	0	0
2	4	0	0	4	4	0	8	0	0	4	4	2	2	0
0	8	4	0	0	2	4	0	0	4	4	4	0	0	4
0	0	4	4	4	0	4	4	0	4	4	2	0	4	0
0	4	4	0	4	0	0	4	6	0	4	0	0	4	4
0	0	0	0	12	0	12	0	0	0	0	0	4	0	6

In [17], two 5-designs ere found by a generalization of the homogenecity of an orbit. However, to find a $t - (v, k, \lambda)$ design for larger t is a much more difficult problem. Not long ago, no 5-designs were known. If we have a t-design, then, as a consequence we will also have the known corresponding derived $(t-s)$-designs for $t > s > 0$. By improvement of the computer control of multiple homogenity, used in the paper, we also obtained the 5-design.

When the Galois field is not prime (e.g. for $q = 25$), the so-called twisted group $(TW(2, 25))$ appears. It is the third linear group, which is a permutation and a projective one. In [16], the projective relations on 26 points, were given, for instance:

1. Designs that are both TW-designs and PGL-designs;
2. TW-designs that are not PGL-designs;

3. PGL-designs that are not TW-designs.

In the continuation of the research in this field, the graphs for the establishing of nonisomorph classes of design will be used. Lemmas 1 and 2 in [29] give the relations between graphs and designs, while Theorem 1 gives the following new results:

Theorem 2.2. *There exist* $12, 295, 1195$ *and* 2368 *pairwise non-isomorphic* $4-(48, 5, \lambda)$ *designs with* $PSL(2, 47)$ *as automorphism group, and with* λ *equal to* $8, 12, 16, 20$ *respectively.*

The last period of our research was exposed in [20], and its emphasis is on the good choice of the combinatorial permutation group. If we wish to obtain a maximal number of $t-(v, k, \lambda)$ designs for fixed t and v, using one permutation group, it must be chosen in accordance with the following principles:

Let we have a sequence of groups

$$M_1 \ \leq^6 \ M_2 \ \leq \ M_3 \ \leq \ M_4,$$

which induces the sequence of Λ matrices:

$$\Lambda(M_1, t, k), \ \ \Lambda(M_2, t, k), \ \ \Lambda(M_3, t, k), \ \ \Lambda(M4, t, k).$$

The advantages and disadvantages when going right are:

1. Λ becomes of smaller dimension (good);
2. Group M obtains a larger order (bad);
3. Most of the orbits are joined together; designs disappear (bad);
4. Homogenicity increases (good).

In view of the above principles, in [20], the following group $M = A_3N$ (7-Sylow)[7] was used.

2.3. Wreath product

Let M be a group and let H be a permutation group on Ω. If $\mathbf{f}$ is a mapping of Ω into M, $h \in H$, and $i \in \Omega$, then the set of all pairs $(\mathbf{f}, h)$ is a group with respect to the composition

$$(\mathbf{f}_1, h_1)(\mathbf{f}_2, h_2) \ = \ (\mathbf{f}_1(i), \mathbf{f}_2(i^{h_1}), \ h_1 h_2).$$

This group is called the wreath product of M and H and is denoted as MH. If M is a permutation group on Γ, then MH is a permutation group on $\Delta = \Gamma \times \Omega$. The group MH acts on Δ as follows:

$$(i, j)(\mathbf{f}, h) \ = \ (i^{\mathbf{f}(j)}, \ j^h).$$

[6] $\leq$ denotes a subgroup

[7] N denotes normaliser; 7-Sylow is a Sylow group of 7 elements; label will be explained later.

This choice of the group turned out to be a very good structure, rich in designs, [20].

Theorem 2.3. *Let M denote the subgroup of order 3 of $PSL(2,2)$ and let H denote the transitive subgroup of order 21 of $PSL(3,2)$. There exist $2-(21,k,\lambda)$ designs with the automorphism group equal to the wreath product MH, with $k \in \{4,5,\dots,10\}$ and with all 4079 possible λ values described in this section. The direct action of this wreath product on the Cartesian product of the projective line of order 2, and the projective plane of order 3 (Fano plane), does not give $2-(21,k,\lambda)$ designs with other values of λ.*

By using the Alltop's extension [23], the obtained $2-(21,10,\lambda)$ designs can be extended to $3-(22,11,\lambda)$ designs. This implies the following:

Corollary 2.1. *There exist $3-(22,11,\lambda)$ designs with 2867 different λ values.*

It is important to note that in the book [27], p. 55 for $3-(22,11,\lambda)$ precisely 154 designs are mentioned. However, in [20] we found 2867 designs. We think that this is the largest 3-designs collection generated by one group.

3. Digital Geometry

Professor Acketa's work in the area of digital image analysis has mostly been focussed on the problems related to the coding of digital curve segments.

Creating efficient coding schemes for digital objects is one of important problems considered in the area of computer vision and image processing. It is worth to mention that such efficient coding schemes preserve low storage complexity, fast transmission and mutual comparison of digital objects, etc.

Digital objects are defined to be a result of subjecting real objects to a certain digitization process. If a planar continuous curve $\rho:\ y=f(x)$ is digitized on the interval $[x_1, x_1+m-1]$ then the associated set of m digital points is called a *digital curve segment*, and it is defined as

$$C_m(\rho,x_1)=\{(i,\lfloor f(i)\rfloor),\ i=x_1,x_1+1,\dots,x_1+(m-1)=x_2\}. \tag{1}$$

Naturally, if ρ is a straight line then $C_m(\rho,x_1)$ is called digital straight line segment, if ρ is a part of a hyperbola, then $C_m(\rho,x_1)$ is called digital hyperbola segment, and so on.

3.1. Digital Straight Line Segments

If $f(x)$ in (1) is of the form $f(x)=a\cdot x+b$ we have a formal definition of a digital straight line segment. The digital straight line segments are digital objects extensively studied in the literature (see [30, 37, 41]). It is important to estimate what is the number of digital straight line segments that can be represented on an (m,n)-integer grid because it shows the capacity of the chosen grid. An asymptotic estimate $\dfrac{3 \cdot m^2 \cdot n^2}{\pi^2} + \mathcal{O}\left(m^2 \cdot n \cdot \log n + m \cdot n^2 \log\log n\right)$ (if $m \le n$) is given in [21]. On the other hand, it turns out that fast and exact computation of such a number is limited by a fast producing of the so–called generalized Farey sequence. Namely, given the natural numbers m and n, so that $m \le n$, then the generalized Farey (m,n)-sequence $F(m,n)$ is a strictly increasing sequence of all fractions of the form $\dfrac{b}{a}$, where the integers a and b satisfy: $\gcd(a,b) = 1$, $b < a \le n$, $b \le m$.

The following two theorems are proved in [21].

Theorem 3.1. *Let* $\dfrac{b}{a}$ *be a member of* $F(m,n)$ *and let* (x_0, y_0) *be an integral solution of the equation* $a \cdot x + b \cdot y = 1$. *Then the (immediate) successor* $\dfrac{b'}{a'}$ *of* $\dfrac{b}{a}$ *in* $F(m,n)$ *is determined by the relations* $b' = x_0 + r \cdot b$ *and* $a' = y_0 + r \cdot a$, *where* $r = \min\left\{ \left\lfloor \dfrac{n - y_0}{a} \right\rfloor, \left\lfloor \dfrac{m - x_0}{b} \right\rfloor \right\}$.

Theorem 3.2. *If* $\dfrac{b'}{a'}$ *is the (immediate) successor of the member* $\dfrac{b}{a}$ *of a generalized Farey sequence, then* $a \cdot b' - b \cdot a' = 1$ *holds.*

A direct application of the above theorems leads to an asymptotically optimal $\mathcal{O}(m \cdot n)$ algorithm for producing all the members of $F(m,n)$ in the increasing order.

3.2. General Coding Scheme

While recognition, reconstruction and coding problems for the digital straight line segments are solved completely, there existed only particular solutions for other types of digital curve segments ([39], [34]). The initial result by Professor Acketa on this topic was given in [22] – the article dealing with digital cubic parabolas. In the subsequent papers this result was generalized to digital polynomial segments of an arbitrary degree ([45]) and moreover, to families of sets of digital curve segments which can consist even of digital curve segments of different kinds ([46]). If digital curve segments from a fixed set are coded, then the number of bits for the coding depends on the maximal number of intersection points between two original curves (which are digitized) and of the size of the observed integer grid (in other words: of the resolution of the observed digital picture). If h is the maximal number of intersection points and the $n \times n$ grid is observed, then the number of sufficient bits is $\mathcal{O}(h^2 \cdot \log n)$. The proposed coding scheme preserves an asymptotically optimal storage if the maximal number of intersection points between two curves is assumed to be a constant (not dependent on the size of the observed integer grid – i.e. of the applied picture resolution).

Theorem 3.3. *Let a set $\mathcal{S}_h$ of digital curve segments be given, and let $C_m(\rho', x_1)$, $C_m(\rho'', x_1) \in \mathcal{S}_h$ such that ρ' and ρ'' have at most h intersection points on the interval $[x_1, x_1 + m - 1]$. Then any digital curve segment $C_m(\rho, x_1) = \{(i, \lfloor f(i) \rfloor), i = x_1, x_1 + 1, \ldots, x_1 + m - 1\}$ from $\mathcal{S}_h$, can be coded uniquely by $h + 3$ integers $(x_1, m, b_0, b_1, \ldots, b_h)$, where:*

- *x_1 and m are the x-coordinate of the left end-point of the observed digital curve segment and the number of its digital points, respectively;*
- $b_j = \sum\limits_{(i,j) \in C_m(\rho, x_1)} i^j \cdot \lfloor f(i) \rfloor \qquad$ *for* $j = 0, 1, \ldots, h$.

The integers $b_0, b_1, \ldots, b_h$ that appear in the code of $C_m(\rho, x_1)$ can be understood as so–called discrete moments of the area bounded by the x-axis, the curve ρ, and the lines $x = x_1$ and $x = x_1 + m - 1$. Since

$$\sum_{(i,j) \in S} i^p \cdot j \; = \; \mathcal{O}\left(n^{p+3}\right)$$

holds for any subset S from an (n, n)-integer grid, and for arbitrary integers p and q, the storage complexity for the coding curves form $\mathcal{S}_h$ is as follows.

Theorem 3.4. *The coding scheme proposed by Theorem 3.3 requires an amount of $\mathcal{O}\left(h^2 \cdot \log n\right)$ bits per coded digital curve segment belonging to a fixed set $\mathcal{S}_h$.*

In order to illustrate the advantage of the proposed coding scheme with respect to the chain coding we cite an example from [46]. The comparison of the code proposed in [46] and the 8-chain code for the digitization of the curve ρ is made under the assumption that ρ belongs to a set $\mathcal{S}_3$.

The "moment–based" proposed code of $C_{82}(1, \rho)$ is

$$(b_0, b_1, b_2, b_3) \; = \; (\ 5030,\ 215522,\ 12020658,\ 736586012\ )\ ,$$

while the 8-chain code is

7 6 7 7 6 7 6 7 6 7 7 6 7 6 7 6 7 7 6 6 6 6 6 6 7 6 6 6 6 6 6 6 6 7 6 6 6 6 6 6 6
6 6 6 7 6 6 6 6 6 6 6 6 6 7 6 6 6 6 6 6 6 6 6 6 7 6 6 6 6 6 6 6 6 6 1 2 1 2 2 2 2 2
1 2 2 2 2 2 1 2 2 2 2 2 2 1 2 2 2 2 2 1 2 2 2 2 2 1 2 2 2 2 2 1 2 2 2 2 2 2 1 2 2 2
2 2 0 0 0 7 0 0 0 0 0 7 0 0 0 0 1 2 2 2 2 1 2 2 2 2 1 2 2 2 1 2 7 7 6 7 7 6 7 7 7 6
7 7 6 1 1 2 2 1 2 2 2 1 2 2 1 2 2 2 1 2 2 1 2 2 2 1 2 7 6 6 7 6 6 6 7 6 6 1 2 1 2 2
2 1 2 2 1 2 2 2 7 6 6 6 6 6 6 7 6 6 6 6 6 6 6 7 6 6 1 2 2 1 2 1 2 2 1 2 2 7 7 6 6 6
6 6 6 6 6 6 6 6 6 6 7 6 6 6 6 6 6 6 6 6 6 6 6 6 7 6 6 6 6 6 6 6 6 6 6 6 7 6 6 6 6 6
6 6 6 6 6 7 6 6 6 6 6 6 6 6 6 6 7 6 6 6 6 6 6 6 6 6 6.

3.3. Encoding of Digital Curve Segments by Least Squares Polynomial Fit

The idea of using the least squares fit lines for a representation of digital lines was proposed in [38]. It is proved that the least squares line fit uniquely determines the digital line on a segment.

In [45] and [46] the result is extended to the coding by least squares polynomial fit. Moreover, it is shown that such coding is a subcase of the proposed general coding scheme.

A finite set of points in the plane is sometimes called a scatter diagram. The least squares curve for a scatter diagram is the curve which minimizes the total sum of the squares of the vertical distances from the curve to the data points. The method for determining such a curve is well-known from statistics. If the scatter diagram is given by $\{(x_i, y_i), i = 1, 2, \dots, m\}$ and the equation of its least squares polynomial is $Y = a_h X^h + a_{h-1} X^{h-1} + \dots + a_0$, then the unknowns $a_h, a_{h-1}, \dots, a_0$ satisfy

$$
\begin{array}{lcccccccl}
S_{2h} \cdot a_h & + & S_{2h-1} \cdot a_{h-1} & + & \dots & + & S_h \cdot a_0 & = & \sum_{i=1}^{m} y_i x_i^h \\
S_{2h-1} \cdot a_h & + & S_{2h-2} \cdot a_{h-1} & + & \dots & + & S_{h-1} \cdot a_0 & = & \sum_{i=1}^{m} y_i x_i^{h-1} \\
 & \dots & & \dots & & \dots & & & \\
S_h \cdot a_h & + & S_{h-1} \cdot a_{h-1} & + & \dots & + & S_0 \cdot a_0 & = & \sum_{i=1}^{m} y_i
\end{array}
\tag{2}
$$

where the coefficients $S_0, S_1, \dots, S_{2h}$ can be calculated recursively by using a well–known technique. The coefficients a_h, a_{h-1}, $\dots$, a_0 of the least squares polynomial fit can be determined by solving the above system. It is shown that the determinant of the system (2) is different from zero for $m \geq h$. Consequently, the system (2) has a unique solution in these cases.

If the scatter diagram is taken to be $C_m(\rho, x_1)$, let us denote the solution of (2) by $a_h(\rho)$, $a_{h-1}(\rho)$, $\dots$, $a_0(\rho)$. A very practical question is:

> *Are there two different digital curve segments, $C_m(\rho_1, x_1)$ and $C_m(\rho_2, x_1)$ that result from digitization of two curves from a set $\mathcal{S}_h$ with the same least squares polynomial fit, i.e. $a_h(\rho_1) = a_h(\rho_2)$, $a_{h-1}(\rho_1) = a_{h-1}(\rho_2)$, $\dots$, $a_0(\rho_1) = a_0(\rho_2)$?*

The answer is negative. This means that digital curve segments from a set $\mathcal{S}_h$ and their least squares polynomial fits of degree h are in one-to-one correspondence. This enables the coding of the digital curve segments with their associated least squares polynomial fit as is stated by the following theorem.

Theorem 3.5. *Let $C_m(\rho_1, x_1)$ and $C_m(\rho_2, x_1)$ be two digital curve segments from a set from $\mathcal{S}_h$ – i.e. ρ_1 and ρ_2 have at most h intersection points* *on* $[x_1, x_1 + m - 1]$. *If* $a_h(\rho_1)$, $a_{h-1}(\rho_1)$, ..., $a_0(\rho_1)$ *and* $a_h(\rho_2)$, $a_{h-1}(\rho_2)$, ..., $a_0(\rho_2)$ *are the coefficients of the least squares polynomial fits associated to* $C_m(\rho_1, x_1)$ *and* $C_m(\rho_2, x_1)$, *respectively, then:*

$$(a_h(\rho_1) = a_h(\rho_2) \quad \text{and} \quad a_{h-1}(\rho_1) = a_{h-1}(\rho_2) \text{ and } \dots \text{ and } a_0(\rho_1) = a_0(\rho_2))$$

is equivalent to

$$C_m(\rho_1, x_1) = C_m(\rho_2, x_1).$$

Determination of the least squares fitting polynomial for a given set of points is a linear problem, and consequently, it is easily solvable. Let us note that the representation of digital curve segments by its least squares curve fits is suitable because it is natural to expect that the least squares fitting curve "looks like" the original curve.

References

[1] Acketa, D.M., On the enumeration of matroids of rank 2, Zbornik radova PMF Novi Sad, 8 (1978), 83–90.

[2] Acketa, D.M., On the essential flats of geometric lattices, Publ. del Inst. Math., 26(40) (1979), 11–17.

[3] Acketa, D.M., On the construction of all matroids on 7 elements at most, Zbor. rad. PMF Novi Sad, 9 (1979), 133–151.

[4] Acketa, D.M., Another construction of rank 4 paving matroids on 8 elements I, Zbor. rad. PMF Novi Sad, 12 (1982), 259–276.

[5] Acketa, D.M., Another construction of rank 4 paving matroids on 8 elements II, Zbor. rad. PMF Novi Sad, 12 (1982), 277–303.

[6] Acketa, D.M., On the positional partitions of simple matroids, Proc. of the 3rd algeb. conf., Belgrade (1982), 1–8.

[7] Acketa, D.M., Some results on 'small' matroids, Coll. Math. Soc. Janos Bolyai, 40, Szeged, 1982.

[8] Acketa, D.M., The catalogue of all non-isomorphic matroids on at most 8 elements, Institute of Mathematics, Novi Sad, spec. iss., 1 (1983), 18–157.

[9] Acketa, D.M., A construction of non-simple matroids on at most 8 elements, J. Combin. Inform. System Sci., 9 (1984), 121–132.

[10] Acketa, D.M., Some Classes on Non–isomorphic Matroids, Ph.D. thesis, University of Novi Sad, 1984 (in Serbian).

[11] Acketa, D.M., On the number of graphic representations of graphic matroids, Proc. of the 6th Yugoslav Seminar on Graph Theory, Dubrovnik (1985), 1–26.

[12] Acketa, D.M., Graphic representations of graphic matroids on 9 elements, Proc. of the 8th Yug. Sem. on Graph Theory, Novi Sad (1987), 1–31.

[13] Acketa, D.M., A construction of all the non–isomorphic non–simple matroids on 9 elements, Zbor. rad. PMF Novi Sad, 17(1) (1987), 269–290.

[14] Acketa, D.M., On binary paving matroids, Discrete Math., 70 (1988), 109–110.

[15] Acketa, D.M., Mudrinski, V., On some 4- and 5- designs on ≤ 49 points, Filomat (Niš), 9(3) (1995), 597–590.

[16] Acketa, D.M., Mudrinski, V., A family of 4-designs on 26 points, Comment. Math. Univ. Carolinae, 37(4) (1996), 843–860.

[17] Acketa, D.M., Mudrinski, V., Two 5-designs on 32 points, Discrete Math., 163 (1997), 209–210.

[18] Acketa, D.M., Mudrinski, V., Some designs with projective symplectic groups as automorphism groups, Novi Sad J. Math., 27(1) (1997), 93–110.

[19] Acketa, D.M., Mudrinski, V., A family of 4-designs on 38 points. Ars Combinatoria, 53 (1999), 283–390.

[20] Acketa, D.M., Mudrinski, V., Matić–Kekić, S., A large collection of designs from wreath product on 21 points, Ars Combinatoria, 54 (2000), 109–118.

[21] Acketa, D.M., Žunić, J., On the Number of Linear Partitions of the (m, n)-grid, Information Processing Letters, 38 (1991), 163–168.

[22] Acketa, D.M., Žunić, J., A Constant Space Representation of Digital Cubic Parabolas, Publications de L'Institut Mathématique, 59(73) (1996), 169–176.

[23] Alltop, W.O., Extending t-designs, J. Comb. Theory (A), 18 (1975), 177–186.

[24] Beth, T., Jungnickel, D., Lenz, B., Design theory, Bibliographsches Institut Manheim/Wien/Zurich, 1985.

[25] Blackburn, J.E., Crapo, H.H., Higgs, D.A., A catalogue of combinatorial geometries, Math. Comp., 27 (1973), 155–166.

[26] Chee, Y.M., Colborn, C.J., Kreher, D.L., Simple t-designs with $t \leq 30$, Ars combinatoria, 29 (1990), 193–258.

[27] Colbourn, C.J., Dinitz, J.H., Combinatorial designs, CRC Press, Boca-Raton, New York, London, Tokyo, 1996.

[28] Crapo, H.H., Erecting geometries, Proc. of 2nd Chapel Hill Conf. on Combinatorial Math. (1970), 74–99.

[29] Dautović, S., Acketa, D.M., Mudrinski, V., Non-isomorphic 4-(48,5,λ) designs from PSL(2,47), Publ. Elektrotehn. Fak., Beograd, 10 (1999), 41–46.

[30] Dorst L., Smeulders, A.W.M., Discrete representation of straight lines, IEEE Trans. Pattern Analysis and Machine Intelligence, 6 (1984), 450–463.

[31] Dukes, W.M.B., Counting and Probability in Matroid Theory, Ph.D. thesis, University of Dublin, 2000.

[32] Huppert, B., Endliche Gruppen, I, Die Grundlehren der matematischen Wissenshaften, Band 134, Springer–Verlag, Berlin Heidelberg, New York xxii+793pp, 1967.

[33] Huppert, B., Blackburn, N., Finite groups, III, Springer–Verlag, Berlin Heidelberg, New York, 1982.

[34] Kim, C.E., Digital disks, IEEE Trans. Pattern Analysis and Machine Intelligence, 6 (1984), 372–374.

[35] Kramer, E.S., Mesner, D.M., t-designs on hypergraphs, Discrete Math., 15 (1976), 263–296.

[36] Koplowitz, J., Lindenbaum M., Bruckstein, A., The number of digital straight lines on $n \times n$ grid, IEEE Trans. on Inf. Theory, 36(1) (1990), 192–197.

[37] Lindenbaum, M., Koplowitz, J., A new parametrization of digital straight lines, IEEE Trans. Pattern Analysis and Machine Intelligence, 13 (1991), 847–852.

[38] Melter, R.A., Rosenfeld, A., New views of linearity and connectedness in digital geometry, Pattern Recognition Letters, 10 (1989), 9–16.

[39] Nakamura, A., Aizawa, K., Digital circles, Computer Vision, Graphics Image Processing, 26 (1984), 242–255.

[40] Oxley, J.G., Matroid Theory, Oxford University Press, 1992.

[41] Rosenfeld, A., Digital straight line segments, IEEE Trans. Comput., 23 (1974), 1264–1269.

[42] Welsh, D.J.A., Matroid Theory, Academic Press, London, 1976.

[43] Whitney, H., On the abstract properties of linear dependence, Amer. J. Math., 57 (1935), 509–533.

[44] Wild, M., Consequences of the Brylawski–Lucas theorem for binary matroids, Europ. J. Combinatorics, 17 (1996), 309–316.

[45] Žunić, J., Acketa, D.M., Least Squares Fitting of Digital Polynomial Segments, Lecture Notes in Computer Science: Discrete Geometry for Computer Imagery, 6th International Workshop, Lyon, France, 1176 (1996), 17–23.

[46] Žunić, J., Acketa, D.M., A General Coding Scheme for Families of Digital Curve Segments, Graphical Models and Image Processing, 60 (1998), 437–460.

Received by the editors June 5, 2002