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$\mathcal{G}$–QUASIASYMPTOTICS AT INFINITY TO SEMILINEAR HYPERBOLIC SYSTEM

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Abstract. We recall the definition of $\mathcal{G}$–quasiasymptotics at infinity in a framework of Colombeau space $\mathcal{G}$ (cf. [8]) and give an application of that notion to a Cauchy problem for a strictly semilinear hyperbolic system. It turns out that quasiasymptotic behaviour at infinity of the solution inherits the quasiasymptotic behaviour at infinity of initial data under suitable assumptions on the nonlinear term.

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1. Introduction

Consider a Cauchy problem for a semilinear strictly hyperbolic $(n\times n)$-system in two independent variables, $(x,t)\in\mathbf{R}^2$,

$$(\partial_t+\Lambda(x,t)\partial_x)u(x,t)=F(x,t,u(x,t)) \tag{1}$$

$$u(x,0)=(u_1(x,0),...,u_n(x,0))=(a_1(x),...,a_n(x))\in(\mathcal{G}(\mathbf{R}))^n,$$

where $\Lambda(x,t)$ is a diagonal matrix with the real smooth functions on the diagonal and $(x,t,u)\mapsto F_i(x,t,u)$, $i=1,...,n$ be smooth functions on $\mathbf{R}^{2+2n}$ such that

(2) $\mathbf{C}^n\ni u\mapsto F_i(x,t,u),\ i=1,...,n,$ is polynomially bounded together with all derivatives uniformly for $(x,t)\in K,$ for any compact set $K\subset\mathbf{R}^2$;

(3) $\mathbf{C}^n\ni u\mapsto\nabla_uF_i(x,t,u),\ i=1,...,n,$ is globally bounded uniformly with respect to $(x,t)\in K,$ for any compact set $K\subset\mathbf{R}^2$.

Under the assumption given above, (1) is uniquely solvable in $(\mathcal{G}(K_T))^n$, for every $T>0$ (cf. [5]).

The paper is an attempt of characterizing generalized solutions to system (1) with singular initial data.

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Let us give a few remarks concerning the "nature" of a solution. If F is linear in u and a is an n-tuple of singular Schwartz distributions, say $a = (\delta(x - x_1), ..., \delta(x - x_n))$ then, there exist a net of smooth solutions which approximate this solution in the setting of Colombeau generalized functions. Moreover, this net can be obtained by the use of Schwartz distributional theory. But if F is nonlinear, the framework of Colombeau generalized functions enables us to consider solutions to the given system with singular data which can not be considered in the framework of Schwartz spaces. Moreover, we can take $a = (\delta_1^2(x-x_1), ..., \delta_n^2(x-x_n))$ and consider the corresponding problem in $\mathcal{G}$ although this n-typle does not have any sense in $(\mathcal{D}')^n$.

The aim of this paper is an application of $\mathcal{G}$–quasiasymptotics at infinity to the given system. We show that under appropriate assumptions on a nonlinear term the quasiasymptotics at infinity of a solution to (1) inherits the quasiasymptotics at infinity of initial data.

2. Preliminaries

Let Ω_k be a sequence of open sets such that $\cup_{k=0}^{\infty}\Omega_k = \Omega$, $\Omega_k \subset\subset \Omega_{k+1}$, $k \in \mathbf{N}_0$. Then, the uniform structure of $C^\infty(\Omega)$ is defined by the sequence of seminorms

$$\mu_k(f) = \sum_{|\alpha|\le k} (\sup_{x\in\overline{\Omega}_k} |\partial^\alpha f(x)|), \quad k \in \mathbf{N}_0, \tag{4}$$

which does not depend on the choice of the sequence Ω_k.

We recall the simplified version of Colombeau theory (cf. [1] [2], [3], [5], [6], [9]).

Let V be a topological vector space whose topology is given by a countable set of seminorms μ_k, $k \in \mathbf{N}$, given by (4).

Then $\mathcal{E}_{M,V}$ is the set of locally bounded functions $R(\varepsilon) = R_\varepsilon : (0, 1) \to V$ such that for every $k \in \mathbf{N}$ there exists $a \in \mathbf{R}$ such that

$$\mu_k(R_\varepsilon) = \mathcal{O}(\varepsilon^a)$$

($\mathcal{O}(\varepsilon^a)$ means that the left side is smaller than or equal to $C\varepsilon^a$ for some $C > 0$ and every $\varepsilon \in (0, \varepsilon_0)$, $\varepsilon_0 > 0$).

The space of all elements $H \in \mathcal{E}_{M,V}$ with the property that for any $k \in \mathbf{N}$ and for any $a \in \mathbf{R}$, $\mu_k(H_\varepsilon) = \mathcal{O}(\varepsilon^a)$ is denoted by $\mathcal{N}_V$.

The quotient space $\mathcal{G}_V = \mathcal{E}_{M,V}/\mathcal{N}_V$ is called the generalized extension of V.

If the space V is an algebra whose products are continuous for all the seminorms, then $\mathcal{N}_V$ is an ideal of the algebra $\mathcal{E}_{M,V}$.

In particular, if $V = \mathbf{C}^\infty(\Omega)$, where Ω is an open set in $\mathbf{R}^n$ and μ_k are given by (4), then $\mathcal{G}_V$ is the algebra of generalized functions on Ω. We denote it by $\mathcal{G}(\Omega)$; $\mathcal{E}_{M,V}$ by $\mathcal{E}_M(\Omega)$ and $\mathcal{N}_V$ by $\mathcal{N}(\Omega)$. If $V = \mathbf{C}$, then $\mathcal{G}_V$ is called the algebra of generalized constants and it is denoted by $\bar{\mathbf{C}}$; $\mathcal{E}_{M,V}$ is denoted by $\mathcal{E}^0$ and $\mathcal{N}_V$ is denoted by $\mathcal{N}^0$.

Let $\psi \in C_c^\infty(\mathbf{R}^n) = \mathcal{D}(\mathbf{R}^n)$ and $\phi \in \mathcal{S}(\mathbf{R}^n)$ such that it is even, $\mathcal{F}(\phi) = \hat{\phi} \in \mathcal{D}(\mathbf{R})$ and $\hat{\phi} \equiv 1$ in a neighbourhood of zero. Put $\phi_\varepsilon(x) = 1/\varepsilon^n \phi(x/\varepsilon)$, $x \in \mathbf{R}^n, \varepsilon \in (0,1)$. Then,

$$N_\varepsilon(x) = (\psi * \phi_\varepsilon(x) - \psi(x)) \text{ belongs to } \mathcal{N}(\Omega),$$

where $*$ denotes a convolution.

Brackets [ ] are used to denote the equivalence class in the quotient space. If $T \in \mathcal{E}'(\Omega)$ then $I_\phi(T) = [T * \phi_\varepsilon]$.

If G is a generalized function with compact support $K \subset\subset \Omega$ $(G \in \mathcal{G}_c(\Omega))$ and $G_\varepsilon(x)$ is a representative of G, then its integral is defined by

$$\int G dx = \left[\int \psi(x) G_\varepsilon(x) dx \right],$$

where $\psi \in C_0^\infty(\mathbf{R}^n)$, $\psi = 1$ on K. This definition does not depend on ψ.

We define $\mathcal{G}_a(\mathbf{R}^n)$, $a = (a_1, ..., a_n) \in \mathbf{R}^n$ as a subspace of $\mathcal{G}(\mathbf{R}^n)$ consisting of $G \in \mathcal{G}(\mathbf{R}^n)$ such that *supp* $G \subset [a_1, \infty) \times ... \times [a_n, \infty)$. The well-known Schwartz spaces $\mathcal{D}'_a(\mathbf{R}^n)$ and $\mathcal{S}'_a(\mathbf{R}^n)$ are defined in an analogous way.

We denote by L Karamata's slowly varying function at zero (cf. [7]). Recall, it is measurable, positive and

$$\lim_{\varepsilon \to 0} \frac{L(\varepsilon t)}{L(\varepsilon)} = 1$$

uniformly for $t \in [a,b] \subset (0,\infty)$ (and $\varepsilon < \varepsilon_0/b$), ε_0 is fixed.

For the time being, C will denote a generic constant which is different in different appearances.

Recall, a notion of quasiasymptotics at infinity in the space $\mathcal{D}'$ is defined by Drozzinov and Zavialov for the elements of S'_+ (cf. [10]). Its modification is given in [7].

Definition 1. *Let $f \in \mathcal{D}'_a(\mathbf{R}^n)$, (resp. $f \in \mathcal{S}'_a(\mathbf{R}^n)$) and c be a positive, measurable function. If*

$$\lim_{k \to \infty} \frac{f(kx)}{c(k)} = g(x) \neq 0, \quad \text{in } \mathcal{D}'(\mathbf{R}^n) \ (\text{resp. in } \ \mathcal{S}'(\mathbf{R}^n)), \tag{5}$$

then it is said that f has the quasiasymptotics at infinity with respect to $c(k)$ in $\mathcal{D}'(\mathbf{R}^n)$, (resp. in $\mathcal{S}'(\mathbf{R}^n)$). We write $f \overset{q}{\sim} g$ at infinity with respect to $c(k)$.

Main characterizations of the quasiasymptotics at infinity in $\mathcal{D}'$ are given in [7]. The extension of this notion to the Colombeau space of generalized functions is given in [8]. We recall the definition.

Let $\mathcal{K}$ be a set of positive measurable functions defined on $(0,1)$ with the property

$$A^{-1}\varepsilon^p \leq c(1/\varepsilon) \leq A\varepsilon^{-p}, \ \varepsilon \in (0,1)$$

for some $A > 0$ and $p > 0$.

Let $a \in \mathbf{R}^n$. We denote by η_a a function of the form $\eta_a(t) = \eta_{a_1}(t_1)...\eta_{a_n}(t_n)$ where $\eta_a(t) \in C^\infty(\mathbf{R})$,

$$\eta_a(t) = \begin{cases} 1 & t > a + m \\ 0 & t < a - m, \end{cases}$$

for some $m > 0$.

Definition 2. *Let $F \in \mathcal{G}_a(\mathbf{R}^n)$. It is said that F has the $\mathcal{G}$–quasiasymptotics at infinity with respect to $c(1/\varepsilon) \in \mathcal{K}$ if there is F_ε, a representative of F, such that for every $\psi \in \mathcal{D}(\mathbf{R}^n)$ there is $C_\psi \in \mathbf{C}$ such that*

$$\lim_{\varepsilon \to 0} \left\langle \frac{(\eta_a F_\varepsilon)(x/\varepsilon)}{c(1/\varepsilon)}, \psi(x) \right\rangle = C_\psi \tag{6}$$

and $C_\psi \neq 0$ for some ψ.

This definition does not depend on the representatives.

3. $\mathcal{G}$–quasiasymptotics at infinity to semilinear hyperbolic system

Integral curves for (1) which pass through (x_0, t_0) at the time $\tau = t_0$ are denoted by $x = \gamma_i(x_0, t_0, \tau)$, $i \in \{1, ..., n\}$, and called characteristic curves of the system. Using them we transform (1) into the system of integral equations

$$u_i(x,t) = a_i(\gamma_i(x,t,0)) + \int_0^t F_i(\gamma_i(x,t,\tau), \tau, u(\gamma_i(x,t,\tau),\tau))d\tau,\ (x,t) \in K_T,$$

$i = 1, ..., n$, where K_T is the domain of determinancy bounded by extremal characteristics emanating from the end points of K_0 and the lines $t = \pm T$, where K_0 be a compact set.

By assumption on matrix ($||\Lambda|| \leq c < \infty$) the characteristic curves globally exist, i.e. for every $(x,t) \in \mathbf{R}^2$ there exists a compact set K such that the characteristic curves that pass through (x,t) start from K.

Denote $l_2(\varepsilon) = \ln|\ln \varepsilon|$. We have the following proposition.

Proposition 1. *(a) Let $c(1/\varepsilon) \in \mathcal{K}$, $\lim_{\varepsilon \to 0} \dfrac{|\ln \varepsilon|^C l_2^2(\varepsilon)}{c(1/\varepsilon)} = 0$ for every $C > 0$. Then,*

$$\lim_{\varepsilon \to 0} \left\langle \frac{u_{i\varepsilon}(xl_2(\varepsilon), tl_2(\varepsilon))}{c(1/\varepsilon)}, \psi(x,t) \right\rangle = 0,\ \psi \in \mathcal{D}(\mathbf{R}^2),\ i = 1, ..., n, \tag{7}$$

provided for every $C > 0$

$$\frac{|\ln \varepsilon|^C l_2(\varepsilon)}{c(1/\varepsilon)} \sup_{(x,t)\in K} |a_\varepsilon(\gamma_i(xl_2(\varepsilon), tl_2(\varepsilon), 0))| \to 0, \ \varepsilon \to 0, \ i = 1, ..., n,$$

for any compact set $K \subset\subset \mathbf{R}^2$ *and*

(8) $\quad F_i(x,t,0)$ *is bounded on* $\mathbf{R}^2$, $\ \nabla F_i(x,t,u)$ *is bounded on* $\mathbf{R}^{2+2n}$.

(b) *Assume that* F *is bounded on* $\mathbf{R}^{2+2n}$ *and* $\dfrac{l_2(\varepsilon)}{c(1/\varepsilon)} \to 0$ *as* $\varepsilon \to 0$. *If*

$$\lim_{\varepsilon\to 0} \frac{a_{i\varepsilon}(\gamma_i(xl_2(\varepsilon), tl_2(\varepsilon), 0)}{c(1/\varepsilon)}, \ i = 1, ..., n$$

exists in $\mathcal{D}'(\mathbf{R}^2)$ *then* $u(x,t) = [(u_{1\varepsilon}(x,t), ..., u_{n\varepsilon}(x,t)]$ *has the quasiasymptotics at infinity with respect to* $c(1/\varepsilon)$, *i.e.*

$$\lim_{\varepsilon\to 0} \left\langle \frac{u_{i\varepsilon}(xl_2(\varepsilon), tl_2(\varepsilon))}{c(1/\varepsilon)}, \psi(x,t) \right\rangle = C_{i,\psi} \in \mathbf{C}, \ \psi \in \mathcal{D}(\mathbf{R}^2), \ i = 1, ..., n.$$

Proof. We will prove only assertion (a), since (b) is proved in [8]. Consider

$$u_{i\varepsilon}(xl_2(\varepsilon), tl_2(\varepsilon)) = a_{i\varepsilon}(\gamma_i(xl_2(\varepsilon), tl_2(\varepsilon), 0))+$$

$$\text{(9)} \qquad \int_0^{tl_2(\varepsilon)} F_i(\gamma_i(xl_2(\varepsilon), tl_2(\varepsilon), \tau), \tau, u_\varepsilon(\gamma_i(xl_2(\varepsilon), tl_2(\varepsilon), \tau), \tau))d\tau,$$

where γ_i is the characteristic curve that passes through $(xl_2(\varepsilon), tl_2(\varepsilon))$, $(x,t) \in K_T$, and $i = 1, ..., n$. Fix $\varepsilon \in (0,1)$ and for given $\psi \in \mathcal{D}(\mathbf{R}^2)$, *supp* $\psi \subset \tilde{K} \subset\subset \mathbf{R}^2$ find K_0 and $T > 0$ such that $\tilde{K} \subset K_T$. First, we shall give the estimate for $u_{i\varepsilon}(xl_2(\varepsilon), tl_2(\varepsilon))$, then the estimate for the integral part in (9), and finally prove assertion (7).

Putting $F(x,t,u) = F(x,t,0) + \nabla_u F(x,t,\theta u)u$, with $\theta = (\theta_1, ..., \theta_n)$, $\theta_i \in [0,1]$ in (9) we obtain

$$u_{i\varepsilon}(xl_2(\varepsilon), tl_2(\varepsilon)) = a_{i\varepsilon}(\gamma_i(xl_2(\varepsilon), tl_2(\varepsilon), 0)) + \int_0^{tl_2(\varepsilon)} F_i(\gamma_i(xl_2(\varepsilon), tl_2(\varepsilon), \tau), \tau, 0)d\tau$$

$$+ \int_0^{tl_2(\varepsilon)} (u_{1\varepsilon}(\gamma_i(xl_2(\varepsilon), tl_2(\varepsilon), \tau), \tau), ..., u_{n\varepsilon}(\gamma_i(xl_2(\varepsilon), tl_2(\varepsilon), \tau), \tau))$$
$$\cdot \nabla_u F_i(\gamma_i(xl_2(\varepsilon), tl_2(\varepsilon), \tau), \tau, \theta(\tau) u_\varepsilon(\gamma_i(xl_2(\varepsilon), tl_2(\varepsilon), \tau), \tau))d\tau, \ 0 \le \theta(\tau) \le 1.$$

This implies

$$\sqrt{\sum_{i=1}^n |u_{i\varepsilon}(xl_2(\varepsilon), tl_2(\varepsilon))|^2} \le \sqrt{\sum_{i=1}^n |a_{i\varepsilon}(\gamma_i(xl_2(\varepsilon), tl_2(\varepsilon), 0))|^2}$$

$$+Tl_2(\varepsilon)\sqrt{\sum_{i=1}^{n}\sup_{(x,t)\in\mathbf{R}^2}|F_i(x,t,0)|^2}$$

$$+\sqrt{\sum_{i=1}^{n}\sup_{\substack{(x,t)\in\mathbf{R}^2\\ u\in\mathbf{C}^n}}|\nabla_u F_i(x,t,u)|^2}\int_0^{tl_2(\varepsilon)}\sqrt{\sum_{i=1}^{n}|u_i(\gamma_i(xl_2(\varepsilon),tl_2(\varepsilon),\tau),\tau)|^2}d\tau,$$

$(x,t)\in K_T$. The Gronwall inequality (cf. [4]) and assumptions (3) and (8) imply that there exist $C>0$ and $\varepsilon_0>0$ such that for

$$\sup_{(x,t)\in K_T}\sqrt{\sum_{i=1}^{n}|u_{i\varepsilon}(xl_2(\varepsilon),tl_2(\varepsilon))|^2}$$

$$\leq\left(\sup_{(x,t)\in\mathbf{R}^2}\sqrt{\sum_{i=1}^{n}|a_{i\varepsilon}(\gamma_i(xl_2(\varepsilon),tl_2(\varepsilon),0))|^2}+CTl_2(\varepsilon)\right)e^{CTl_2(\varepsilon)}$$

$$\leq C|\ln\varepsilon|^{CT}\left(\sup_{(x,t)\in\mathbf{R}^2}\sqrt{\sum_{i=1}^{n}|a_{i\varepsilon}(\gamma_i(xl_2(\varepsilon),tl_2(\varepsilon),0))|^2}+l_2(\varepsilon)\right).\tag{10}$$

Let us estimate the integral part to (9). We have

$$\int_0^{tl_2(\varepsilon)}|F_i(\gamma_i(xl_2(\varepsilon),tl_2(\varepsilon),\tau),\tau,u_\varepsilon(\gamma_i(xl_2(\varepsilon),tl_2(\varepsilon),\tau),\tau))|d\tau$$

$$\leq\int_0^{tl_2(\varepsilon)}|F_i(\gamma_i(xl_2(\varepsilon),tl_2(\varepsilon),\tau),\tau,0))|d\tau+$$

$$\int_0^{tl_2(\varepsilon)}|u_\varepsilon(\gamma_i(xl_2(\varepsilon),tl_2(\varepsilon),\tau),\tau)|d\tau\sup_{\substack{(x,t)\in\mathbf{R}^2\\ u\in\mathbf{C}^n}}|\nabla_u F_i(x,t,u)|\leq C|l_2(\varepsilon)|$$

$$\{\sup_{(x,t)\in\mathbf{R}^2}|F_i(\gamma_i(xl_2(\varepsilon),tl_2(\varepsilon),\tau),\tau,0)|+C\sup_{(x,t)\in K_T}|u_\varepsilon(\gamma_i(xl_2(\varepsilon),tl_2(\varepsilon),\tau),\tau))|\}.$$

Now, (10) implies that there exists $C>0$ such that

$$\int_0^{tl_2(\varepsilon)}|F_i(\gamma_i(xl_2(\varepsilon),tl_2(\varepsilon),\tau),\tau,u_\varepsilon(\gamma_i(xl_2(\varepsilon),tl_2(\varepsilon),\tau),\tau))|d\tau\tag{11}$$

$$\leq Cl_2(\varepsilon)\left(|\ln\varepsilon|^C\left(\sup_{(x,t)\in K_T}|a_\varepsilon(\gamma_i(xl_2(\varepsilon),tl_2(\varepsilon),0))|+l_2(\varepsilon)\right)+1\right).$$

We have

$$\left\langle\frac{u_{i\varepsilon}(xl_2(\varepsilon),tl_2(\varepsilon))}{c(1/\varepsilon)},\psi(x,t)\right\rangle=\left\langle\frac{a_{i\varepsilon}(\gamma_i(xl_2(\varepsilon),tl_2(\varepsilon),0))}{c(1/\varepsilon)},\psi(x,t)\right\rangle+\int\int\frac{1}{c(1/\varepsilon)}$$

$$\int_0^{tl_2(\varepsilon)} F_i(\gamma_i(xl_2(\varepsilon), tl_2(\varepsilon), \tau), \tau, u_\varepsilon(\gamma_i(xl_2(\varepsilon), tl_2(\varepsilon), \tau), \tau))d\tau)\psi(x,t)dxdt.$$

By (11) we have

$$\frac{1}{c(1/\varepsilon)} \int_0^{tl_2(\varepsilon)} |F_i(\gamma_i(xl_2(\varepsilon), tl_2(\varepsilon), \tau), \tau, u_\varepsilon(\gamma_i(xl_2(\varepsilon), tl_2(\varepsilon), \tau), \tau))| d\tau$$

$$\leq C \left(l_2(\varepsilon) \frac{\sup_{(x,t)\in K_T} |a_\varepsilon(\gamma_i(xl_2(\varepsilon), tl_2(\varepsilon), 0))| |\ln \varepsilon|^C}{c(1/\varepsilon)} + \frac{|\ln\varepsilon|^C l_2^2(\varepsilon)}{c(1/\varepsilon)} + \frac{l_2(\varepsilon)}{c(1/\varepsilon)} \right)$$

$\to 0$,

as $\varepsilon \to 0$. Thus,

$$\lim_{\varepsilon\to 0} \left\langle \frac{u_{i\varepsilon}(xl_2(\varepsilon), tl_2(\varepsilon))}{c(1/\varepsilon)}, \psi(x,t) \right\rangle = \lim_{\varepsilon\to 0} \left\langle \frac{a_{i\varepsilon}(\gamma_i(xl_2(\varepsilon), tl_2(\varepsilon), 0))}{c(1/\varepsilon)}, \psi(x,t) \right\rangle = 0,$$

since

$$\frac{a_{i\varepsilon}(\gamma_i(xl_2(\varepsilon), tl_2(\varepsilon), 0))}{c(1/\varepsilon)} \leq \frac{|\ln\varepsilon|^C l_2(\varepsilon)}{c(1/\varepsilon)} \sup_{(x,t)\in K} |a_\varepsilon(\gamma_i(xl_2(\varepsilon), tl_2(\varepsilon), 0))| \to 0$$

as $\varepsilon \to 0$. □

Consider a Cauchy problem

$$u'(t) = F(t,u),\ u(0) = a = [a_\varepsilon] \in \bar{\mathbf{C}}, \tag{12}$$

where $(t,u) \mapsto F(t,u)$ is a smooth function on $\mathbf{R}^2$ such that (2) and (3) hold for F. It is uniquely solvable in $\mathcal{G}(\mathbf{R})$ (cf. [5]).

For the behaviour of the solution to (12) we use a stronger concept of asymptotic behaviour at infinity since the initial data does not depend on x.

Definition 3. *Let $F \in \mathcal{G}(\mathbf{R}^n)$. It is said that F has the strong $\mathcal{G}$–quasiasymptotics at infinity with the limit $g \in C^\infty(\mathbf{R}^n)$ with respect to $c(1/\varepsilon) \in \mathcal{K}$ if there exists F_ε, a representative of F, such that for every $K \subset\subset \mathbf{R}^n$,*

$$\lim_{\varepsilon\to 0} \frac{F_\varepsilon(x/\varepsilon)}{c(1/\varepsilon)} = g(x) \ \text{ uniformly for } x \in K.$$

Proposition 2. *(a) Let $c(1/\varepsilon) \in \mathcal{K}$, and $a = [a_\varepsilon] \in \bar{\mathbf{C}}$ such that for every $C > 0$,*

$$\lim_{\varepsilon\to 0} \frac{|\ln\varepsilon|^C l_2^2(\varepsilon)}{c(1/\varepsilon)} = 0.$$

Then the solution $u(t) \in \mathcal{G}(\mathbf{R})$ to the Cauchy problem (12) satisfies

$$\lim_{\varepsilon\to 0} \frac{u_\varepsilon(tl_2(\varepsilon))}{c(1/\varepsilon)} = 0,$$

uniformly in $t \in K$, where K is an arbitrary compact set of $\mathbf{R}$, *provided for every $C > 0$*

$$\frac{|\ln \varepsilon|^C l_2(\varepsilon)}{c(1/\varepsilon)} |a_\varepsilon| \to 0 \text{ as } \varepsilon \to 0,$$

and

$$F(x,0) \text{ is bounded in } \mathbf{R}^2, \quad \nabla F(t,u) \text{ is bounded on } \mathbf{R}^{1+2n}. \tag{13}$$

(*b*) *Let F be bounded on $\mathbf{R}^{1+2n}$. Assume that*

$$\lim_{\varepsilon\to 0} \frac{l_2(\varepsilon)}{c(1/\varepsilon)} \to 0 \quad \text{and} \quad \lim_{\varepsilon\to 0} \frac{a_\varepsilon}{c(1/\varepsilon)} = 1.$$

Then, $\lim_{\varepsilon\to 0} \frac{u_\varepsilon(tl_2(\varepsilon))}{c(1/\varepsilon)} = 1.$

Proof. We will prove (*a*). Refer to [8] for (*b*). We have

$$u_\varepsilon(tl_2(\varepsilon)) = a_\varepsilon + \int_0^{tl_2(\varepsilon)} F(\tau, u_\varepsilon(\tau))d\tau, \ t \in \mathbf{R}. \tag{14}$$

Condition (13) and Gronwall inequality imply that there exist $C > 0$ such that

$$\sup_{t\in K} |u_\varepsilon(tl_2(\varepsilon))| \le C(|a_\varepsilon| + l_2(\varepsilon)), \ \varepsilon \in (0, \varepsilon_0).$$

This implies that there exists $C > 0$ such that

$$\left| \int_0^{tl_2(\varepsilon)} F(\tau, u_\varepsilon(\tau))d\tau \right| \le CTl_2(\varepsilon)\left(1 + |\ln \varepsilon|^{CT}(|a_\varepsilon| + l_2(\varepsilon))\right), \ t \in (-T, T).$$

The last summand in (14) is then for $t \in (-T, T)$

$$\frac{1}{c(1/\varepsilon)} \left| \int_0^{tl_2(\varepsilon)} F(\tau, u_\varepsilon(\tau))d\tau \right| \le C\left(l_2(\varepsilon)\frac{|a_\varepsilon||\ln \varepsilon|^C}{c(1/\varepsilon)} + \frac{|\ln \varepsilon|^C l_2^2(\varepsilon)}{c(1/\varepsilon)} + \frac{l_2(\varepsilon)}{c(1/\varepsilon)} \right).$$

Because $\frac{|\ln \varepsilon|^C l_2^2(\varepsilon)}{c(1/\varepsilon)}|a_\varepsilon| \to 0$ and $\frac{l_2(\varepsilon)}{c(1/\varepsilon)} \to 0$, as $\varepsilon \to 0$, the second summand in (14) tends to zero and

$$\lim_{\varepsilon\to 0} \frac{u_\varepsilon(tl_2(\varepsilon))}{c(1/\varepsilon)} = \lim_{\varepsilon\to 0} \frac{a_\varepsilon}{c(1/\varepsilon)} = 0.$$

□

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