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SOME PROPERTIES OF THE DISCONTINUOUS GALERKIN METHOD FOR ONE–DIMENSIONAL SINGULARLY PERTURBED PROBLEMS

Hans–Görg Roos[1], **Helena Zarin**[2]

Abstract. A nonsymmetric discontinuous Galerkin method with interior penalties is considered for one–dimensional reaction–diffusion and convection–diffusion equations. Discrete problems are analyzed and some properties of the corresponding matrices are given. Beside first–order error estimate for linear elements, an L_2–error bound is also derived.

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1. Introduction

Although continuous Galerkin finite element methods are traditionally used for solving elliptic problems, recently new interest has arisen in applying discontinuous Galerkin finite element methods (DGFEMs). For instance, in [5] it is emphasized that the DGFEMs allow the discretization of convection–diffusion problems without invoking streamline–diffusion stabilization in order to reduce the amount of numerical dissipation. The DGFEMs belong to the class of nonconforming methods and they approximate exact solutions by piecewise polynomial functions over a finite element space without any requirement on interelement continuity – however, continuity on interelement boundaries together with boundary conditions is weakly enforced through the bilinear form.

So far not much is known on properties of the discrete problems generated by the DGFEMs. In the sequel we shall study the following one–dimensional linear singularly perturbed problem

$$\begin{cases} -\varepsilon u'' + bu' + cu = f & \text{in} \quad \Omega = (0,1)\,, \\ u(0) = u(1) = 0\,, \end{cases} \tag{1}$$

assuming b, c, f to be smooth on $\overline{\Omega} = [0,1]$ and $b(x) \geq \beta > 0$, $c(x) \geq 0$, $x \in \overline{\Omega}$. Here $0 < \varepsilon \ll 1$ represents a small perturbation parameter. Using linear finite elements for the discretization of (1), we shall compare some properties of a specially chosen DGFEM with the standard version of the finite element method.

[1]Institute of Numerical Mathematics, Dresden University of Technology, Germany, e-mail: roos@math.tu-dresden.de

[2]Department of Mathematics and Informatics, Faculty of Science and Mathematics, University of Novi Sad, Serbia and Montenegro, e-mail: helena@im.ns.ac.yu

2. The discontinuous Galerkin method

Although there are many variants of the DGFEMs that are developed for elliptic problems (see [1] for a survey), we choose the nonsymmetric version of the DGFEM with interior penalties (the NIPG method, [5]) because of its good stability and consistency properties. For simplicity of presentation, let us consider an equidistant mesh Ω_h with the step size h and the mesh points $x_i = ih$, $i = 0, 1, \ldots, N$, $Nh = 1$. We shall use the following broken Sobolev space $H^1(\Omega_h)$ that is given by

$$H^1(\Omega_h) = \{u \in L^2(\Omega) : u|_{(x_{i-1},x_i)} \in H^1(x_{i-1}, x_i), \text{ for all } i\},$$

where $H^1(x_{i-1}, x_i)$ denotes the Sobolev space of order one defined on (x_{i-1}, x_i).

Let us first consider reaction–diffusion case ($b \equiv 0$). Then, the weak form of the problem (1) that is associated with the NIPG method reads

$$\text{(2)} \qquad \begin{cases} \text{find} \quad u \in V = H^1(\Omega_h) \ \text{ such that} \\ a_1(u, v) = l(v)\,, \ \text{ for all } \ v \in V\,, \end{cases}$$

where

$$\text{(3)} \qquad \begin{aligned} a_1(u, v) \;&=\; \sum_{i=1}^{N} \int_{x_{i-1}}^{x_i} (\varepsilon u'v' + cuv) + \varepsilon\,(uv')\,\big|_0^1 - \varepsilon\,(u'v)\,\big|_0^1 \\ &\quad + \varepsilon \sum_{i=1}^{N-1} ([v]_i \langle u' \rangle_i - \langle v' \rangle_i [u]_i) \\ &\quad + \sigma_0 (uv)(0) + \sigma_N (uv)(1) + \sum_{i=1}^{N-1} \sigma_i [u]_i [v]_i\,, \\ l(v) \;&=\; \sum_{i=1}^{N} \int_{x_{i-1}}^{x_i} fv\,. \end{aligned}$$

Here $[\,\cdot\,]_i$ denotes the jump and $\langle\,\cdot\,\rangle_i$ the average of the corresponding discontinuous function at the mesh point x_i. Let us choose the penalty parameter σ_i as in [5], namely $\sigma_i = \sigma = \varepsilon/h$, $i = 0, 1, \ldots, N$.

For the convective part $bu' = f$ of (1), the discontinuous Galerkin method is associated with the bilinear form

$$\text{(4)} \qquad a_2(u, v) = \sum_{i=1}^{N} \int_{x_{i-1}}^{x_i} bu'v + (buv)(0) + \sum_{i=1}^{N-1} b_i [u]_i v_i^+\,.$$

Here v_i^+ denotes $v(x_i + 0)$ and $b_i = b(x_i)$.

If $a(\cdot, \cdot)$ represents the sum of the two bilinear forms a_1 and a_2 defined above, the NIPG method for the problem (1) is

$$\text{(5)} \qquad \begin{cases} \text{find} \quad u_h \in V_h \subset H^1(\Omega_h) \ \text{ such that} \\ a(u_h, v_h) = l(v_h)\,, \ \text{ for all } \ v_h \in V_h\,. \end{cases}$$

We shall study the case when V_h consists of (discontinuous) polynomials of degree one over the given mesh Ω_h.

Based on the consistency of the method that is studied in [5] in a much more general situation, the error of the NIPG method is analyzed in the following DG–norm:

$$
\begin{aligned}
\|w\|_{\mathrm{DG}}^2 \;=\; & \sum_{i=1}^{N} \int_{x_{i-1}}^{x_i} \left(\varepsilon (w')^2 + w^2 \right) + \frac{1}{2} \sum_{i=1}^{N-1} b_i [w]_i^2 \\
& + \frac{1}{2}\, b_0 w^2(0) + \frac{1}{2}\, b_N w^2(1) \\
& + \sigma_0 w^2(0) + \sum_{i=1}^{N-1} \sigma_i [w]_i^2 + \sigma_N w^2(1)\,.
\end{aligned} \tag{6}
$$

Applying the technique from [5] on the singularly perturbed problem (1), for the NIPG method with linear elements on the uniform mesh Ω_h we obtain the following error estimate

$$
\|u - u_h\|_{DG} \le C \left(\varepsilon^{1/2} h + h^{3/2} + h^2 \right) \|u\|_{H^2(\Omega)}\,, \tag{7}
$$

with the discontinuity–penalization parameter $\sigma_i = \sigma = \varepsilon / h$.

Remark 2.1. *From (7), in the nonsingular case $\varepsilon = 1$, we get*

$$
\|u - u_h\|_{\mathrm{DG}} \le C h\,,
$$

and therefore $[u_h]_i = \mathcal{O}(h^{3/2})$, but numerical experiments indicate that the order of the jumps is two (see also [3]).

3. Some properties of the discrete problem

For the given partitioning Ω_h we can choose

$$
\varphi_{i-1}^{+}(x) = \frac{x_i - x}{h}\,, \qquad \varphi_i^{-}(x) = \frac{x - x_{i-1}}{h}\,, \qquad x \in [x_{i-1}, x_i]\,,
$$

for the basis functions and introduce

$$
u_h(x) = u_{i-1}^{+} \varphi_{i-1}^{+}(x) + u_i^{-} \varphi_i^{-}(x)\,, \qquad x \in [x_{i-1}, x_i]\,. \tag{8}
$$

The discontinuous Galerkin method generates a difference scheme for a vector–valued grid function. In the nonsingular case, for the reaction–diffusion equation (1) with $b \equiv 0$ and with a constant function $c(x) = c$, the generated discretization stencil reads

$$
\begin{array}{cc|cc|cc}
-\dfrac{1}{h^2} & -\dfrac{2}{h^2} & \dfrac{4}{h^2} + \dfrac{2c}{3} & -\dfrac{2}{h^2} + \dfrac{c}{3} & \dfrac{1}{h^2} & \\[2ex]
 & \dfrac{1}{h^2} & -\dfrac{2}{h^2} + \dfrac{c}{3} & \dfrac{4}{h^2} + \dfrac{2c}{3} & -\dfrac{2}{h^2} & -\dfrac{1}{h^2}
\end{array}\;. \tag{9}
$$

Remark 3.1. *In [4], the same discretization stencil (9) is obtained for the Poisson equation and symmetric and nonsymmetric version of the discontinuous Galerkin method with interior penalties.*

In comparison with the standard Galerkin method with continuous linear elements, (9) shows that using the NIPG method we loose the nice M–matrix property when $\varepsilon = 1$. For examining inverse–monotonicity, let A_h denote the stiffness matrix related to the stencil (9). If we do not choose $u_0 = u_N = 0$ (that means, if we choose not to satisfy the boundary conditions, which is possible for the discontinuous Galerkin method), then we do not have $A_h^{-1} \geq 0$ as practical calculations show. Now let us choose $u_0 = u_N = 0$ and denote the corresponding modified matrix by $\widetilde{A}_h$.

Lemma 3.1. *If* $\varepsilon = 1$ *and* $c \equiv 0$*, then* $\tilde{A}_h$ *is an inverse monotone matrix.*

Proof. For $c \equiv 0$, the matrix $\tilde{A}_h$ can be written as

$$\tilde{A}_h = \frac{1}{h^2}\begin{bmatrix} 4 & -2 & -1 & & & & \\ -2 & 4 & -2 & 1 & & & \\ 1 & -2 & 4 & -2 & -1 & & \\ & \ddots & \ddots & \ddots & \ddots & \ddots & \\ & & -1 & -2 & 4 & -2 & 1 \\ & & & 1 & -2 & 4 & -2 \\ & & & & -1 & -2 & 4 \end{bmatrix}.$$

First, let us decompose $\tilde{A}_h$ as in [8]

$$\tilde{A}_h = D + \tilde{A}_h^+ - \tilde{A}_h^- ,$$

where D is a diagonal matrix with $D_{ii} = 4/h^2$,

$$(\tilde{A}_h^+)_{ij} = \begin{cases} (\tilde{A}_h)_{ij}\,, & (\tilde{A}_h)_{ij} > 0\,, \\ 0\,, & (\tilde{A}_h)_{ij} \leq 0\,, \end{cases}$$

and

$$(\tilde{A}_h^-)_{ij} = \begin{cases} 0\,, & (\tilde{A}_h)_{ij} > 0\,, \\ -(\tilde{A}_h)_{ij}\,, & (\tilde{A}_h)_{ij} \leq 0\,. \end{cases}$$

Using the theory from [8], we prove that $\tilde{A}_h^{-1} \geq 0$ by showing the existence of a decomposition $\widetilde{A}_h^- = B + C$ with the following properties:

1. there exists a vector $\delta > 0$ such that $D\delta > B\delta$;
2. there exists a vector $e > 0$ such that $\widetilde{A}_h e \geq 0$ and B or C connects $\tau^0(\widetilde{A}_h e)$ with $\tau^+(\widetilde{A}_h e)$, where

$$\tau^0(\widetilde{A}_h e) = \{i : (\widetilde{A}_h e)_i = 0\}\,, \qquad \tau^+(\widetilde{A}_h e) = \{i : (\widetilde{A}_h e)_i > 0\}\,;$$

3. for each $(\widetilde{A}_h)_{ij} > 0$, $i \neq j$, we have

$$(\widetilde{A}_h)_{ij} \leq \sum_k B_{ik} (\widetilde{A}_h)^{-1}_{kk} C_{kj} \,.$$

With the choice

$$B = \frac{1}{h^2} \begin{bmatrix} 0 & 2 & 1 & & & & & \\ 0 & 0 & 2 & 0 & & & & \\ 0 & 2 & 0 & 0 & 1 & & & \\ & 1 & 0 & 0 & 2 & 0 & & \\ & & \ddots & \ddots & \ddots & \ddots & \ddots & \\ & & & 1 & 0 & 0 & 2 & 0 \\ & & & & 0 & 2 & 0 & 0 \\ & & & & & 1 & 2 & 0 \end{bmatrix},$$

$$C = \frac{1}{h^2} \begin{bmatrix} 0 & 0 & 0 & & & & & \\ 2 & 0 & 0 & 0 & & & & \\ 0 & 0 & 0 & 2 & 0 & & & \\ & 0 & 2 & 0 & 0 & 0 & & \\ & & \ddots & \ddots & \ddots & \ddots & \ddots & \\ & & & 0 & 2 & 0 & 0 & 0 \\ & & & & 0 & 0 & 0 & 2 \\ & & & & & 0 & 0 & 0 \end{bmatrix}$$

and $\delta = e = [1, 1, \ldots, 1]^T$, the desired properties can be easily confirmed. $\square$

Remark 3.2. *The statement of the previous Lemma also holds if* $c(x) \geq 0$, $x \in \overline{\Omega}$, *and* $h \leq h_0(c)$.

It is interesting to note that for a fixed h we loose the property $\tilde{A}_h^{-1} \geq 0$ if we increase c. This means that for the reaction–diffusion problem and its discretization with the NIPG method, we cannot expect inverse–monotonicity.

Lemma 3.2. *For* $\varepsilon = 1$, *let us choose* $u_0 = u_N = 0$ *and let* $h \leq h_0(c)$. *Then,*

$$\left| u_i^{\pm} - u(x_i) \right| \leq C h^2 \,. \tag{10}$$

Proof. It can easily be shown that the difference scheme (9) is consistent of order two. As usual for the central difference scheme, [9], there exists a vector e with $\widetilde{A}_h e \geq \varrho$, with some $\varrho > 0$. Together with $\widetilde{A}_h^{-1} \geq 0$ this leads to

$$\|\widetilde{A}_h^{-1}\|_\infty \leq C \,.$$

At the end, stability and consistency of order two lead to the estimate (10). $\square$

Under the assumptions of Lemma 3.2, we have proved as a consequence that the jumps of the approximate solution u_h at the mesh points are of order $\mathcal{O}(h^2)$ when $\varepsilon = 1$.

Let us now consider the discretization of $bu' + cu$, where for simplicity we assume $b(x) = 1$, $c(x) = c$, $x \in [0,1]$. The NIPG method on uniform mesh with $\varepsilon = 1$ generates the stencil

$$-\frac{2}{h}\left|\begin{array}{cc} \frac{1}{h}+\frac{2c}{3} & \frac{1}{h}+\frac{c}{3} \\ -\frac{1}{h}+\frac{c}{3} & \frac{1}{h}+\frac{2c}{3} \end{array}\right| . \tag{11}$$

Elimination of u_i^+ shows that the generated scheme is related to the midpoint upwind scheme, [7],

$$\frac{u_{i+1}^- - u_i^-}{h} + \frac{c}{2}\frac{u_i^- + (1+h/3)u_{i+1}^-}{1+h/6} = \frac{1}{2}\frac{f_i + (1+h/3)f_{i+1}}{1+h/6}.$$

But, unfortunately, it is not possible to achieve the inverse–monotonicity with the stencil (11). This shows the inverse of the matrix

$$B_h = \frac{1}{2}\begin{bmatrix} 1 & 0 & 0 & & & & & \\ -2 & 1 & 1 & 0 & & & & \\ 0 & -1 & 1 & 0 & 0 & & & \\ & 0 & -2 & 1 & 1 & 0 & & \\ & & \ddots & \ddots & \ddots & \ddots & \ddots & \\ & & & 0 & -2 & 1 & 1 & 0 \\ & & & & 0 & -1 & 1 & 0 \\ & & & & & 0 & -2 & 1 \end{bmatrix}$$

given by

$$B_h^{-1} = \begin{bmatrix} 2 & & & & & & \\ 2 & 1 & -1 & & & & \\ 2 & 1 & 1 & & & & \\ 2 & 1 & 1 & 1 & -1 & & \\ 2 & 1 & 1 & 1 & 1 & & \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 4 & 2 & 2 & 2 & 2 & \cdots & 2 \end{bmatrix}.$$

In case of constant functions $b(x) = b$ and $c(x) = c$ in the problem (1) with $\varepsilon = 1$, the property of $\widetilde{A}_h$ dominates the influence of $(b/h)\widetilde{B}_h$, where $\widetilde{B}_h$ represents the stiffness matrix related to the stencil (11) when the boundary conditions are satisfied. This implies that we could repeat Lemmas 3.1 and 3.2 for these data functions and $h \le h_0$. But, unfortunately, in the singularly perturbed case (1) the discretization of the convective part cannot help to preserve the inverse–monotonicity.

3.1. A different basis

Now, let us use the shape functions

$$\psi_1(x) = 1\,, \quad \psi_{2,i}(x) = \frac{x - x_{i+1/2}}{h}\,,$$

instead of φ_{i-1}^+, φ_i^-. To simplify the following, let us denote the mesh points by $x_{i+1/2}$. Then,

$$u_h(x) = u_i + u_i' \frac{x - x_i}{h}\,, \qquad x \in [x_{i-1/2}, x_{i+1/2}]\,. \tag{12}$$

The NIPG method then generates a difference scheme for the approximation of the function values as well as for the derivatives.

As a first example we take (1) with $\varepsilon = 1$, $b \equiv 0$ and $c \equiv 0$. Then, the generated scheme is

$$\begin{aligned} -\frac{u_{i-1} - 2u_i + u_{i+1}}{h^2} &= f_i\,, \\ \frac{u_{i-1}' + 6u_i' + u_{i+1}'}{4h^2} - \frac{u_{i+1} - u_{i-1}}{h^3} &= \frac{1}{12} f_i'\,. \end{aligned} \tag{13}$$

Remark 3.3. *The decoupling of the system takes only place for an equidistant mesh. But this property also holds in 2D on a tensor–product mesh or a mesh from equilateral triangles.*

It is easy to check that the equations in (13) are consistent approximations of the second order of the equations

$$-u'' = f \qquad \text{and} \qquad -u''' = f'\,.$$

The generated scheme for $u' = f$, however, reads

$$\frac{u_i - u_{i-1}}{h} + \frac{u_i' - u_{i-1}'}{2} = f_i\,, \tag{14}$$

$$\frac{u_i' + u_{i-1}'}{4h} - \frac{u_i - u_{i-1}}{2h^2} = \frac{1}{12} f_i'\,. \tag{15}$$

The expression (14) is a second order approximation of $u' = f$. But Taylor expansion in (15) yields

$$\frac{1}{2} u'''(x_j) h + \mathcal{O}(h^2) = f_j'\,,$$

whereby, surprisingly, this equation makes no sense.

4. A remark on L_2–error estimates

From the definition of the DG–norm we can easily obtain that the L^2–error estimate satisfies the inequality (7). The question is: Is it possible to obtain an improved estimate for $\|u - u_h\|_{L^2(\Omega)}$ that is of order $\mathcal{O}(h^2)$?

In the nonsingular case and for reaction–diffusion problems, the NIPG method does not satisfy the adjoint consistency condition (see [1]) that allows the application of Nitsche's trick. Numerical experiments in [2] nevertheless indicate $\mathcal{O}(h^2)$ errors in L^2–norm for linear/bilinear elements, but error reduction for elements of even order.

In the singularly perturbed case we cannot apply Nitsche's trick, but we try to apply superconvergent techniques. Let u^I be the continuous piecewise linear interpolating function of u, $\xi = u^I - u_h$ and $\eta = u - u^I$ the interpolation error. Then,

$$\|\xi\|^2_{\mathrm{DG}} = -a(\eta, \xi)\,.$$

Since η is continuous, $a(\eta, \xi)$ consists of only a few terms:

$$a(\eta, \xi) = \sum_{i=1}^{N} \int_{x_{i-1}}^{x_i} c\eta\xi + \varepsilon \sum_{i=1}^{N-1} \langle \eta' \rangle_i [\xi]_i + \sum_{i=1}^{N} \int_{x_{i-1}}^{x_i} b\eta'\xi\,.$$

Again we are assuming that the boundary conditions are fulfilled. Because the first term gives already the desired $\mathcal{O}(h^2)$ estimate, we have to estimate the both remaining terms. First we have

$$\begin{aligned}
\langle \eta' \rangle_i &= \frac{1}{2}\left(\eta'(x_i + 0) + \eta'(x_i - 0)\right) \\
&= \frac{1}{2}\left(u'(x_i) - \frac{u(x_{i+1}) - u(x_i)}{h} + u'(x_i) - \frac{u(x_i) - u(x_{i-1})}{h}\right) \\
&= u'(x_i) - \frac{u(x_{i+1}) - u(x_{i-1})}{2h} = \mathcal{O}(h^2)\,,
\end{aligned}$$

for the smooth u. Thus,

$$\varepsilon \left| \sum_{i=1}^{N-1} \langle \eta' \rangle_i [\xi]_i \right| \le \varepsilon \left(\frac{1}{\sigma} \sum_{i=1}^{N-1} \langle \eta' \rangle_i^2 \right)^{1/2} \left(\sum_{i=1}^{N-1} \sigma [\xi]_i^2 \right)^{1/2} \le C\sqrt{\varepsilon} h^2 \|\xi\|_{\mathrm{DG}}\,.$$

Finally we consider

$$\int_{x_{i-1}}^{x_i} b(u - u^I)'\xi\,. \tag{16}$$

In the nonsingular case, integration by parts results into an $\mathcal{O}(h^2)$ estimate. In the sequel it suffices to assume a constant b on (x_{i-1}, x_i), for otherwise we have $b(x) = \tilde{b}_i + \mathcal{O}(h)$ and further have no difficulties with the additional term. Introducing

$$\xi(x) = \xi(x_{i-1/2}) + (x - x_{x-1/2})\xi'(x) = \xi(x_{i-1/2}) + E'(x)\xi'(x)\,,$$

with $E(x) = ((x - x_{i-1/2})^2 - h^2)/2$, Lin's technique from [6] results in

$$\int_{x_{i-1}}^{x_i} (u - u^I)'\xi = \frac{h^2}{3}\,(u''\xi)\,\Big|_{x_{i-1}}^{x_i} - \frac{h^2}{3}\int_{x_{i-1}}^{x_i} u'''\xi + \frac{1}{6}\int_{x_{i-1}}^{x_i} u'''(E^2)'\xi'\,. \tag{17}$$

For the continuous function ξ, summation results in an estimate of the type $\mathcal{O}(h^2)\|u\|_{H^3(\Omega)}\|\xi\|_{L^2(\Omega)}$. But for a discontinuous ξ we loose an $h^{1/2}$ due to

$$\xi(x_i) \le \frac{C}{h^{1/2}}\left(\int_{x_{i-1}}^{x_i} \xi^2\right)^{1/2}.$$

Therefore we get

$$\left|\sum_{i=1}^{N}\int_{x_{i-1}}^{x_i} b\eta'\xi\right| \le Ch^{3/2}\|\xi\|_{L^2(\Omega)} \le Ch^{3/2}\|\xi\|_{\mathrm{DG}}\,.$$

Alternatively, we can use the part $\varepsilon^{1/2}\|\nabla\xi\|_{L^2(\Omega)}$ of the DG–norm or also

$$\sigma^{1/2}\left(\sum_{i=1}^{N-1}[\xi]_i^2\right)^{1/2} = \frac{\varepsilon^{1/2}}{h^{1/2}}\left(\sum_{i=1}^{N-1}[\xi]_i^2\right)^{1/2}$$

to obtain

$$\left|\sum_{i=1}^{N}\int_{x_{i-1}}^{x_i} b\eta'\xi\right| \le C\frac{h^2}{\varepsilon^{1/2}}\|\xi\|_{\mathrm{DG}}\,.$$

Combining both estimates, we conclude the following:

Lemma 4.1. *For a smooth solution u, in the singularly perturbed case the discretization with discontinuous linear elements leads to an L^2–error estimate of the type*

$$\|u - u_h\|_{L^2(\Omega)} \le C\frac{h^{\gamma/2+3/2}}{h^{\gamma/2} + \varepsilon^{\gamma/2}}\,,$$

with some $\gamma \in [0, 1]$.

Nevertheless, numerically we observe second–order convergence in the L^2–norm independently of the perturbation parameter for the test example

$$\begin{cases} -\varepsilon u'' + (2 - x)u' + u = f \quad \text{in} \quad (0,1)\,, \\ u(0) = u(1) = 0\,, \end{cases} \tag{18}$$

with the exact solution $u(x) = (1 - x)\sin x$ and for $\varepsilon = 1, 10^{-1}, \ldots, 10^{-8}$, see Table 1.

	$\varepsilon = 1$		$\varepsilon = 10^{-4}$		$\varepsilon = 10^{-8}$	
N	*error*	*rate*	*error*	*rate*	*error*	*rate*
8	2.073(−3)	2.088	4.318(−3)	1.993	4.322(−3)	1.992
16	4.876(−4)	2.047	1.085(−3)	2.000	1.087(−3)	1.998
32	1.180(−4)	2.024	2.713(−4)	2.003	2.721(−4)	1.999
64	2.900(−5)	2.012	6.770(−5)	2.007	6.805(−5)	2.000
128	7.188(−6)	2.006	1.684(−5)	2.014	1.701(−5)	2.000
256	1.789(−6)	2.003	4.170(−6)	2.027	4.254(−5)	2.000
512	4.463(−7)	–	1.023(−6)	–	1.063(−6)	–

Table 1: L^2–norm of the error for the test problem (18)

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