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SOME CHARACTERIZATIONS OF RECTIFYING CURVES IN THE MINKOWSKI 3–SPACE

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Abstract. Some characterizations of the Euclidean rectifying curves, i.e. the curves in E^3 which have a property that their position vector always lies in their rectifying plane, are given in [3]. In this paper, we characterize non–null and null rectifying curves, lying fully in the Minkowski 3–space E^3_1. Also, in considering a causal character of a curve we give some parametrizations of rectifying curves.

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1. Introduction

In the Euclidean space E^3, to each regular unit speed curve $\alpha : I \to E^3$, $I \subset R$, with at least four continuous derivatives, it is possible to associate three mutually orthogonal unit vector fields T, N and B, called respectively the tangent, the principal normal and the binormal vector field. The planes spanned by the vector fields $\{T, N\}$, $\{T, B\}$ and $\{N, B\}$ are known as the *osculating plane*, the *rectifying plane* and the *normal plane*, respectively. The Euclidean curves that have a property that their position vector α always lies in their rectifying plane, are called in [3] *rectifying curves.* Therefore, the position vector α of a rectifying curve satisfies by definition of Chen [3] the equation $\alpha(s) = \lambda(s)T(s)+\mu(s)B(s)$, for some differentiable functions $\lambda(s)$ and $\mu(s)$. One of the most interesting characteristics of such curves is that the ratio of their torsion and curvature is a non–constant linear function of the arclength parameter s. In [3], rectifying curves, lying fully in the space E^3, are determined explicitely.

In this paper, we give some characterizations of rectifying curves lying fully in the Minkowski 3–space E^3_1. In particular, we prove that the ratio of torsion and curvature of any regular rectifying curve in E^3_1 is a non–constant linear function of the pseudo arclength parameter s. We emphasize that this property is invariant with respect to the causal character of a curve and its rectifying plane. Also, we find some parametrizations of non–null and null rectifying curves that lie fully in the Minkowski 3–space.

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2. Preliminaries

The Minkowski 3–space E_1^3 is the real vector space R^3 provided with the standard flat metric given by

$$g = -dx_1^2 + dx_2^2 + dx_3^2,$$

where (x_1, x_2, x_3) is a rectangular coordinate system of E_1^3. Since g is an indefinite metric, recall that a vector $v \neq 0$ in E_1^3 can be a *spacelike*, a *timelike* or a *null (lightlike)*, if respectively holds $g(v,v) > 0$, $g(v,v) < 0$ or $g(v,v) = 0$. In particular, the vector $v = 0$ is a spacelike. The norm (length) of a vector v is given by $||v|| = \sqrt{|g(v,v)|}$ and two vectors v and w are said to be orthonormal when $g(v,w) = 0$. We also recall that an arbitrary curve $\alpha = \alpha(s)$ can locally be a *spacelike, timelike* or *null (lightlike)*, if all of its velocity vectors $\alpha'(s)$ are respectively *spacelike, timelike* or *null.* A non–null or a null curve $\alpha(s)$ is said to be parameterized by the pseudo arclength parameter s, if respectively hold $g(\alpha'(s), \alpha'(s)) = \pm 1$ or $g(\alpha''(s), \alpha''(s)) = 1$ (see [6], [1]). In both of these cases, the curve α is said to be of unit speed. Recall that an arbitrary plane π in E_1^3 is by definition a spacelike, timelike or lightlike, if $g|_\pi$ is respectively positive definite, nondegenerate of index 1, or degenerate. Recall that when α is a non–null curve in E_1^3 with spacelike or timelike rectifying plane, then the Frenet equations are of the form [4]:

$$(*) \qquad \begin{aligned} T' &= kN, \\ N' &= -\epsilon_0\epsilon_1 kT + \tau B, \\ B' &= -\epsilon_1\epsilon_2 \tau N, \end{aligned}$$

where $\epsilon_0 = g(T,T) = \pm 1$, $\epsilon_1 = g(N,N) = \pm 1$, $\epsilon_2 = g(B,B) = \pm 1$ and $\epsilon_0\epsilon_1\epsilon_2 = -1$. Further, when α is a spacelike curve with lightlike rectifying plane or a null curve (with timelike rectifying plane), then the Frenet formulae are given respectively by [7]:

$$(**) \qquad \begin{aligned} T' &= kN, \\ N' &= \tau N, \\ B' &= -kT - \tau B, \end{aligned}$$

and

$$(***) \qquad \begin{aligned} T' &= kN, \\ N' &= \tau T - kB, \\ B' &= -\tau N. \end{aligned}$$

In both cases (**) and (***), there are only two values of the first curvature $k(s)$: $k(s) = 0$ when α is a straight line, or $k(s) = 1$ in all other cases.

We also recall that the pseudosphere of radius 1 and center at the origin is the hyperquadric in E_1^3 defined by $S_1^2(1) = \{v \in E_1^3 : g(v,v) = 1\}$, and the pseudohyperbolic space of radius 1 and center at the origin is the hyperquadric in E_1^3 defined by $H_0^2(1) = \{v \in E_1^3 : g(v,v) = -1\}$.

3. Some characterizations of rectifying curves in $\mathbf{E}_1^3$

In this section we characterize non–null (spacelike and timelike) and null rectifying curves lying fully in the Minkowski 3–space. Accordingly, we first characterize unit speed non–null rectifying curves.

Theorem 1. *Let $\alpha = \alpha(s)$ be a unit speed non–null rectifying curve in E_1^3 with spacelike or timelike rectifying plane, the curvature $k(s) > 0$ and $g(T,T) = \epsilon_0 = \pm 1$. Then the following statements hold:*

(i) The distance function $\rho = ||\alpha||$ satisfies $\rho^2 = |\epsilon_0 s^2 + c_1 s + c_2|$, for some $c_1 \in R, c_2 \in R_0$.

(ii) The tangential component of the position vector of α is given by $g(\alpha, T) = \epsilon_0 s + c$, where $c \in R$.

(iii) The normal component α^N of the position vector of the curve has a constant length and the distance function ρ is non–constant.

(iv) The torsion $\tau(s) \neq 0$ and the binormal component of the position vector of the curve is constant, i.e. $g(\alpha, B)$ is constant.

Conversely, if $\alpha(s)$ is a unit speed non–null curve in E_1^3, with spacelike or timelike rectifying plane, the curvature $k(s) > 0$, $g(T,T) = \epsilon_0 = \pm 1$ and one of the statements (i), (ii), (iii) and (iv) holds, then α is a rectifying curve.

Proof. Let us first suppose that $\alpha = \alpha(s)$ is a unit speed non–null rectifying curve. Then the position vector α of a curve satisfies the equation

$$\alpha(s) = \lambda(s)T(s) + \mu(s)B(s), \tag{1}$$

where $\lambda(s)$ and $\mu(s)$ are some differentiable functions of the pseudo arclength parameter s. Differentiating the relation (1) with respect to s, and by applying the Frenet equations (*), we obtain

$$\lambda'(s) = 1, \quad \lambda(s)k(s) - \epsilon_1\epsilon_2\mu(s)\tau(s) = 0, \quad \mu'(s) = 0, \tag{2}$$

whereby $\epsilon_1 = g(N,N) = \pm 1$ and $\epsilon_2 = g(B,B) = \pm 1$. Therefore, it follows that

$$\lambda(s) = s + j, \quad j \in R, \quad \mu(s) = l, \quad l \in R, \quad \mu(s)\tau(s) = \lambda(s)k(s) \neq 0, \tag{3}$$

and hence $\mu(s) = l \neq 0$, $\tau(s) \neq 0$. From the equation (1) we easily find $\rho^2 = |g(\alpha,\alpha)| = |\epsilon_0\lambda^2 + \epsilon_2\mu^2|$. Substituting (3) into the last equation, we obtain statement (i). Further, from (1) we obtain $g(\alpha, T) = \epsilon_0\lambda$, which together with (3) implies (ii). Next, from the relation (1) it follows that the normal component α^N of the position vector α is given by $\alpha^N = \mu B$. Therefore, $||\alpha^N|| = |l| \neq 0$. Thus we proved statement (iii). Finally, from (1) we easily get $g(\alpha, B) = \mu\epsilon_2 =$ constant and since $\tau(s) \neq 0$, the statement (iv) is proved.

Conversely, assume that statement (i) or statement (ii) holds. Then there holds the equation $g(\alpha(s), T(s)) = s + c$, $c \in R$. Differentiating this equation with respect to s, we get $k(s)g(\alpha(s), N(s)) = 0$. Since $k(s) > 0$, it follows that $g(\alpha, N) = 0$. Hence α is a rectifying curve.

Next, suppose that statement (*iii*) holds. Let us put $\alpha(s) = m(s)T(s) + \alpha^N$, $m(s) \in R$. Then we easily find that $g(\alpha^N, \alpha^N) = C = \text{constant} = g(\alpha, \alpha) - \frac{1}{\epsilon_0} g(\alpha, T)^2$. Differentiating this equation with respect to s gives

$$g(\alpha, T) = \frac{1}{\epsilon_0} g(\alpha, T)(\epsilon_0 + k g(\alpha, N)). \tag{4}$$

Since $\rho \neq$ constant, we have $g(\alpha, T) \neq 0$. Moreover, since $k(s) > 0$ and from (4) we obtain $g(\alpha, N) = 0$, which means that α is a rectifying curve.

Finally, if statement (*iv*) holds, then by applying Frenet equations (*), we easily obtain that the curve α is a rectifying curve. □

In the next theorem, we prove that the ratio of torsion and curvature of a unit speed non–null rectifying curve is a non–constant linear function of the pseudo arclength parameter s.

Theorem 2. *Let $\alpha = \alpha(s)$ be a unit speed non–null curve in E_1^3, with a spacelike or a timelike rectifying plane and with the curvature $k(s) > 0$. Then up to isometries of E_1^3, the curve α is a rectifying if and only if there holds $\tau(s)/k(s) = c_1 s + c_2$, where $c_1 \in R_0$, $c_2 \in R$.*

Proof. Let us first suppose that the curve $\alpha(s)$ is rectifying. By the proof of Theorem 1 and by the relations (2) and (3), it follows that

$$\frac{\tau(s)}{k(s)} = \frac{s + j}{\epsilon_1 \epsilon_2 l}, \tag{5}$$

whereby $j \in R$, $l \in R_0$. Consequently, $\tau(s)/k(s) = c_1 s + c_2$, whereby $c_1 \in R_0$, $c_2 \in R$.

Conversely, let us suppose that $\tau(s)/k(s) = c_1 s + c_2$, $c_1 \in R_0$, $c_2 \in R$. Then we may choose $c = 1/(\epsilon_1 \epsilon_2 l)$, $c_2 = j/(\epsilon_1 \epsilon_2 l)$, where $j \in R$, $l \in R_0$, $\epsilon_1 = \pm 1$, $\epsilon_2 = \pm 1$. Hence $\tau(s)/k(s) = (s + j)/(\epsilon_1 \epsilon_2 l)$. Applying the Frenet equations (*), we easily find that

$$\frac{d}{ds}(\alpha(s) - (s + j)T(s) - lB(s)) = 0,$$

which means that up to isometries of E_1^3, the curve α is rectifying. □

In the next theorem we determine some parametrizations of a unit speed non–null rectifying curves in E_1^3.

Theorem 3. *Let $\alpha = \alpha(s)$ be a unit speed non–null curve in E_1^3. Then the following statements hold:*

(i) α is a rectifying curve with a spacelike rectifying plane if and only if, up to a parametrization, α is given by

$$\alpha(t) = y(t)\frac{l}{\cos t}, \quad l \in R_0^+, \tag{a}$$

where $y(t)$ is a unit speed spacelike curve lying in the pseudosphere $S_1^2(1)$.

(ii) α is a spacelike (timelike) rectifying curve with a timelike rectifying plane and a spacelike (timelike) position vector if and only if, up to a parametriza-

tion, α is given by

$$\alpha(t) = y(t)\frac{l}{\sinh t}, \quad l \in R_0^+, \tag{b}$$

where $y(t)$ is a unit speed timelike (spacelike) curve lying in the pseudosphere $S_1^2(1)$ (pseudohyperbolic space $H_0^2(1)$).

(iii) α is a spacelike (timelike) rectifying curve with a timelike rectifying plane and a timelike (spacelike) position vector if and only if, up to a parametrization, α is given by

$$\alpha(t) = y(t)\frac{l}{\cosh t}, \quad l \in R_0^+, \tag{c}$$

where $y(t)$ is a unit speed spacelike (timelike) curve lying fully in the pseudohyperbolic space $H_0^2(1)$ (pseudosphere $S_1^2(1)$).

Proof. (i) Let us first assume that $\alpha(s)$ is a unit speed non–null rectifying curve with spacelike rectifying plane in E_1^3. Since the position vector lies in the spacelike rectifying plane, we have $g(\alpha, \alpha) > 0$, $g(T, T) = \epsilon_0 = 1$ and $g(B, B) = \epsilon_2 = 1$. By the proof of Theorem 1, it follows that $\rho^2 = ||\alpha||^2 = (s + j)^2 + l^2, j \in R, l \in R_0$. We may choose $l \in R_0^+$. Also, we may apply a translation with respect to s , such that $\rho^2 = s^2 + l^2$. Next, we define a curve y lying in the pseudosphere $S_1^2(1)$ by

$$y(s) = \frac{\alpha(s)}{\rho(s)}. \tag{6}$$

Then we have

$$\alpha(s) = y(s)\sqrt{s^2 + l^2}. \tag{7}$$

Differentiating (7) with respect to s, we get

$$T(s) = y(s)\frac{s}{\sqrt{s^2 + l^2}} + y'(s)\sqrt{s^2 + l^2}. \tag{8}$$

Since $g(y, y) = 1$, it follows that $g(y, y') = 0$. From (8) we obtain

$$1 = g(T, T) = g(y', y')(s^2 + l^2) + \frac{s^2}{s^2 + l^2},$$

and hence

$$g(y', y') = l^2/(s^2 + l^2)^2, \tag{9}$$

which means that y is a spacelike curve. From (9) we get $||y'(s)|| = l/(s^2 + l^2)$. Let $t = \int_0^s ||y'(u)||du$ be the pseudo acrlength parameter of the curve y. Then we have

$$t = \int_0^s \frac{l}{u^2 + l^2}du,$$

and therefore $s = l \tan t$. Substituting this into (7) we obtain the parametrization (a).

Conversely, assume that α is a curve defined by (a), where $y(t)$ is a unit speed spacelike curve lying in the pseudosphere $S_1^2(1)$. Differentiating the equation (a) with respect to t, we get

$$\alpha'(t) = \frac{l}{\cos^2 t}(y(t)\sin t + y'(t)\cos t).$$

By assumption we have $g(y', y') = 1$, $g(y, y) = 1$ and consequently $g(y, y') = 0$. Therefore, it follows that

$$g(\alpha, \alpha') = \frac{l^2 \sin t}{\cos^3 t}, \quad g(\alpha', \alpha') = \frac{l^2}{\cos^4 t}, \tag{10}$$

and consequently $||\alpha'(t)|| = \frac{l}{\cos^2 t}$. Let us put $\alpha(t) = m(t)\alpha'(t) + \alpha^N$, where $m(t) \in R$ and α^N is a normal component of the position vector α. Then we easily find that $m = g(\alpha, \alpha')/g(\alpha', \alpha')$, and therefore

$$g(\alpha^N, \alpha^N) = g(\alpha, \alpha) - \frac{g(\alpha, \alpha')^2}{g(\alpha', \alpha')}.$$

Since $g(\alpha, \alpha) = \frac{l^2}{\cos^2 t}$ and by using (10), the last equation becomes $g(\alpha^N, \alpha^N) = l^2 =$ constant. It follows that $||\alpha^N|| =$ constant and since $\rho = ||\alpha|| = \frac{l}{\cos t} \neq$ constant, Theorem 1 implies that α is a rectifying curve.

(ii) Let us first suppose that α is a spacelike rectifying curve with a timelike rectifying plane and a spacelike position vector. Then we have $g(\alpha, \alpha) > 0$, $g(T, T) = \epsilon_0 = 1$ and $g(B_2, B_2) = \epsilon_2 = -1$. By the proof of Theorem 1, we obtain $\rho^2 = ||\alpha||^2 = g(\alpha, \alpha) = (s + j)^2 - l^2$, where $j \in R$, $l \in R_0$. We may choose $l \in R_0^+$ and apply a translation with respect to s, such that $\rho^2 = s^2 - l^2$, $|s| > l$. Further, define a curve $y(s)$ lying in the pseudosphere $S_1^2(1)$ by

$$y(s) = \frac{\alpha(s)}{\rho(s)}. \tag{11}$$

It follows that

$$\alpha(s) = y(s)\sqrt{s^2 - l^2}, \tag{12}$$

and differentiating the previous equation with respect to s, we find

$$T(s) = y'(s)\sqrt{s^2 - l^2} + y(s)\frac{s}{\sqrt{s^2 - l^2}}. \tag{13}$$

Since $g(y, y) = 1$, it follows that $g(y, y') = 0$. Consequently,

$$g(T, T) = g(y', y')(s^2 - l^2) + \frac{s^2}{s^2 - l^2} = 1.$$

The previous equation implies that

$$g(y', y') = -\frac{l^2}{s^2 - l^2}, \tag{14}$$

which means that y is a timelike curve. By using (14), we easily get that

$||y'(s)|| = \frac{l}{s^2-l^2}$, $l \in R_0^+$, $|s| > l$. Let $t = \int_0^s ||y'(u)||du$ be the pseudo arclength parameter of the curve y. Then we have

$$t = \int_0^s \frac{l}{u^2 - l^2} du,$$

and thus $t = -\coth^{-1}(\frac{s}{l})$. Hence $s = -l\coth(t)$. Substituting this into equation (12), we obtain parametrization (b).

Conversely, let us assume that α is a curve defined by (b), where $y(t)$ is a unit speed timelike curve lying in the pseudosphere $S_1^2(1)$. Differentiating the equation (b) with respect to t, we get

$$\alpha'(t) = \frac{l}{\sinh^2(t)}(y'(t)\sinh t - y(t)\cosh t). \tag{15}$$

By assumption we have $g(y', y') = -1$, $g(y, y) = 1$ and therefore $g(y, y') = 0$. Then the equation (15) implies that

$$g(\alpha, \alpha') = -\frac{l^2\cosh t}{\sinh^3 t}, \quad g(\alpha', \alpha') = \frac{l^2}{\sinh^4 t} \tag{16}$$

and therefore $||\alpha'(t)|| = \frac{l}{\sinh^2 t}$. Let us put $\alpha(t) = m(t)\alpha'(t) + \alpha^N$, where $m(t) \in R$ and α^N is a normal component of the position vector α. Then we easily find that $m = g(\alpha', \alpha')/g(\alpha, \alpha')$, and hence

$$g(\alpha^N, \alpha^N) = g(\alpha, \alpha) - \frac{g(\alpha, \alpha')^2}{g(\alpha', \alpha')}. \tag{17}$$

Since $g(\alpha, \alpha) = \frac{l^2}{\sinh^2 t}$ and by using (16), the equation (17) becomes $g(\alpha^N, \alpha^N) = -l^2 = \text{constant}$. Hence $||\alpha^N|| = \text{constant}$ and since $\rho = \frac{l}{\sinh t} \neq \text{constant}$, Theorem 1 impies that the curve α is rectifying.

The proof in the case when α is a timelike rectifying curve with a timelike rectifying plane and a timelike position vector is analogous.

(iii) The proof is analogous to the proofs of the statements (i) and (ii). □

Theorem 4. *There are no unit speed non–null rectifying curves in E_1^3 with the curvature $k(s) = 1$ and a lightlike rectifying plane.*

Proof. Let us suppose that there exists a unit speed non–null rectifying curve α in E_1^3, with the curvature $k(s) = 1$ and a lightlike rectifying plane. Then α is a spacelike curve with the position vector satisfying the following equation:

$$\alpha(s) = \lambda(s)T(s) + \mu(s)B(s),$$

where $\lambda(s)$ and $\mu(s)$ are arbitrary functions. Differentiating the previous equation with respect to s and by using the Frenet equations (***), we obtain

$$\lambda(s) = 0, \quad \mu(s) = -1, \quad \tau(s) = 0.$$

Consequently, $\alpha(s) = -B(s)$. Further, since $\tau(s) = 0$ and by using the Frenet equations (***), we find $\alpha' = T$, $\alpha'' = N$, $\alpha''' = 0$. On the other hand, by using the MacLaurin expansion for α given by

$$\alpha(s) = \alpha(0) + \alpha'(0)\frac{s}{1!} + \alpha''(0)\frac{s^2}{2!} + \alpha'''(0)\frac{s^3}{3!} + \dots,$$

we conclude that α lies fully in the osculating plane, spanned by $\{\alpha'(0), \alpha''(0)\}$, which is a contradiction. □

In the next three theorems we characterize all null rectifying curves in E_1^3.

Theorem 5. *Let $\alpha(s)$ be a unit speed null rectifying curve in E_1^3 with the curvature $k(s) = 1$. Then the following statements hold:*

(i) The distance function $\rho = ||\alpha||$ satisfies $\rho^2 = |c_1 s + c_2|$, where $c_1 \in R_0$, $c_2 \in R$.

(ii) The tangential component $g(\alpha, T)$ of the position vector of the curve is constant.

(iii) The torsion $\tau(s) \neq 0$ and the binormal component of the position vector of the curve is given by $g(\alpha, B) = s + c$, whereby $c \in R$.

Conversely, if $\alpha(s)$ is a unit speed null curve in E_1^3 with the first curvature $k(s) = 1$ and one of the statements (i), (ii) or (iii) holds, then α is a rectifying curve.

Proof. Let us suppose that $\alpha(s)$ is a unit speed null rectifying curve in E_1^3 with the curvature $k(s) = 1$. Then the position vector of the curve satisfies the equation

$$\alpha(s) = \lambda(s)T(s) + \mu(s)B(s), \tag{18}$$

where $\lambda(s)$ and $\mu(s)$ are arbitrary functions of the pseudo arclength parameter s. Differentiating the equation (18) with respect to s and using the Frenet equations (***), we obtain

$$\lambda'(s) = 1, \quad \lambda(s) - \mu(s)\tau(s) = 0, \quad \mu'(s) = 0. \tag{19}$$

From the previous equation, we find

$$\lambda(s) = s + j, \quad j \in R, \quad \mu(s) = l, \quad l \in R, \quad \mu(s)\tau(s) = \lambda(s) \neq 0. \tag{20}$$

Therefore, it follows that $\mu(s) = l \in R_0$ and $\tau(s) \neq 0$. Next the equation (18) implies $g(\alpha, \alpha) = 2l(s + j)$ and hence $\rho^2 = ||\alpha||^2 = |c_1 s + c_2|$, where $c_1 \in R_0$, $c_2 \in R$. This proves statement (i). Next, from the equation (18) we get $g(\alpha, T) = l$, $l \in R_0$ and $g(\alpha, B) = s + j$, $j \in R$. This proves statements (ii) and (iii).

Conversely, assume that α is a unit speed null rectifying curve in E_1^3 with the curvature $k(s) = 1$ and let statement (i) holds.Then $\rho^2 = |g(\alpha, \alpha) = |c_1 s + c_2|$, where $c_1 \in R_0$, $c_2 \in R$, and hence $g(\alpha, \alpha) = \pm(c_1 s + c_2)$. Differentiating the last equation two times with respect to s and using the Frenet equations (***), we obtain $g(\alpha, N) = 0$. Therefore, α is a rectifying curve. Next, suppose that statement (*ii*) holds. By differentiating with respect to s the equation $g(\alpha, T) =$ constant and by applying the Frenet equations (***), we easily find that $g(\alpha, N) = 0$, which means that α is a rectifying curve. Finally, assume that statement (*iii*) holds. Since $\tau(s) \neq 0$ and by taking the derivative with respect to s of the equation $g(\alpha, B) = s + c$, $c \in R$, we get $g(\alpha, N) = 0$. Thus the curve α is rectifying. □

In Theorem 2 we have proved that the ratio of torsion and curvature of a non–null rectifying curve is a non–constant linear function of the pseudo arclength parameter s. The same property holds for the null rectifying curves. We omit the proof of the following theorem, since it is analogous to the proof of Theorem 2.

Theorem 6. *Let $\alpha = \alpha(s)$ be a unit speed null curve in E_1^3 with the first curvature $k(s) = 1$. Then up to isometries of E_1^3 the curve α is rectifying if and only if there holds $\tau(s)/k(s) = c_1 s + c_2$, where $c_1 \in R_0, c_2 \in R$.*

In Theorem 7 we determine explicitly all unit speed null rectifying curves, lying fully in the Minkowski 3–space.

Theorem 7. *Let $\alpha = \alpha(s)$ be a unit speed null curve in E_1^3 with the first curvature $k(s) = 1$. Then α is a rectifying curve with a spacelike (timelike) position vector if and only if, up to a parameterization, α is given by*

$$\alpha(t) = e^t y(t), \tag{d}$$

where $y(t)$ is a unit speed timelike (spacelike) curve lying in the pseudosphere $S_1^2(1)$ (pseudohyperbolic space $H_0^2(1)$).

Proof. Let us assume first that $\alpha(s)$ is a unit speed null rectifying curve in E_1^3 with the first curvature $k(s) = 1$ and a spacelike position vector. Then we have $g(\alpha, \alpha) > 0$. By the proof of Theorem 5, it follows that $g(\alpha, \alpha) = c_1 s + c_2$, where $c_1 \in R_0$, $c_2 \in R$, and thus $\rho^2(s) = ||\alpha||^2 = c_1 s + c_2$. We may take $c_1 \in R_0^+$.

Further, define a curve y lying in the pseudosphere $S_1^2(1)$ by

$$y(s) = \frac{\alpha(s)}{\rho(s)}.$$

Then we have

$$\alpha(s) = y(s)\sqrt{c_1 s + c_2}. \tag{21}$$

Differentiating the previous equation with respect to s we get

$$T(s) = \frac{c_1}{2\sqrt{c_1 s + c_2}} y(s) + \sqrt{c_1 s + c_2}\, y'(s), \tag{22}$$

Since $g(y,y)=1$, it follows that $g(y,y')=0$. From (22) we obtain

$$0 = g(T,T) = g(y',y')(c_1 s + c_2) + \frac{c_1^2}{4(c_1 s + c_2)}$$

and thus

$$g(y',y') = -\frac{c_1^2}{4(c_1 s + c_2)^2}, \tag{23}$$

which means that y is a timelike curve. Equation (23) implies that $||y'(s)|| = c_1/2(c_1 s + c_2)$. Let $t = \int_0^s ||y'(u)||du$ be the pseudo arclength parameter of the curve y. Then we obtain

$$t = \int_0^s \frac{c_1}{2(c_1 s + c_2)} du$$

and hence $t = \frac{1}{2}\ln(c_1 s + c_2)$. From the last equation we get $c_1 s + c_2 = e^{2t}$. Substituting this into (21), we obtain the parametrization (d).

Conversely, assume that α is a curve defined by (d), where $y(t)$ is a unit speed timelike curve lying in the pseudosphere $S_1^2(1)$. Then we may reparameterize the curve $\alpha(t)$ by $t = (1/2)\ln(c_1 s + c_2)$, where s is the pseudo arclength parameter of the null curve α, $c_1 s + c_2 > 0$, and $c_1 \in R_0$, $c_2 \in R$. Then we have $\alpha(s) = y(s)\sqrt{c_1 s + c_2}$. Consequently, we obtain that $\rho^2 = ||\alpha||^2 = g(\alpha,\alpha) = c_1 s + c_2$. Finally, Theorem 5 implies that α is a rectifying curve.

The proof in the case when α is a unit speed null rectifying curve in E_1^3 with the timelike position vector is analogous. □

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