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VOLTERRA INTEGRAL EQUATIONS WITH ITERATIONS OF LINEAR MODIFICATION OF THE ARGUMENT

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Abstract. We consider some integral equations with iterations of linear modification of the argument which provide us Picard operators and weakly Picard operators. We study the existence, existence and uniqueness, and data dependence for the solutions of these equations.

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1. Introduction

In the last thirty years there has been a great deal of work in the field of differential equations with modified argument. These equations arise in a wide variety of scientific and technical applications, including the modelling of problems from the natural and social sciences such as physics, biological sciences and economics.

A special class is represented by the differential equations with affine modification of the argument which can be delay differential equations or differential equations with linear modification of the argument. Many results concerning these equations are given in the papers [2]–[7], [18].

Another class of differential equations with modified arguments are the differential equations with iteration such as equation $x'(t) = x(x(t))$, considered by Petukhov in [8]. In [1], results concerning the existence, uniqueness and data dependence for the solutions of some Cauchy problems for nonlinear equation with iteration $x'(t) = f(t, x(x(t)))$, are given.

The Cauchy problem for an equation with iterations of linear modification of the argument:

$$x'(t) = f(t, x(t), x(\lambda t), x(\lambda x(\lambda t))), \quad t \in [0, b],\ b > 0,\ 0 < \lambda < 1$$
$$x(0) = u_0,$$

where $f \in C([0, b]^4)$, $u_0 \in \mathbb{R}$, is equivalent to the following integral equation

$$(1) \quad x(t) = u_0 + \int_0^t f(s, x(s), x(\lambda s), x(\lambda x(\lambda s)))ds,\ t \in [0, b],\ b > 0,\ 0 < \lambda < 1.$$

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In this paper, by using the Picard and weakly Picard operators' technique, due to I. A. Rus (see [11]–[15]), we obtain the existence, existence and uniqueness, and data dependence results for the solution of equation (1). Our results generalize those obtained in [1] and [7].

2. Preliminaries

Let (X,d) be a metric space and $A : X \to X$ an operator. We shall use the following notations:

$P(X) := \{Y \subseteq X |\ Y \neq \emptyset\}$;

$P_{b,cl}(X) := \{Y \in P(X)|\ Y$ is bounded and closed$\}$;

$F_A := \{x \in X|\ A(x) = x\}$ – the fixed point set of A;

$I(A) := \{Y \in P(X)|\ A(Y) \subseteq Y\}$;

$O_A(x) := \{x, A(x), A^2(x), \dots, A^n(x), \dots\}$ – the A-orbit of $x \in X$;

$\delta(Y) := \sup\{d(a,b)|\ a,b \in Y\}$ – the diameter of $Y \in P(X)$;

$H : P(X) \times P(X) \to \mathbb{R}_+ \cup \{\infty\}$;

$H(Y,Z) = \max\left(\sup_{a\in Y}\inf_{b\in Z} d(a,b), \sup_{b\in Z}\inf_{a\in Y} d(a,b)\right)$ – the Pompeiu–Hausdorff functional on $P(X)$.

Definition 2.1. [11] *Let (X,d) be a metric space. An operator $A : X \to X$ is a Picard operator if there exists $x^* \in X$ such that:*

(i) $F_A = \{x^*\}$;

(ii) the sequence $(A^n(x_0))_{n\in\mathbb{N}}$ converges to x^* for all $x_0 \in X$.

Definition 2.2. [12] *Let (X,d) be a metric space. An operator $A : X \to X$ is a weakly Picard operator if the sequence $(A^n(x_0))_{n\in\mathbb{N}}$ converges for all $x_0 \in X$ and its limit (which may depend on x_0) is a fixed point of A.*

If A is a weakly Picard operator then we consider the following operator

$$A^\infty : X \to X, \quad A^\infty(x) := \lim_{n\to\infty} A^n(x).$$

The following results will be useful in what follows:

Theorem 2.1. [10] *Let (X,d) be a complete metric space and $A, B : X \to X$ two operators. We suppose that:*

(i) A is a contraction with the constant α and $F_A = \{x_A^*\}$;

(ii) B has fixed points and $x_B^* \in F_B$;

(iii) there exists $\eta > 0$ such that $d(A(x), B(x)) \leq \eta$, for all $x \in X$.

Then

$$d(x_A^*, x_B^*) \leq \frac{\eta}{1-\alpha}.$$

Theorem 2.2. [14] *Let (X, d) be a metric space and $A : X \to X$ an operator. The operator A is a weakly Picard operator if and only if there exists a partition of X,*

$$X = \bigcup_{\lambda \in \Lambda} X_\lambda, \text{ where } \Lambda \text{ is the indices' set of partition,}$$

such that

(a) $X_\lambda \in I(A)$, *for all* $\lambda \in \Lambda$;

(b) $A|_{X_\lambda} : X_\lambda \to X_\lambda$ *is a Picard operator for all* $\lambda \in \Lambda$.

Theorem 2.3. [15] *Let (X, d) be a complete metric space and $A, B : X \to X$ two orbitally continuous operators. We suppose that:*

(i) there exists $\alpha \in [0, 1[$ *such that*

$$d(A^2(x), A(x)) \leq \alpha d(x, A(x)), \text{ for all } x \in X$$

and

$$d(B^2(x), B(x)) \leq \alpha d(x, B(x)), \text{ for all } x \in X;$$

(ii) there exists $\eta > 0$ *such that* $d(A(x), B(x)) \leq \eta$, *for all* $x \in X$.

Then

$$H(F_A, F_B) \leq \frac{\eta}{1-\alpha}.$$

3. A Volterra integral equation with iterations of linear modification of the argument

We consider the integral equation (1).

We need the following sets:

$$C_L[0, b] := \{x \in C([0, b], [0, b]) |\ |x(t_1) - x(t_2)| \leq L|t_1 - t_2|, \text{ for all } t_1, t_2 \in [0, b]\},$$

and

$$C_{L,\theta}[0, b] := \{x \in C_L[0, b] |\ x(t) \leq \theta t, \text{ for all } t \in [0, b]\},$$

where $L, \theta \in \mathbb{R}_+^*$. Here $\mathbb{R}_+^* = \{a \in \mathbb{R} |\ a > 0\}$.

Let $\|\cdot\|_B : C[0, b] \to \mathbb{R}_+$ be the Bielecki norm, defined by

$$\|x\|_B = \max_{t \in [0,b]} |x(t)| e^{-\tau t}, \text{ where } \tau > 0,$$

and let $\|\cdot\|_C$ be the Chebyshev norm on $C[0, b]$, defined by $\|x\|_C = \max_{t \in [0,b]} |x(t)|$.

We denote by d_B, respectively by d_C, their corresponding metrics.

We remark that (see [1]):

If $d \in \{d_C, d_B\}$ then $(C([0,b],[0,b]),d)$, $(C_L[0,b],d)$ and $(C_{L,\theta}[0,b],d)$ are complete metric spaces.

If $\|\cdot\| \in \{\|\cdot\|_B, \|\cdot\|_C\}$ then $C_L[0,b]$ and $C_{L,\theta}[0,b]$ are convex, compact subsets of the Banach space $(C([0,b],[0,b]), \|\cdot\|)$.

The main results of this paper are the following

Theorem 3.1. *Suppose that:*

(i) $f \in C([0,b]^4)$ *and* $\max\limits_{s,u,v,w \in [0,b]} |f(s,u,v,w)| \leq M$*, where* $M \in \mathbb{R}_+^*$;

(ii) $M \leq L$.

Then the equation (1) has solutions in $C_L[0,b]$.

Proof. Let $A : C_L[0,b] \to C_L[0,b]$ be defined by

$$A(x)(t) := u_0 + \int_0^t f(s, x(s), x(\lambda s), x(\lambda x(\lambda s)))ds, \quad t \in [0,b],\ 0 < \lambda < 1,$$

that is a continuous operator.

We have

$$\begin{aligned} |A(x)(t_1) - A(x)(t_2)| &= \left| \int_{t_2}^{t_1} f(s, x(s), x(\lambda s), x(\lambda x(\lambda s)))ds \right| \\ &\leq M|t_1 - t_2| \leq L|t_1 - t_2|, \text{ for all } t_1, t_2 \in [0,b]. \end{aligned}$$

So, A is a continuous operator which applies the compact, convex set $C_L[0,b]$ into itself. By using Schauder's fixed point theorem we obtain that $F_A \neq \emptyset$. □

Theorem 3.2. *Suppose that:*

(i) there exists $T > 0$ *such that*

$$|f(s,u_1,v_1,w_1) - f(s,u_2,v_2,w_2)| \leq T(|u_1 - u_2| + |v_1 - v_2| + |w_1 - w_2|),$$

for all $s, u_i, v_i, w_i \in [0,b]$, $i = 1,2$;

(ii) $M \leq L$;

(iii) $\lambda^2 \theta < 1$;

(iv) $u_0 \in \mathbb{R}$ *is such that* $|u_0| + Mt \leq \theta t$*, for all* $t \in [0,b]$.

Then equation (1) has a unique solution x^* *in* $C_{L,\theta}[0,b]$ *and this solution can be obtained by successive approximation method, starting from any* $x_0 \in C_{L,\theta}[0,b]$.

Proof. We consider $A : C_{L,\theta}[0,b] \to C_{L,\theta}[0,b]$, defined by

$$A(x)(t) := u_0 + \int_0^t f(s, x(s), x(\lambda s), x(\lambda x(\lambda s)))ds, \quad t \in [0,b],\ 0 < \lambda < 1.$$

Because of (ii) and (iv) we have that $C_{L,\theta}[0,b] \in I(A)$.
Let $x, z \in C_{L,\theta}[0,b]$. By using (i) and (iii) we obtain

$$\begin{aligned}
&|A(x)(t) - A(z)(t)| \\
&\leq \int_0^t |f(s, x(s), x(\lambda s), x(\lambda x(\lambda s))) - f(s, z(s), z(\lambda s), z(\lambda z(\lambda s)))|ds \\
&\leq T \int_0^t [|x(s) - z(s)| + |x(\lambda s) - z(\lambda s)| + |x(\lambda x(\lambda s)) - z(\lambda z(\lambda s))|]ds \\
&\leq T \left[\int_0^t |x(s) - z(s)|e^{-\tau s} e^{\tau s} ds + \int_0^t |x(\lambda s) - z(\lambda s)|e^{-\tau\lambda s} e^{\tau\lambda s} ds \right. \\
&\quad + \int_0^t |x(\lambda x(\lambda s)) - z(\lambda x(\lambda s))|e^{-\tau\lambda x(\lambda s)} e^{\tau\lambda x(\lambda s)} ds \\
&\quad \left. + \int_0^t |z(\lambda x(\lambda s)) - z(\lambda z(\lambda s))|ds \right] \\
&\leq T \left[\int_0^t \|x - z\|_B e^{\tau s} ds + \int_0^t \|x - z\|_B e^{\tau\lambda s} ds \right. \\
&\quad \left. + \int_0^t \|x - z\|_B e^{\tau\lambda\theta\lambda s} ds + L\lambda \int_0^t |x(\lambda s) - z(\lambda s)|e^{-\tau\lambda s} e^{\tau\lambda s} ds \right] \\
&\leq T\|x - z\|_B \left(\frac{e^{\tau t} - 1}{\tau} + \frac{e^{\tau\lambda t} - 1}{\tau\lambda} + \frac{e^{\tau\lambda^2\theta t} - 1}{\tau\lambda^2\theta} + L\lambda \frac{e^{\tau\lambda t} - 1}{\tau\lambda} \right) \\
&\leq e^{\tau t} \frac{T}{\tau} \left(1 + \frac{1}{\lambda} + \frac{1}{\lambda^2\theta} + L \right) \|x - z\|_B, \text{ for all } t \in [0,b].
\end{aligned}$$

It follows that

$$|A(x)(t) - A(z)(t)|e^{-\tau t} \leq \frac{T}{\tau} \left(1 + \frac{1}{\lambda} + \frac{1}{\lambda^2\theta} + L \right) \|x - z\|_B,$$

for all $t \in [0,b]$.
Therefore,

$$\|A(x) - A(z)\|_B \leq \frac{T}{\tau} \left(1 + \frac{1}{\lambda} + \frac{1}{\lambda^2\theta} + L \right) \|x - z\|_B,$$

for all $x, z \in C_{L,\theta}[0,b]$. So, A is a Lipschitz operator with the Lipschitz constant $\frac{T}{\tau} \left(1 + \frac{1}{\lambda} + \frac{1}{\lambda^2\theta} + L \right)$.

Choosing $\tau = T\left(1+\frac{1}{\lambda}+\frac{1}{\lambda^2\theta}+L\right)+1$ we have that A is a contraction. We denote

$$L_A = \frac{T\left(1+\frac{1}{\lambda}+\frac{1}{\lambda^2\theta}+L\right)}{T\left(1+\frac{1}{\lambda}+\frac{1}{\lambda^2\theta}+L\right)+1}.$$

So $0 < L_A < 1$.

By applying Contraction principle we obtain that A is a Picard operator. □

Now, we consider both (1) and

$$(2) \qquad x(t) = v_0 + \int_0^t g(s, x(s), x(\lambda s), x(\lambda x(\lambda s)))ds, \ t \in [0,b], \ 0 < \lambda < 1,$$

where $g \in C([0,b]^4)$ and $v_0 \in \mathbb{R}$.

Let $M_1 > 0$ be such that $\max\limits_{s,u,v,w \in [0,b]} |g(s,u,v,w)| \le M_1$.

We have

Theorem 3.3. *We suppose that:*

(i) the conditions of Theorem 3.2 are satisfied and $x^* \in C_{L,\theta}$ is the unique solution of equation (1);

(ii) there exists $T_1 > 0$ such that

$$|g(s,u_1,v_1,w_1) - g(s,u_2,v_2,w_2)| \le T_1(|u_1-u_2| + |v_1-v_2| + |w_1-w_2|),$$

for all $s, u_i, v_i, w_i \in [0,b]$, $i = 1,2$;

(iii) $M_1 \le L$;

(iv) there exists $\eta > 0$ such that

$$|f(s,u,v,w) - g(s,u,v,w)| \le \eta, \ \text{for all } s,u,v,w \in [0,b].$$

If z^* is a solution of equation (2), then

$$\|x^* - z^*\|_B \le \frac{\eta b + |u_0 - v_0|}{1 - L_A},$$

where

$$L_A = \frac{T\left(1+\frac{1}{\lambda}+\frac{1}{\lambda^2\theta}+L\right)}{T\left(1+\frac{1}{\lambda}+\frac{1}{\lambda^2\theta}+L\right)+1}.$$

Proof. We consider the operators $A, B : C_L[0,b] \to C_L[0,b]$ defined by

$$A(x)(t) := u_0 + \int_0^t f(s, x(s), x(\lambda s), x(\lambda x(\lambda s)))ds, \quad t \in [0,b],\ 0 < \lambda < 1,$$

$$B(x)(t) := v_0 + \int_0^t g(s, x(s), x(\lambda s), x(\lambda x(\lambda s)))ds, \quad t \in [0,b],\ 0 < \lambda < 1$$

in which λ is the same.

We have $|A(x)(t) - B(x)(t)| \le |u_0 - v_0| + \eta b,$ for all $t \in [0,b]$.

It follows that

$$\|A(x) - B(x)\|_B \le |u_0 - v_0| + \eta b.$$

So, we apply Theorem 2.1. □

Remark 3.1. *Our results given above are more general than those obtained by A. Buică in [1].*

4. Another integral equation with iterations of linear modification of the argument

Now, we consider the following integral equation:

$$(3) \quad x(t) = x(0) + \int_0^t f(s, x(s), x(\lambda s), x(\lambda x(\lambda s)))ds,\ t \in [0,b],\ 0 < \lambda < 1,$$

where $f \in C([0,b]^4)$.

Let $M > 0$ be such that $\max\limits_{s,u,v,w \in [0,b]} |f(s,u,v,w)| \le M$.

We can write $C([0,b],[0,b]) = \bigcup\limits_{\alpha \in [0,b]} X_\alpha$, where

$$X_\alpha := \{\varphi \in C([0,b],[0,b]) |\ \varphi(0) = \alpha\}.$$

We consider the operator $A_* : C_{L,\theta}[0,b] \to C_{L,\theta}[0,b]$ defined by

$$A_*(x)(t) := x(0) + \int_0^t f(s, x(s), x(\lambda s), x(\lambda x(\lambda s)))ds,\ t \in [0,b],\ 0 < \lambda < 1,$$

that is a continuous operator but it is not a Lipschitz operator.

We have that $X_\alpha \in I(A_*)$ and $A_*|_{X_\alpha}$ is a Picard operator. But $A_*|_{X_\alpha}$ is the operator which appears in the proof of Theorem 3.1. By applying Theorem 2.2 we obtain that if the conditions of Theorem 3.1 are satisfied then A_* is a weakly Picard operator.

We denote $A_*^\infty(x) = \lim\limits_{n\to\infty} A_*^n(x)$.

From $A_*^{n+1}(x) = A_*(A_*^n(x))$ and the continuity of A_* we have that $A_*^\infty(x) \in F_{A_*}$, that is $F_{A_*} \neq \emptyset$. So, we have

Theorem 4.1. *If the conditions of Theorem 3.1 are satisfied then equation (3) has solutions in* $C_{L,\theta}[0,b]$*, that is* $F_{A_*}\neq\emptyset$ *and* $card\,F_{A_*}=card[0,b]$*.*

In order to examine the data dependence of the solutions set for the equation (3), we consider the equation:

$$x(t)=x(0)+\int_0^t g(s,x(s),x(\lambda s),x(\lambda x(\lambda s)))ds,\quad t\in[0,b],$$

in which λ is the same as in (3), and $g\in C([0,b]^4)$.

Let $M_1>0$ be such that $\max\limits_{s,u,v,w\in[0,b]}|g(s,u,v,w)|\leq M_1$.

We consider the operators

$$A_*,B_*:(C_{L,\theta}[0,b],\|\cdot\|_C)\to(C_{L,\theta}[0,b],\|\cdot\|_C)$$

defined by

$$A_*(x)(t):=x(0)+\int_0^t f(s,x(s),x(\lambda s),x(\lambda x(\lambda s)))ds,\ t\in[0,b],\ 0<\lambda<1,$$

$$B_*(x)(t):=x(0)+\int_0^t g(s,x(s),x(\lambda s),x(\lambda x(\lambda s)))ds,\quad t\in[0,b],\ 0<\lambda<1,$$

in which λ is the same.

We have

Theorem 4.2. *Suppose that*

(i) there exists $T>0$ *such that*

$$|f(s,u_1,v_1,w_1)-f(s,u_2,v_2,w_2)|\leq T(|u_1-u_2|+|v_1-v_2|+|w_1-w_2|),$$

and

$$|g(s,u_1,v_1,w_1)-g(s,u_2,v_2,w_2)|\leq T(|u_1-u_2|+|v_1-v_2|+|w_1-w_2|),$$

for all $s,u_i,v_i,w_i\in[0,b],\ i=1,2$*;*

(ii) $M\leq L$ *and* $M_1\leq L$*;*

(iii) $|x(0)|+Mt\leq\theta t$ *and* $|x(0)|+M_1t\leq\theta t$*, for all* $x\in C_{L,\theta}[0,b]$ *and all* $t\in[0,b]$*;*

(iv) there exists $\eta_1>0$ *such that*

$$|f(s,u,v,w)-g(s,u,v,w)|\leq\eta_1,\ \textit{for all}\ s,u,v,w\in[0,b],$$

(v) $3Tb<1$*.*

Then

(a) $F_{A_*} \neq \emptyset$ *and* $F_{B_*} \neq \emptyset$;

(b) $H_{\|\cdot\|_C}(F_{A_*}, F_{B_*}) \leq \dfrac{\eta_1 b}{1-3Tb}$, *where by* $H_{\|\cdot\|_C}$ *we denote the Pompeiu–Hausdorff metric with respect to* $\|\cdot\|_C$ *on* $C_{L,\theta}[0,b]$.

Proof. (a) By using the results of Theorem 4.1 we have that $F_{A_*} \neq \emptyset$ and $F_{B_*} \neq \emptyset$ and $card\, F_{A_*} = card\, F_{B_*} = card[0,b]$.

(b) We have

$$\begin{aligned} A_*^2(x)(t) &= A_*(A_*(x))(t) \\ &:= A_*(x)(0) + \int_0^t f(s, A_*(x)(s), A_*(x)(\lambda s), A_*(x)(\lambda x(\lambda s)))ds \\ &= x(0) + \int_0^t f(s, A_*(x)(s), A_*(x)(\lambda s), A_*(x)(\lambda x(\lambda s)))ds. \end{aligned}$$

Because of (i) we obtain

$$\begin{aligned} |A_*^2(x)(t) - A_*(x)(t)| &\leq T\int_0^t (|A_*(x)(s) - x(s)| + |A_*(x)(\lambda s) - x(\lambda s)| \\ &\qquad + |A_*(x)(\lambda x(\lambda s)) - x(\lambda x(\lambda s))|)\, ds \\ &\leq 3T\int_0^t \left(\max_{u\in[0,b]} |A_*(x)(u) - x(u)|\right) ds \\ &\leq 3Tb\|A_*(x) - x\|_C, \text{ for all } t \in [0,b]. \end{aligned}$$

So,

$$\|A_*^2(x) - A_*(x)\|_C \leq 3Tb\|A_*(x) - x\|_C, \text{ for all } x \in C_{L,\theta}[0,b].$$

Similarly,

$$\|B_*^2(x) - B_*(x)\|_C \leq 3Tb\|B_*(x) - x\|_C, \text{ for all } x \in C_{L,\theta}[0,b].$$

From (iv) we obtain that

$$\|A_*(x) - B_*(x)\|_C \leq \eta_1 b, \text{ for all } x \in C_{L,\theta}[0,b].$$

By applying Theorem 2.3 we have that

$$H_{\|\cdot\|_C}(F_{A_*}, F_{B_*}) \leq \frac{\eta_1 b}{1-3Tb}.$$

□

Remark 4.1. *The previous results generalized those obtained in [7].*

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