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# MATRIX METHOD APPROACH TO CAFIERO THEOREM

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**Abstract.** We give a simple proof of the Cafiero theorem based on a matrix method approach in the form of Lemma 2.4 in the $\sigma$-additive context. Based on a version of Drewnowski lemma for an SCP-ring we obtain an extension of Cafiero theorem for exhaustive finitely additive set functions defined on an SCP-ring. As consequences, the well-known Nikodým and Brooks-Jewett convergence theorems are obtained.

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## 1. Introduction

In a paper of 1952 F. Cafiero [3] has characterized $\sigma$-additive set functions defined on a $\sigma$-ring that are uniformly additive, see Cafiero [3, 4]. The notion of the uniform additiveness was introduced by R. Caccioppoli [2] and independently by Dubrovskii [15], and it was utilized by many authors to obtain a Lebesgue type theorem for the convergence of integrals. Since the time when R. Rickart [22] introduced the notion of exhaustivity or s-boundedness for additive set functions defined on a ring it has been clear that for $\sigma$-additive set functions the uniform additivity of Caccioppoli and Dubrovskii is equivalent to the uniform exhaustivity. Therefore the Cafiero lemma was reformulated for the case of the uniform exhaustivity (see d'Andrea, de Lucia [7], H. Weber [25]).

In this paper we give a simple proof of the Cafiero theorem based on a matrix method approach (diagonal theorems), as a further extension of sliding hump method, and which was initiated by Mikusiński [18]. Many results in measure theory and functional analysis can be found in Antosik and Swartz [1], Pap [19, 20, 21], Swartz [23] (see also Diestel and Uhl Jr [13], Weber [26]), where the matrix method is used instead of the Baire Category Theorem, which is

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unsuitable for obtaining more general results. We start with a matrix method lemma, Lemma 2.4, which we use in the proof of Cafiero theorem 2.5 in the $\sigma$-additive context. After that, by a version of Drewnowski lemma 3.3 for an SCP-ring, we arrive at an extension of the Cafiero theorem 3.4 for exhaustive finitely additive set functions defined on a SCP-ring. Finally, in section 4 we obtain immediately, by the Cafiero theorem 3.4, the well known Nikodým and Brooks-Jewett convergence theorems.

For more information on Cafiero theorem see d'Andrea, de Lucia [8], de Lucia [9], de Lucia, Salvati [11], de Lucia, Traynor [12], Traynor [24], Weber [25].

## 2. Cafiero uniform exhaustivity theorem

Let $\mathcal{R}$ be a ring and $\mathcal{M}$ a family of finitely additive set functions defined on $\mathcal{R}$. A finitely additive set function $m : \mathcal{R} \to \mathbb{R}$ is *exhaustive* (strongly additive or strongly bounded) if $\lim_{j\to\infty} m(E_j) = 0$ for every sequence $\{E_j\}_{j\in\mathbb{N}}$ of pairwise disjoint elements from $\mathcal{R}$. A family $\mathcal{M}$ of additive set functions is *uniformly exhaustive* if $\lim_{j\to\infty} m(E_j) = 0$ uniformly in $m \in \mathcal{M}$ for every sequence $\{E_j\}_{j\in\mathbb{N}}$ of pairwise disjoint elements from $\mathcal{R}$. It is obvious that a finite measure on a $\sigma$-ring is exhaustive.

The following propositions are well known (see de Lucia, Pap [10]).

**Proposition 2.1.** *Let $\mathcal{M}$ be a family of finite additive exhaustive set functions on $\mathcal{R}$. The following statements are equivalent*

*(i) $\mathcal{M}$ is uniformly exhaustive,*

*(ii) for every increasing sequence $\{E_i\}_{i\in\mathbb{N}}$ the sequence $\{m(E_i)\}_{i\in\mathbb{N}}$ is a convergent sequence, uniformly for $m \in \mathcal{M}$*

*(iii) for every disjoint sequence $\{E_i\}_{i\in\mathbb{N}}$ of $\mathcal{R}$ the series $\sum_i m(E_i)$ is convergent uniformly for $m \in \mathcal{M}$.*

**Proposition 2.2.** *Let $\mathcal{M}$ be a family of finite additive exhaustive set functions on $\mathcal{R}$. Then*

*(i) $\mathcal{M}$ is uniformly exhaustive,*

*implies*

*(ii) for every decreasing sequence $\{E_i\}_{i\in\mathbb{N}}$ the sequence $\{m(E_i)\}_{i\in\mathbb{N}}$ is a convergent sequence uniformly for $m \in \mathcal{M}$.*

*If $\mathcal{R}$ is an algebra then (i) and (ii) are equivalent.*

Let $\Sigma$ be a $\sigma$-algebra and $\mathcal{M}$ family of $\sigma$-additive set functions defined on $\Sigma$. A family $\mathcal{M}$ of countable additive measures $\mu : \Sigma \to \mathbb{R}$ is *uniformly countable additive* if

$$\lim_{n\to\infty} \sum_{j=n}^{\infty} \mu(E_j) = 0$$

uniformly in $\mu \in \mathcal{M}$ for every sequence $\{E_j\}_{j\in\mathbb{N}}$ of pairwise disjoint elements from $\Sigma$. We have by Propositions 2.1 and 2.2

**Proposition 2.3.** *Let $\Sigma$ be a $\sigma$-algebra and $\mathcal{M}$ family of $\sigma$-additive set functions defined on $\Sigma$. The following statements are equivalent*

*(i) $\mathcal{M}$ is uniformly exhaustive,*

*(ii) $\mathcal{M}$ is uniformly countable additive,*

*(iii) for every decreasing sequence $\{E_i\}_{i\in\mathbb{N}}$ of $\Sigma$ the sequence $\{\mu(E_i)\}_{i\in\mathbb{N}}$ is a convergent sequence uniformly for $\mu \in \mathcal{M}$.*

First we shall give an elementary matrix type lemma.

**Lemma 2.4.** *Let $[x_{ni}]_{n,i\in\mathbb{N}}$ be an infinite matrix of real numbers such that*

*1) for every $n \in \mathbb{N}$ and every subset $I$ of $\mathbb{N}$ there exists $\sum_{i\in I} x_{ni}$;*

*2) for every sequence $\{I_k\}_{k\in\mathbb{N}}$ of pairwise disjoint subsets of $\mathbb{N}$ and for every $\varepsilon > 0$ there exists $\bar{k} \in \mathbb{N}$ and $n_0 \in \mathbb{N}$ such that*

$$\left| \sum_{i\in I_{\bar{k}}} x_{ni} \right| < \varepsilon \text{ for every } n > n_0.$$

*Then*

$$\lim_{i\to\infty} x_{ni} = 0 \text{ uniformly in } n \in \mathbb{N}.$$

*Proof.* We note that if the matrix $[x_{ni}]_{n,i\in\mathbb{N}}$ has the properties 1) and 2) then also every its submatrix has the same properties. By 1) we have

$$\lim_{i\to\infty} x_{ni} = 0 \quad (n \in \mathbb{N}). \tag{1}$$

Then we need to prove that: for every $\varepsilon > 0$ there exist $k, m \in \mathbb{N}$ such that $| x_{ni} | < \varepsilon$ for every $i > k$ and every $n > m$. Suppose that this is not true. Then there exists $\sigma > 0$ such that for every $k, m \in \mathbb{N}$ there exist $i > k$ and $n > m$ such that $| x_{ni} | \geq \sigma$. By induction we can then construct two strictly increasing sequences $\{i_k\}_{k\in\mathbb{N}}$ and $\{n_k\}_{k\in\mathbb{N}}$ of natural numbers such that $| x_{n_k i_k} | \geq \sigma$

for every $k \in \mathbb{N}$. Therefore there exists a submatrix of the starting matrix, which we denote by the same symbol $[x_{ni}]_{n,i\in\mathbb{N}}$, such that it satisfies 1), 2) and for some $\sigma > 0$

3) $| x_{kk} | \geq \sigma$ for every $k \in \mathbb{N}$.

Let $\{\sigma_r\}_{r\in\mathbb{N}}$ be a decreasing sequence of real numbers such that

$$\sum_{r\in\mathbb{N}} \sigma_r < \frac{\sigma}{2},$$

from 2) we have that for every $r \in \mathbb{N}$ there exist $i_r$ and $m_r$ such that

$$| x_{ni_r} | < \sigma_r$$

for every $n \geq m_r$. We can suppose that the sequences $\{i_r\}$ and $\{m_r\}$ are strictly increasing and so to construct a new submatrix of $[x_{ni}]$, which we denote by the same symbol $[x_{ni}]_{n,i\in\mathbb{N}}$, such that it satisfies 1), 2), 3) and

$$\text{for every } i \in \mathbb{N},\ | x_{ni} | < \sigma_i \text{ for all } n \geq m_i. \tag{2}$$

Now we construct by induction two strictly increasing sequences $\{\rho_n\}_{n\in\mathbb{N}}$ and $\{r_n\}_{n\in\mathbb{N}}$ of natural numbers with the property

$$\rho_n > r_n > \max\{\rho_{n-1}, m_{r_{n-1}}\} \text{ for every } n \in \mathbb{N},$$

where $\rho_0 = m_0 = 0$, such that we have for every $n \in \mathbb{N}$,

$$| x_{r_k i} | < \sigma_{r_n} \text{ for every } i > \rho_n \text{ and } k = 1, \ldots, n. \tag{3}$$

Suppose that $r_1, \ldots, r_{n-1}, ; \rho_1, \ldots, \rho_{n-1}$ are determined and let be $r_n$ an element of $\mathbb{N}$ such that

$$r_n > \max\{\rho_{n-1}, m_{r_{n-1}}\},$$

by (1) we can find $\rho_n$ such that $\rho_n > r_n$ and (3) is true.

Consider now the submatrix $[x_{r_n r_i}]_{n,i\in\mathbb{N}}$, for $n \in \mathbb{N}$ and $i = 1, \ldots, n-1$, we have $r_n > m_{r_i}$ and then, by (2)

$$| x_{r_n r_i} | < \sigma_{r_i},$$

and for $i \geq n+1$ it results

$$r_i > \rho_{i-1},\ n \leq i-1$$

and then, by (3)

$$| x_{r_n r_i} | < \sigma_{r_{i-1}}.$$

Hence for an infinite subset $I$ of $\mathbb{N}$ and $n \in I$ we have by 3)

$$\begin{aligned}\left|\sum_{i\in I} x_{r_n r_i}\right| &\geq \mid x_{r_n r_n} \mid - \sum_{i\in I, i\neq n} \mid x_{r_n r_i} \mid \\ &\geq \sigma - \sum_{i\in\mathbb{N}} \sigma_i \\ &= \frac{\sigma}{2}.\end{aligned}$$

If $\{I_k\}_{k\in\mathbb{N}}$ is a disjoint sequence of infinite subsets of $\mathbb{N}$ we have for every $k \in \mathbb{N}$

$$\left|\sum_{i\in I_k} x_{r_n r_i}\right| > \frac{\sigma}{2} \text{ for all } n \in I_k$$

and this contradicts 2). □

**Theorem 2.5.** [Cafiero] *Let $\Sigma$ be a $\sigma$-algebra. A sequence $\{\mu_n\}_{n\in\mathbb{N}}$ of countable additive real measures defined on $\Sigma$ is uniformly exhaustive if and only if the following condition holds*
*$\alpha$) for every sequence $\{E_n\}_{n\in\mathbb{N}}$ of pairwise disjoint elements of $\Sigma$ and every $\varepsilon > 0$ there exist $\bar{k}, n_0 \in \mathbb{N}$ such that*

$$|\mu_n(E_{\bar{k}})| < \varepsilon \textit{ for every } n \geq n_0.$$

*Proof.* Let $\{E_n\}_{n\in\mathbb{N}}$ be a sequence of pairwise disjoint elements of $\Sigma$. Then $(\mu_n(E_i))_{n,i\in\mathbb{N}}$ is an infinite matrix of real numbers. By countable additivity of every $\mu_n$ we have that for every $n \in \mathbb{N}$ and $I \subset \mathbb{N}$ there exists $\sum_{i\in I} \mu_n(E_i)$. If $\{I_k\}_{k\in\mathbb{N}}$ is a sequence of pairwise disjoint subsets of $\mathbb{N}$, then $\{\cup_{i\in I_k} E_i\}_{k\in\mathbb{N}}$ is a sequence of pairwise disjoint elements of $\Sigma$ and then by $\alpha$) for every $\varepsilon > 0$ there exist $\bar{k}, n_0 \in \mathbb{N}$ such that

$$\left|\mu_n\left(\bigcup_{i\in I_{\bar{k}}} E_i\right)\right| = \left|\sum_{i\in I_{\bar{k}}} \mu_n(E_i)\right| < \varepsilon \text{ for every } n \geq n_0.$$

Therefore the conditions 1) and 2) of Lemma 2.4 are satisfied and we have that

$$\lim_{i\to\infty} \mu_n(E_i) = 0 \text{ uniformly in } n \in \mathbb{N}.$$

□

## 3. Cafiero theorem for additive set functions and SCP

To extend the previous results to the finitely additive case we will prove a generalization of a very useful lemma obtained by Drewnowski [14] for a $\sigma$-ring, now for the case of an SCP-ring. We have the following definition by Constantinescu [5, 6] and Haydon [17], see also Freniche [16] and H. Weber [25].

**Definition 3.1.** *A ring $\mathcal{R}$ has the Sequential Completeness Property, and will be called SCP-ring, if each disjoint sequence $\{E_n\}_{n\in\mathbb{N}}$ from $\mathcal{R}$ has a subsequence $\{E_{n_j}\}_{j\in\mathbb{N}}$, whose union is in $\mathcal{R}$.*

**Lemma 3.2.** [Drewnowski lemma with SCP] *Let $\mathcal{R}$ be an SCP-ring. If $m : \mathcal{R} \to \mathbb{R}$ is an exhaustive monotone set function with $m(\emptyset) = 0$ and $\{E_n\}_{n\in\mathbb{N}}$ is a sequence of pairwise disjoint sets from $\mathcal{R}$, then there exist a subsequence $\{E_{k_n}\}_{n\in\mathbb{N}}$ of $\{E_n\}_{n\in\mathbb{N}}$ and a SCP-ring $\mathcal{R}'$ with SCP such that $\mathcal{R}' \subseteq \mathcal{R}$, $E_{k_n} \in \mathcal{R}'$ for every $n \in \mathbb{N}$, and $m$ is order continuous on the ring $\mathcal{R}'$.*

*Proof.* Let $\{J_i\}_{i\in\mathbb{N}}$ be a sequence of pairwise disjoint infinite subsets of $\mathbb{N}$. Then for every $i \in \mathbb{N}$ there exists an infinite subset $J_i'$ of $J_i$ such that $\{\cup_{k\in J_i'} E_k\}_{i\in\mathbb{N}}$ is a sequence of pairwise disjoint subsets of $\mathcal{R}$. By the exhaustivity of $m$ it is possible to find an infinite subset $N_1$ of $\mathbb{N}$ such that

$$m\left(\bigcup_{k\in N_1} E_k\right) < \frac{1}{2}.$$

In the same way, if $\{J_i\}_{i\in\mathbb{N}}$ is a sequence of pairwise disjoint infinite subsets of $N_1 \setminus \{\min N_1\}$, it is possible to find an infinite subset $N_2$ of $N_1 \setminus \{\min N_1\}$ such that

$$m\left(\bigcup_{k\in N_2} E_k\right) < \frac{1}{2^2}.$$

In this way, we construct, by induction, a decreasing sequence $\{N_i\}_{i\in\mathbb{N}}$ of infinite subsets of $\mathbb{N}$ such that

$$N_{i+1} \subseteq N_i \setminus \{\min N_i\}, \quad m\left(\bigcup_{k\in N_i} E_k\right) < \frac{1}{2^i} \text{ for every } i \in \mathbb{N}.$$

Let $k_i = \min N_i$ and let $\mathcal{R}'$ be the set

$$\left\{X \in \mathcal{R} : \text{ there exists } I \subseteq \mathbb{N} \text{ such that } X = \bigcup_{i\in I} E_{k_i}\right\}. \tag{4}$$

We claim that $\mathcal{R}'$ is a ring with the SCP. It is clear that the supremum of two elements of $\mathcal{R}'$ belongs to $\mathcal{R}'$. Let $X_1, X_2$ be two elements of $\mathcal{R}'$ such that $X_2 \subseteq X_2$. Then there exist $I_1$ and $I_2$ subsets of $\mathcal{N}$ such that

$$X_p = \bigcup_{j\in I_p} E_{k_j} \text{ for every } p \in \mathbb{N}.$$

Then we have

$$X_1 \setminus X_2 = \bigcup_{j\in I_1\setminus I_2} E_{k_j} \in \mathcal{R},$$

i.e., $X_1 \setminus X_2 \in \mathcal{R}'$. Therefore $\mathcal{R}'$ is a ring. Let $\{X_p\}_{p\in\mathbb{N}}$ be a disjoint sequence of elements of $\mathcal{R}'$, then there exists a disjoint sequence $\{I_p\}_{p\in\mathbb{N}}$ of subsets of $\mathbb{N}$ such that

$$X_p = \bigcup_{j\in I_p} E_{k_j}.$$

By the property of SCP of $\mathcal{R}$ there exists a subsequence $\{X_{p_r}\}_{r\in\mathbb{N}}$ of $\{X_p\}_{p\in\mathbb{N}}$ such that

$$\bigcup_{r\in\mathbb{N}} X_{p_r} = \bigcup_{j\in\cup_{r\in\mathbb{N}} I_{p_r}} E_{k_j}$$

belongs to $\mathcal{R}$ and then also to $\mathcal{R}'$ that the restriction of $m$ to $\mathcal{R}'$ is order continuous is proved in the same way as it was proved in the case of the $\sigma$-ring, see de Lucia, Pap (2002). □

We shall call the ring $\mathcal{R}'$ defined by (4) the SCP-ring generated by $\{E_{k_n}\}$.

**Lemma 3.3.** [Drewnowski lemma for additive set functions] *Let $\mathcal{R}$ be a SCP-ring. If $\mu_i : \mathcal{R} \to \mathbb{R}$ is a sequence of exhaustive additive set functions and $\{E_n\}$ is a sequence of pairwise disjoint sets from $\mathcal{R}$, then there exists a subsequence $\{E_{k_n}\}_{n\in\mathbb{N}}$ of $\{E_n\}_{n\in\mathbb{N}}$ such that $\mu_i$ is countable additive on the SCP-ring $\mathcal{R}'$ generated by $\{E_{k_n}\}_{n\in\mathbb{N}}$.*

*Proof.* Let for every $n \in \mathbb{N}$ denote by $|\mu_n|$ the total variation of $\mu_n$. We introduce the set function $m : \mathcal{R} \to \mathbb{R}$ by

$$m = \sum_{n\in\mathbb{N}} \frac{1}{2^n} \frac{|\mu_n|}{1 + M_n},$$

where $M_n = \sup\{|\mu_n|(A) : A \in \mathcal{R}\}$. Since the function $m$ satisfies the conditions of Drewnowski lemma 3.2 there exists a SCP-ring $\mathcal{R}'$ generated by a subsequence $\{E_{k_n}\}_{n\in\mathbb{N}}$ of $\{E_n\}_{n\in\mathbb{N}}$ such that the restriction of $m$ to $\mathcal{R}'$ is order continuous. Then it easily follows that the restrictions of $|\mu_n|$ to $\mathcal{R}'$ are order continuous and therefore every $\mu_n$ is countable additive on $\mathcal{R}'$. □

We will generalize Theorem 2.5 to the additive case.

**Theorem 3.4.** [Cafiero theorem for additive set functions] *Let $\mathcal{R}$ be an SCP-ring. A sequence $\{\mu_n\}_{n\in\mathbb{N}}$ of finite additive bounded set functions defined on $\mathcal{R}$ is uniformly exhaustive if and only if the following condition holds*
*$\alpha$) for every sequence $\{E_n\}_{n\in\mathbb{N}}$ of pairwise disjoint elements of $\mathcal{R}$ and every $\varepsilon > 0$ there exist $\bar{k}, n_0 \in \mathbb{N}$ such that*

$$|\mu_n(E_{\bar{k}})| < \varepsilon \textit{ for every } n \geq n_0.$$

*Proof.* Suppose that the $\{\mu_n\}_{n\in\mathbb{N}}$ are not uniformly exhaustive. Then there exists a disjoint sequence $\{E_i\}_{i\in\mathbb{N}}$ of $\mathcal{R}$ such that $\lim_{n\to\infty} \mu_n(E_i) = 0$ but not uniformly in $n$. Then there exists $\varepsilon > 0$ such that we can construct a subsequence of $\{E_i\}_{i\in\mathbb{N}}$ and a subsequence of $\{\mu_n\}_{n\in\mathbb{N}}$, that for simplicity, we will denote yet by the same symbols so that

$$|\mu_n(E_n)| > \varepsilon \text{ for every } n \in \mathbb{N}. \tag{5}$$

By Drewnowski lemma 3.3 there exists a subsequence $\{E_{i_k}\}_{k\in\mathbb{N}}$ of $\{E_i\}_{i\in\mathbb{N}}$ such that if $\mathcal{R}'$ is a SCP-ring generated by $\{E_{i_k}\}_{k\in\mathbb{N}}$ the restriction of $\mu_n$ to $\mathcal{R}'$ are $\sigma$-additive. From Theorem 3.4 it follows that these restrictions are uniformly exhaustive but by (5) we have

$$|\mu_{i_k}(E_{i_k})| > \varepsilon \text{ for every } k \in \mathbb{N}.$$

A contradiction. □

## 4. Applications

### 4.1. Nikodým convergence theorem

**Theorem 4.1.** [Nikodým convergence theorem] *Let $\mathcal{R}$ be an SCP-ring. Let $\{\mu_n\}_{n\in\mathbb{N}}$ be a pointwise convergent sequence of countable additive measures defined on $\mathcal{R}$, i.e.,*

$$\lim_{n\to\infty} \mu_n(E) = \mu(E),\ E \in \Sigma, \tag{6}$$

*then*

*(i) $\{\mu_n\}_{n\in\mathbb{N}}$ converges to a countable additive measure $\mu$,*

*(ii) $\{\mu_n\}_{n\in\mathbb{N}}$ is uniformly $\sigma$-additive.*

*Proof.* We consider first a special case. If $\{\mu_n\}_{n\in\mathbb{N}}$ is pointwise convergent to zero then condition $\alpha$) in Theorem 2.5 is satisfied. Then $\{\mu_n\}_{n\in\mathbb{N}}$ is uniformly exhaustive and, by Proposition 2.3, it is uniformly $\sigma$-additive.

The general case for (ii), i.e., under the condition (6), it easily follows by the fact that by (6) $\{\mu_n(E)\}_{n\in\mathbb{N}}$ is a Cauchy sequence. Namely, suppose that (ii) does not hold for $\{\mu_n\}_{n\in\mathbb{N}}$, i.e., that there is a sequence of pairwise disjoint sets $\{E_n\}_{n\in\mathbb{N}}$ from $\Sigma$, a subsequence $\{\mu_{k_n}\}_{n\in\mathbb{N}}$ of $\{\mu_n\}_{n\in\mathbb{N}}$ and $\varepsilon > 0$ such that $|\mu_{k_n}(E_{k_n})| \geq 2\varepsilon$. By exhaustivity of $\mu_{k_n}$ there exists a subsequence $\{p_n\}_{n\in\mathbb{N}}$ of $\{k_n\}_{n\in\mathbb{N}}$ such that $|\mu_{p_n}(E_{p_{n+1}})| \leq \varepsilon$. Taking $m_n = \mu_{p_{n+1}} - \mu_{p_n}$ we obtain a sequence $\{m_n\}_{n\in\mathbb{N}}$ of countable additive set functions which is pointwise convergent to zero and therefore by the previously proved part it is uniformly countable additive, but this is in contradiction with

$$|m_n(E_{p_{n+1}})| \geq |\mu_{p_{n+1}}(E_{p_{n+1}})| - |\mu_{p_n}(E_{p_{n+1}})| \geq \varepsilon$$

for all $n \in \mathbb{N}$.

To prove (i), we have to use (ii). Namely, by (ii) we have

$$\begin{aligned} \mu\left(\bigcup_{j=1}^{\infty} E_j\right) &= \lim_{i\to\infty} \mu_i \left(\bigcup_{j=1}^{\infty} E_j\right) \\ &= \lim_{i\to\infty}\lim_{n\to\infty}\sum_{j=1}^{n}\mu_i(E_j) \\ &= \lim_{n\to\infty}\mu\left(\bigcup_{j=1}^{n} E_j\right) \\ &= \sum_{j=1}^{\infty}\mu(E_j). \end{aligned}$$

□

### 4.2. Brooks-Jewett theorem and related results

A generalization of the Nikodým convergence theorem was obtained by Brooks and Jewett.

**Theorem 4.2.** [Brooks-Jewett] *Let $\mathcal{R}$ be an SCP-ring. A pointwise convergent sequence $\{m_n\}_{n\in\mathbb{N}}$ of finitely additive scalar and exhaustive set functions (strongly additive) defined on an $\mathcal{R}$, i.e., $\lim_{n\to\infty} m_n(E) = m(E),\ E \in \mathcal{R}$,*

*(i) converges to an additive and exhaustive set function $m$,*

*(ii) $\{m_n\}_{n\in\mathbb{N}}$ is uniformly exhaustive.*

*Proof.* If $\{m_n\}_{n\in\mathbb{N}}$ is pointwise convergent to 0 then the condition $\alpha$) of Theorem 3.4 is verified and so in this special case (ii) is true. The general case for (ii) follows in the same manner as in the proof of Nikodým theorem.

Then (i) follows by (ii)

$$\begin{aligned} \lim_{j\to\infty} m(E_j) &= \lim_{j\to\infty}\lim_{i\to\infty} m_i(E_j) \\ &= \lim_{i\to\infty}\lim_{j\to\infty} m_i(E_j) \\ &= 0. \end{aligned}$$

□

**Theorem 4.3.** [Nikodým boundedness theorem for additive case] *Let $\mathcal{R}$ be an SCP-ring. A family $\mathcal{M}$ of finitely additive bounded set functions $m$, defined on* *an $\mathcal{R}$, which is pointwise bounded, i.e., for each $E \in \mathcal{R}$ there exists $M_E > 0$ such that*

$$|m(E)| < M_E \quad (m \in \mathcal{M}),$$

*is uniformly bounded, i.e., there exist $M > 0$ such that*

$$|m(E)| < M \quad (m \in \mathcal{M}, E \in \mathcal{R})$$

The proof is the same as the proof of Nikodým boundedness theorem, only using Brooks-Jewett theorem 4.2 instead of Nikodým convergence theorem (see de Lucia, Pap [10]). □

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