Novi Sad J. Math.
Vol. 33, No. 2, 2003, 77–88

EXPONENTIAL FORMULA FOR ONE–TIME INTEGRATED SEMIGROUPS

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Abstract. In this paper we prove that

$$\lim_{n\to\infty}\int_0^T \left(\frac{n+1}{t}\right)^{n+1} R^{n+1}\left(\frac{n+1}{t};A\right)dt = S(T), T>0$$

where $S(T)$ is one–time integrated exponentially bounded semigroup and limit is uniform in $T>0$ on any bounded interval.

AMS Mathematics Subject Classification (2000): 47D03

Key words and phrases: C_0–semigroup, one–time integrated semigroup, exponential formula

1. Introduction

One–time and n–times, $n>1$, integrated exponentially bounded semigroups $n\in\mathbf{N}$ of operators in Banach space were introduced by Arendt [1] and studied by Arendt, Kellermann, Hieber [2], Thieme [4] and many others.

We study here one–time integrated semigroups and prove the result from Abstract. The main motivation of our investigations is exponential formula for a C_0–semigroup of bounded operators [3].

2. Preliminaries from the semigroup theory

We denote by X a Banach space with the norm $\|\cdot\|$; $L(X)=L(X,X)$ is the space of bounded linear operators from X into X. A family $(T(t))_{t\geq 0}$ in $L(X)$ is a *semigroup* of bounded linear operators on X if

(i) $T(0)=I$, (I is the identity operator on X),

(ii) $T(t+s)=T(t)T(s)$, for every $t,s\geq 0$ (the semigroup property).

If for a semigroup $(T(t))_{t\geq 0}$ the following condition holds

(iii) $\lim\limits_{t\downarrow 0} T(t)x=x$, for every $x\in X$,

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then $(T(t))_{t\geq 0}$ is said to be a *strongly continuous semigroup* or, simply, C_0- semigroup. A linear operator A, defined on the set

$$D(A) = \{x \in X : \lim_{t\downarrow 0} \frac{T(t)x - x}{t} \quad \text{exists}\}$$

by

$$Ax = \lim_{t\downarrow 0} \frac{T(t)x - x}{t} = \frac{d^+T(t)x}{dt}|_{t=0}, \quad x \in D(A)$$

is the *infinitesimal generator* of the semigroup $(T(t))_{t\geq 0}$; $D(A)$ is the domain of A.

Let A be a linear operator on X and let $(T(t))_{t\geq 0}$ be a C_0-semigroup. It is well known that A is the infinitesimal generator of this semigroup iff there exist $\omega \in \mathbf{R}$ such that $(\omega, \infty) \subset \rho(A)$ and $R : \{\lambda \in \mathbf{C} : Re\lambda > \omega\} \to L(X)$, defined by $R(\lambda) = (\lambda I - A)^{-1} = \mathcal{L}(T)(\lambda)$, $Re\lambda > \omega$, where $\mathcal{L}(T)$ is the Laplace transformation of $(T(t))_{t\geq 0}$.

The following theorem holds [3]

Theorem 1. [The exponential formula] *Let $(T(t))_{t\geq 0}$ be a C_0-semigroup on X. If A is the infinitesimal generator of $(T(t))_{t\geq 0}$ then*

$$T(t)x = \lim_{n\to\infty} \left(I - \frac{t}{n}A\right)^{-n} x = \lim_{n\to\infty} [\frac{n}{t}R\left(\frac{n}{t}; A\right)]^n x, \quad t \geq 0, \quad x \in X$$

and the limit is uniform on any bounded interval $[a, b] \subset [0, \infty)$.

3. Preliminaries from the theory of n–times integrated semigroup

Let $\{S(t), t \geq 0\}$ be a strongly continuous exponentially bounded family in $L(X)$. It is called *$n-$times integrated semigroup* if

(i) $S(0) = 0$;

(ii) $S(t)S(s) = \frac{1}{(n-1)!}\left[\int_t^{t+s}(t+s-r)^{n-1}S(r)dr - \int_0^s(t+s-r)^{n-1}S(r)dr\right]$, $t, s \geq 0$, $x \in X$.

In the case $n = 1$, one obtains one–time integrated semigroup.

Recall, a family $\{S(t), t \geq 0\}$ is said to be exponentially bounded if there are constants $\omega \geq 0$ and $M \geq 0$ such that

$$\|S(t)\| \leq Me^{\omega t}, \quad t \geq 0.$$

4. Exponential formula for one–time integrated semigroup

Before we state our result, we will recall some known results.

It is known that there exists a subspace $\mathbf{X}_\omega$ of $\mathbf{X}$ with a norm that generates a stronger topology on $\mathbf{X}_\omega$ then the topology inheriting from the space $\mathbf{X}$. Also, it is known that the restriction of $S(t)$ to $\mathbf{X}_\omega$ forms a $\mathbf{C_0}$–semigroup $(T(t))_{t\geq 0}$ on $\mathbf{X}_\omega$ such that

$$S(t)x = \int_0^t T(s)xds, \quad t \geq 0, \quad x \in \mathbf{X}_\omega.$$

It is known that for all $x \in \mathbf{X}_\omega$ and $t > 0$ the following holds

$$\left(\frac{n}{t}\right)^n R^n \left(\frac{n}{t}; A\right) x \to T(t)x, \quad n \to \infty,$$

where the limit is taken in $\mathbf{X}_\omega$.

In particular, the limit holds in the sense of topology inheriting from the space $\mathbf{X}$ that is

$$S(t)x = \int_0^t T(s)xds = \lim_{n\to\infty} \int_0^t \left(\frac{n}{t}\right) R^n \left(\frac{n}{t}; A\right) xds, \quad x \in \mathbf{X}, \quad \mathbf{t > 0}.$$

On the other hand, we note the following: If the space $\mathbf{X}_\omega$ is dense in $\mathbf{X}$ for a non–degenerate exponentially bounded integrated semigroup $(S(t))_{t\geq 0}$, then the theory of such semigroups would be the trivial consequence of the theory of $\mathbf{C_0}$–semigroups. In this case our result can be obtained by using analogous results which hold on $\mathbf{X}_\omega$.

Further, it is known that for an exponentially bounded integrated semigroup $(S(t))_{t\geq 0}$ there exists a larger space $\mathbf{X}_\lambda$ with a weaker norm $\|\cdot\|$, and a $\mathbf{C_0}$–semigroup $(T_\lambda(t))_{t\geq 0}$ on $\mathbf{X}_\lambda$ such that $(S(t))_{t\geq 0}$ can be obtained as an integral of the restriction of $T_\lambda(t)$ to $\mathbf{X}$. This implies that the following holds

$$\lim_{n\to\infty} \int_0^t \left(\frac{n}{s}\right)^n R^n \left(\frac{n}{s}; A\right) xds = S(t)x, \quad t > 0, \quad x \in \mathbf{X},$$

where integral and limit are taken in $\mathbf{X}_\lambda$.

But we know that this norm is weaker, so we conclude that our result cannot be obtained as a trivial consequence of the known result.

Now we are ready to state our theorem.

Theorem 2. *Let $(S(t))_{t\geq 0}$ be one–time exponentially bounded integrated semigroup. Then*

$$\lim_{n\to\infty} \int_0^T \left(\frac{n+1}{t}\right)^{n+1} R^{n+1} \left(\frac{n+1}{t}; A\right) dt = S(T), \quad T > 0,$$

and the limit is uniform in $T > 0$ on any bounded interval $[a, b] \subset [0, \infty)$.

Proof. We will use the resolvent formula for one–time integrated semigroups

$$R(\lambda; A) = \int_0^\infty \lambda e^{-\lambda s} S(s) ds, \quad \lambda \in \rho(A). \tag{1}$$

Differentiating (1) n times, with respect to λ, we obtain

$$\begin{aligned} \frac{d^n}{d\lambda^n} R(\lambda; A) &= \int_0^\infty [(-1)^n \lambda s^n e^{-\lambda s} + (-1)^{n-1} n e^{-\lambda s} s^{n-1}] S(s) ds \\ &= (-1)^n \int_0^\infty [\lambda s^n - n s^{n-1}] e^{-\lambda s} S(s) ds, \quad n \in \mathbf{N}, \quad \lambda \in \rho(\mathbf{A}). \end{aligned} \tag{2}$$

But $\frac{d^n}{d\lambda^n} R(\lambda; A) = (-1)^n n! R^{n+1}(\lambda; A)$, $n \in \mathbf{N}$, $\lambda \in \rho(A)$, and therefore, from (2) we obtain

$$R^{n+1}(\lambda; A) = \frac{1}{n!} \int_0^\infty [\lambda s^n - n s^{n-1}] e^{-\lambda s} S(s) ds, \quad n \in \mathbf{N}, \quad \lambda \in \rho(\mathbf{A}). \tag{3}$$

For sufficiently large n and $\lambda = \frac{n+1}{t}$, (3) takes the form

$$R^{n+1}(\frac{n+1}{t}; A) = \frac{1}{n!} \int_0^\infty \left(\frac{n+1}{t} s^n - n s^{n+1} \right) e^{-(n+1)\frac{s}{t}} S(s) ds. \tag{4}$$

Take $\delta \in (0, T)$, $N > 0$ and consider the integral

$$I_{\delta,N} = \int_\delta^T \left(\frac{n+1}{t} \right)^{n+1} \frac{1}{n!} \int_0^N \left[\frac{n+1}{t} s^n - n s^{n-1} \right] e^{-\frac{n+1}{t} s} S(s) ds.$$

We interchange the order of integration and obtain

$$I_{\delta,N} = \frac{n+1}{n!} \int_0^N s^n \left[\left(\frac{n+1}{T} \right)^n e^{-\frac{n+1}{T} s} - \left(\frac{n+1}{\delta} \right) e^{-\frac{n+1}{\delta} s} \right] S(s) ds. \tag{5}$$

Recall that the function $S(s)$ is exponentially bounded. Letting $N \to \infty$ in (5), we obtain that the right side in (5), for sufficiently large n, tends to

$$\frac{n+1}{n!} \int_0^\infty s^{n-1} \left[\left(\frac{n+1}{T} \right)^n e^{-\frac{n+1}{T} s} - \left(\frac{n+1}{\delta} \right)^n e^{-\frac{n+1}{\delta} s} \right] S(s) ds.$$

Notice that

$$\int_0^N \left[\frac{n+1}{t} s^n - n s^{n-1} \right] e^{-\frac{n+1}{t}} S(s) ds$$

tends to

$$\int_0^\infty \left[\frac{n+1}{t} s^n - n s^{n-1} \right] e^{-\frac{n+1}{t}} S(s) ds \quad \text{as} \quad N \to \infty$$

and the limit is uniform in $t \in [\delta, T]$, then the left side in (5) tends to

$$\frac{1}{n!}\int_\delta^T \left(\frac{n+1}{t}\right)^{n+1} dt \int_0^\infty \left[\frac{n+1}{t}s^n - ns^{n-1}\right] e^{-\frac{n+1}{t}} S(s)ds.$$

So, we obtain

$$(6) \qquad \frac{1}{n!}\int_\delta^T \left(\frac{n+1}{t}\right)^{n+1} R^{n+1}\left(\frac{n+1}{t}; A\right) dt =$$

$$= \frac{n+1}{n!}\int_0^\infty s^{n-1}\left[\left(\frac{n+1}{T}\right)^n e^{-\frac{n+1}{T}s} - \left(\frac{n+1}{\delta}\right)^n e^{-\frac{n+1}{\delta}s}\right] S(s)ds.$$

Letting $\delta \downarrow 0$ in (6), then the right side in (6) tends to

$$\frac{n+1}{n!}\int_0^\infty s^{n-1}\left(\frac{n+1}{T}\right)^n e^{-\frac{n+1}{T}s} S(s)ds,$$

because

$$\frac{n+1}{n!}\int_0^\infty s^{n-1}\left(\frac{n+1}{\delta}\right)^n e^{-\frac{n+1}{\delta}s} S(s)ds =$$

$$= \frac{n+1}{n!}\int_0^\infty \sigma^{n-1} e^{\sigma} S\left(\frac{\delta\sigma}{n+1}\right) d\sigma \to 0, \quad \text{as} \quad \delta \downarrow 0.$$

So, we conclude that

$$\frac{1}{n!}\int_0^T \left(\frac{n+1}{t}\right)^{n+1} R^{n+1}\left(\frac{n+1}{t}; A\right) dt$$

exists and

$$\frac{1}{n!}\int_0^T \left(\frac{n+1}{t}\right)^{n+1} R^{n+1}\left(\frac{n+1}{t}; A\right) dt =$$

$$= \frac{(n+1)^{n+1}}{n!}\int_0^\infty u^{n-1} e^{-(n+1)u} S(uT)du.$$

Multiplying the left and the right side of (4) by $\left(\frac{n+1}{t}\right)^{n+1}$, we obtain

$$(7) \qquad \left(\frac{n+1}{t}\right)^{n+1} R^{n+1}\left(\frac{n+1}{t}; A\right) =$$

$$= \left(\frac{n+1}{t}\right)^{n+1} \frac{1}{n!}\int_0^\infty \left(\frac{n+1}{t}s^n - ns^{n-1}\right) e^{-(n+1)\frac{s}{t}} S(s)ds.$$

Integrating (7) from 0 to T, we obtain

$$(8) \qquad \int_0^T \left(\frac{n+1}{t}\right)^{n+1} R^{n+1}\left(\frac{n+1}{t}; A\right) dt =$$

$$= \int_0^T \left(\frac{n+1}{t}\right)^{n+1} \frac{1}{n!} \int_0^\infty \left(\frac{n+1}{t} s^n - n s^{n-1}\right) e^{-(n+1)\frac{s}{t}} S(s) ds dt.$$

We interchange the order of integration and obtain

$$\int_0^T \left(\frac{n+1}{t}\right)^{n+1} R^{n+1}\left(\frac{n+1}{t}; A\right) dt = \tag{9}$$

$$= \int_0^\infty S(s) s^{n-1} ds \int_0^T \left(\frac{n+1}{t}\right)^{n+1} \frac{1}{n!} \left[\frac{n+1}{t} s - n\right] e^{-(n+1)\frac{s}{t}} S(s) ds.$$

From the right side of (9), consider

$$\begin{aligned}
&\int_0^T \left(\frac{n+1}{t}\right)^{n+1} \left[(n+1)\frac{s}{t} - n\right] e^{-(n+1)\frac{s}{t}} dt = \left|\frac{s}{t} = v\right| \\
= \; & \int_{\frac{s}{T}}^\infty (n+1)^{n+1} \frac{v^{n+1}}{s^{n+1}} [(n+1)v - n] e^{-(n+1)v} \frac{s dv}{v^2} \\
= \; & \frac{(n+1)^{n+1}}{s^n} \int_{\frac{s}{T}}^\infty [(n+1)v^n - n v^{n-1}] e^{-(n+1)v} dv \\
= \; & \frac{(n+1)^{n+1}}{s^n} \left(\int_{\frac{s}{T}}^\infty (n+1) v^n e^{-(n+1)v} dv - \int_{\frac{s}{T}}^\infty n v^{n-1} e^{-(n+1)v} dv\right).
\end{aligned}$$

Integrating by parts the second integral from the last relation, we obtain

$$\begin{aligned}
&\int_0^T \left(\frac{n+1}{t}\right)^{n+1} [(n+1)\frac{s}{t} - n] e^{-(n+1)\frac{s}{t}} dt \\
&\qquad = \frac{(n+1)^{n+1}}{s^n} \left(\frac{s}{T}\right)^n e^{-(n+1)\frac{s}{T}}.
\end{aligned} \tag{10}$$

Now, (9) becomes

$$\begin{aligned}
&\int_0^T \left(\frac{n+1}{t}\right)^{n+1} R^{n+1}\left(\frac{n+1}{t}; A\right) dt \\
= \; & \frac{(n+1)^{n+1}}{n!} \int_0^\infty S(s) s^{n-1} \left(\frac{s}{T}\right)^n e^{-(n+1)\frac{s}{T}} \frac{1}{s^n} ds \\
= \; & \frac{(n+1)^{n+1}}{n!} \int_0^\infty \frac{S(s) \left(\frac{s}{T}\right)^n e^{-(n+1)\frac{s}{T}}}{s} ds \\
= \; & \frac{(n+1)^{n+1}}{n!} \int_0^\infty \frac{S(s) s^{n-1} e^{-(n+1)\frac{s}{T}}}{T^n} ds.
\end{aligned} \tag{11}$$

So, we obtain

$$\int_0^T \left(\frac{n+1}{t}\right)^{n+1} R^{n+1}\left(\frac{n+1}{t}; A\right) dt \tag{12}$$

$$= \frac{(n+1)^{n+1}}{n!} \int_0^\infty \frac{S(s)s^{n-1}e^{-(n+1)\frac{s}{T}}}{T^n} ds.$$

Using substitution $\frac{s}{T} = u$, (12) takes the form

$$\int_0^T \left(\frac{n+1}{t}\right)^{n+1} R^{n+1}\left(\frac{n+1}{t}; A\right) dt \tag{13}$$

$$= \frac{(n+1)^{n+1}}{n!} \int_0^\infty S(uT)u^{n-1}e^{-(n+1)u} du.$$

Take $z = (n+1)u$. Then (13) takes the form

$$\int_0^T \left(\frac{n+1}{t}\right)^{n+1} R^{n+1}\left(\frac{n+1}{t}; A\right) dt \tag{14}$$

$$= \frac{n+1}{n!} \int_0^\infty S\left(\frac{z}{n+1}T\right) e^{-z}z^{n-1} dz.$$

For every $\epsilon > 0$, choose $\delta > 0$ (small enough) such that for

$$(n+1)\left(1 - \frac{\delta}{T}\right) < z < (n+1)\left(1 + \frac{\delta}{T}\right), \quad T > 0, \quad n \in \mathbf{N}$$

we have

$$\| S\left(\frac{z}{n+1}T\right) x - S(T)x \| < \epsilon, \quad x \in X.$$

We have (with $x \in \mathbf{X}$, $T > 0$, $n \in \mathbf{N}$)

$$I = \frac{n+1}{n!} \int_0^\infty \left[S\left(\frac{z}{n+1}T\right) x - S(T)x\right] e^{-z}z^{n-1} dz = I_1 + I_2 + I_3,$$

where

$$I_1 = \frac{n+1}{n!} \int_0^{(n+1)\left(1-\frac{\delta}{T}\right)} \left[S\left(\frac{z}{n+1}T\right) x - S(T)x\right] e^{-z}z^{n-1} dz$$

$$I_2 = \frac{n+1}{n!} \int_{(n+1)\left(1-\frac{\delta}{T}\right)}^{(n+1)\left(1+\frac{\delta}{T}\right)} \left[S\left(\frac{z}{n+1}T\right) x - S(T)x\right] e^{-z}z^{n-1} dz$$

$$I_3 = \frac{n+1}{n!} \int_{(n+1)\left(1+\frac{\delta}{T}\right)}^\infty \left[S\left(\frac{z}{n+1}T\right) x - S(T)x\right] e^{-z}z^{n-1} dz.$$

We will estimate each of these integrals. We have

$$\| I_1 \| \leq \frac{n+1}{n!} \int_0^{(n+1)\left(1-\frac{\delta}{T}\right)} \| S\left(\frac{z}{n+1}T\right) - S(T)x \| e^{-z}z^{n-1} dz.$$

The assumption $\|S(t)\| \le Me^{\omega t}$, $t \ge 0$, implies

$$\begin{aligned} I_1 &\le M \parallel x \parallel \frac{n+1}{n!} \int_0^{(n+1)\left(1-\frac{\delta}{T}\right)} \left[e^{\frac{\omega T}{n+1} z} + e^{\omega T}\right] e^{-z} z^{n-1} dz \\ &= M \parallel x \parallel \frac{n+1}{n!} \int_0^{(n+1)\left(1-\frac{\delta}{T}\right)} e^{-z\left(1-\frac{\omega T}{n+1}\right)} z^{n-1} dz \\ &\quad + M \parallel x \parallel \frac{n+1}{n!} \int_0^{(n+1)\left(1-\frac{\delta}{T}\right)} e^{\omega T} e^{-z} z^{n-1} dz. \end{aligned}$$

Put

$$\begin{aligned} S_1 &= M \parallel x \parallel \frac{n+1}{n!} \int_0^{(n+1)\left(1-\frac{\delta}{T}\right)} e^{-z\left(1-\frac{\omega T}{n+1}\right)} z^{n-1} dz, \\ S_2 &= M \parallel x \parallel \frac{n+1}{n!} \int_0^{(n+1)\left(1-\frac{\delta}{T}\right)} e^{\omega T} e^{-z} z^{n-1} dz. \end{aligned}$$

Let us estimate S_1. Take $z\frac{n+1-\omega T}{n+1} = u$. Then the integral S_1 becomes

$$S_1 = M \parallel x \parallel \frac{n+1}{n!} \frac{(n+1)^n}{(n+1-\omega T)^n} \int_0^{(n+1-\omega T)\left(1-\frac{\delta}{T}\right)} e^{-u} u^{n-1} du.$$

Let $f_1(u) = e^{-u} u^{n-1}$, $u \in \mathbf{R}$. Differentiating f_1 with respect to u, we obtain

$$\frac{df_1}{du} = e^{-u} u^{n-2} (n-1-u).$$

Function f_1 takes the maximum at the point $(n-1)$. It easy to see that for a large enough n and fixed δ, $n-1$ is greater than $(n+1-\omega T)(1-\frac{\delta}{T})$. Also, f_1 is increasing in the interval $\left[0, (n+1-\omega T)\left(1-\frac{\delta}{T}\right)\right]$. Using these facts, we obtain

$$\begin{aligned} & M \parallel x \parallel \frac{n+1}{n!} \frac{(n+1)^n}{(n+1-\omega T)^n} \int_0^{(n+1-\omega T)\left(1-\frac{\delta}{T}\right)} e^{-u} u^{n-1} du \\ & \quad \le M \parallel x \parallel \frac{n+1}{n!} \frac{(n+1)^n}{(n+1-\omega T)^n} \frac{(n+1-\omega T)^{n-1}\left(1-\frac{\delta}{T}\right)^{n-1}}{e^{(n+1-\omega T)\left(1-\frac{\delta}{T}\right)}}. \end{aligned}$$

Stirling's formula implies

$$M \parallel x \parallel \frac{1}{\sqrt{2\pi}\left(1-\frac{\delta}{T}\right) e^{(1-\omega T)\left(1-\frac{\delta}{T}\right)}} \left(1+\frac{1}{n}\right)^n \frac{1}{1-\frac{\omega T}{n+1}} \frac{\left[\left(1-\frac{\delta}{T}\right) e^{\frac{\delta}{T}}\right]^n}{\sqrt{n}}.$$

Function $\mathbf{R} \ni x \mapsto (1-x)e^x$ has a maximum at point $x = 0$ equals 1. This implies $(1-x)e^x < 1$, $x \in \mathbf{R}$. Take $\delta < T$. Then using the last inequality, we obtain $(1-\delta/T)e^{\frac{\delta}{T}} < 1$. So we obtain that $S_1 \to 0$ as $n \to \infty$, and the limit is uniform in $T > 0$ on any bounded interval.

Let us estimate S_2:

$$S_2 = M \parallel x \parallel \frac{n+1}{n!} e^{\omega T} \int_0^{(n+1)\left(1-\frac{\delta}{T}\right)} e^{-z} z^{n-1} dz.$$

Function $\mathbf{R} \ni z \mapsto e^{-z}z^{n-1}$ has a maximum at the point $n-1$. It is easy to see that for a large enough n and fixed δ, $n-1$ belongs to the interval $\left[(n+1)\left(1-\frac{\delta}{T}\right), (n+1)\left(1+\frac{\delta}{T}\right)\right]$. We have that the function $z \mapsto e^{-z}z^{n-1}$ is increasing for $z < n-1$. Thus,

$$\begin{aligned} S_2 &\leq M \parallel x \parallel \frac{n+1}{n!} e^{\omega T} \frac{(n+1)^{n-1}\left(1-\frac{\delta}{T}\right)^{n-1}}{e^{(n+1)\left(1-\frac{\delta}{T}\right)}} \\ &= \frac{M \parallel x \parallel e^{\omega T}}{\left(1-\frac{\delta}{T}\right) e^{\left(1-\frac{\delta}{T}\right)}} \frac{(n+1)^n \left(1-\frac{\delta}{T}\right)^n}{n! e^{n\left(1-\frac{\delta}{T}\right)}}. \end{aligned}$$

Using Stirling's formula, we obtain

$$S_2 \leq \frac{M \parallel x \parallel e^{\omega T}}{\left(1-\frac{\delta}{T}\right) e^{\left(1-\frac{\delta}{T}\right)} \sqrt{2\pi}} \left(1+\frac{1}{n}\right) \frac{\left[\left(1-\frac{\delta}{T}\right) e^{\frac{\delta}{T}}\right]^n}{\sqrt{n}}.$$

Similarly, for $\left(1-\frac{\delta}{T}\right) e^{\frac{\delta}{T}} < 1$, we obtain $S_2 \to \infty$ as $n \to \infty$, and the limit is uniform in $T > 0$ on any bounded interval.

Now, we will estimate the integral I_2:

$$\begin{aligned} \parallel I_2 \parallel &\leq \frac{n+1}{n!} \int_{(n+1)\left(1-\frac{\delta}{T}\right)}^{(n+1)\left(1+\frac{\delta}{T}\right)} \parallel S\left(\frac{z}{n+1}T\right) x - S(T)x \parallel e^{-z} z^{n-1} dz \\ &\leq \frac{n+1}{n!} \epsilon \int_{(n+1)\left(1-\frac{\delta}{T}\right)}^{(n+1)\left(1+\frac{\delta}{T}\right)} e^{-z} z^{n-1} dz < \epsilon \frac{n+1}{n!} \int_0^{\infty} e^{-z} z^{n-1} dz \\ &= \epsilon \frac{n+1}{n!} (n-1)! = \epsilon \frac{n+1}{n} < 2\epsilon. \end{aligned}$$

Let us estimate the integral I_3:

$$\begin{aligned} \parallel I_3 \parallel &\leq M \parallel x \parallel \frac{n+1}{n!} \int_{(n+1)\left(1+\frac{\delta}{T}\right)}^{\infty} \left[e^{\frac{\omega T}{n+1} z} + e^{\omega T}\right] e^{-z} z^{n-1} dz \\ &= M \parallel x \parallel \frac{n+1}{n!} \int_{(n+1)\left(1+\frac{\delta}{T}\right)}^{\infty} e^{-z\left(1-\frac{\omega T}{n+1}\right)} z^{n-1} dz \\ &\quad + M \parallel x \parallel \frac{n+1}{n!} \int_{(n+1)\left(1+\frac{\delta}{T}\right)}^{\infty} e^{\omega T} e^{-z} z^{n-1} dz = S_3 + S_4, \end{aligned}$$

where

$$\begin{aligned} S_3 &= M \parallel x \parallel \frac{n+1}{n!} \int_{(n+1)\left(1+\frac{\delta}{T}\right)}^{\infty} e^{-z\left(1-\frac{\omega T}{n+1}\right)} z^{n-1} dz \\ S_4 &= M \parallel x \parallel \frac{n+1}{n!} \int_{(n+1)\left(1+\frac{\delta}{T}\right)}^{\infty} e^{\omega T} e^{-z} z^{n-1} dz. \end{aligned}$$

Take $z\frac{n+1-\omega T}{n+1} = u$. Then S_3 becomes

$$S_3 = M \parallel x \parallel \frac{n+1}{n!} \frac{(n+1)^n}{(n+1-\omega T)^n} \int_{(n+1-\omega T)\left(1+\frac{\delta}{T}\right)}^{\infty} e^{-u} u^{n-1} du.$$

Consider the integral

$$\int_{(n+1-\omega T)\left(1+\frac{\delta}{T}\right)}^{\infty} e^{-u} u^{n-1} du.$$

We have

$$\int_{(n+1-\omega T)\left(1+\frac{\delta}{T}\right)}^{\infty} e^{-u} u^{n-1} du = \int_{(n+1-\omega T)\left(1+\frac{\delta}{T}\right)}^{\infty} e^{-u(1-\eta)} e^{-u\eta} u^{n-1} du,$$

for $0 < \eta < 1$. We notice that the function $h(u) = e^{-u\eta} u^{n-1}$, $u \in \mathbf{R}$, has a maximum at the point $\frac{n-1}{\eta}$:

$$h\left(\frac{n-1}{\eta}\right) = \frac{e^{-(n-1)}(n-1)^{n-1}}{\eta^{n-1}}.$$

Now, we obtain

$$\begin{aligned} \int_{(n+1-\omega T)\left(1+\frac{\delta}{T}\right)}^{\infty} e^{-u} u^{n-1} du &= \int_{(n+1-\omega T)\left(1+\frac{\delta}{T}\right)}^{\infty} e^{-u(1-\eta)} e^{-u\eta} u^{n-1} du \\ &< \frac{e^{-(n-1)}(n-1)^{n-1}}{\eta^{n-1}} \frac{1}{1-\eta} e^{-(n+1-\omega T)\left(1+\frac{\delta}{T}\right)(1-\eta)} \\ &= \frac{\eta e e^{-(1-\omega T)\left(1+\frac{\delta}{T}\right)(1-\eta)}}{1-\eta} \frac{(n-1)^{n-1}}{\eta^n e^n e^{n\left(1+\frac{\delta}{T}\right)(1-\eta)}}. \end{aligned}$$

Using Stirling's formula, we obtain

$$S_3 \leq M \frac{\eta}{(1-\eta) e^{(1-\omega T)\left(1+\frac{\delta}{T}\right)(1-\eta)-1} \sqrt{2\pi n}} \frac{n+1}{n-1} \cdot \left(\frac{1}{1-\frac{\omega T}{n+1}}\right)^n \left(1-\frac{1}{n}\right)^n \frac{1}{\eta^n e^{n\left(1+\frac{\delta}{T}\right)(1-\eta)}}. \tag{15}$$

Notice that $\eta^n e^{n\left(1+\frac{\delta}{T}\right)(1-\eta)} = e^{n\ln\eta + n\left(1+\frac{\delta}{T}\right)(1-\eta)}$. Letting $n\to\infty$ in (15), we obtain

$$\frac{n+1}{n-1}\to 1, \quad \left(\frac{n+1}{n+1-\omega T}\right)^n \to e^{\omega T}, \quad \left(1-\frac{1}{n}\right)^n \to e^{-1}.$$

In order to obtain that (15) tends to zero as $n\to\infty$, one must prove the following inequality

$$\ln\eta + \left(1+\frac{\delta}{T}\right)(1-\eta) > 0. \tag{16}$$

Since, $\ln\eta = \ln(1+(\eta-1))$ and the following inequalities hold

$$\frac{\eta-1}{\eta} < \ln(1+(\eta-1)) < \eta - 1,$$

we obtain

$$\ln\eta + \left(1+\frac{\delta}{T}\right)(1-\eta) > \frac{\eta-1}{\eta} + \left(1+\frac{\delta}{T}\right)(1-\eta).$$

On the other hand

$$\frac{\eta-1}{\eta} + \left(1+\frac{\delta}{T}\right)(1-\eta) = \frac{-\eta^2\left(1+\frac{\delta}{T}\right) + \eta\left(2+\frac{\delta}{T}\right) - 1}{\eta}. \tag{17}$$

In order to have that the expression (17) is greater than zero, since $\eta > 0$, we must have

$$-\eta^2\left(1+\frac{\delta}{T}\right) + \eta\left(2+\frac{\delta}{T}\right) - 1 > 0.$$

This inequality holds for

$$\frac{1}{1+\frac{\delta}{T}} < \eta < 1. \tag{18}$$

We obtain that if (18) holds, then (17) tends to zero as $n\to\infty$, and the limit is uniform in $T>0$ on any bounded interval.

Let us estimate S_4:

$$S_4 = M \parallel x \parallel e^{\omega T}\frac{n+1}{n!}\int_{(n+1)\left(1+\frac{\delta}{T}\right)}^{\infty} e^{-z} z^{n-1}dz.$$

Consider

$$\int_{(n+1)\left(1+\frac{\delta}{T}\right)}^{\infty} e^{-z} z^{n-1}dz.$$

We have

$$\begin{aligned}\int_{(n+1)\left(1+\frac{\delta}{T}\right)}^{\infty} e^{-z(1-\psi)}e^{-z\Psi}z^{n-1}dz &< \frac{e^{-(n-1)}(n-1)^{n-1}}{\Psi^{n-1}(1-\Psi)}e^{-(n+1)\left(1+\frac{\delta}{T}\right)(1-\Psi)} \\ &= \frac{e\Psi}{(1-\Psi)e^{\left(1+\frac{\delta}{T}\right)}}\frac{(n-1)^{n-1}}{e^{n\left(1+\frac{\delta}{T}\right)(1-\Psi)}}\frac{1}{\Psi^n}.\end{aligned}$$

Using Stirling's formula, we obtain

$$S_4 < M \parallel x \parallel \frac{e\Psi}{(1-\Psi)e^{\left(1+\frac{\delta}{T}\right)(1-\Psi)}} \frac{n+1}{n-1} \left(1-\frac{1}{n}\right)^n \frac{1}{e^{n\ln\Psi + n\left(1+\frac{\delta}{T}\right)(1-\Psi)}}.$$

Similarly, we have that the right side of the last inequality tends to zero for $1/(1+\frac{\delta}{T}) < \Psi < 1$ when $n \to \infty$, and the limit is uniform in $T > 0$ on any bounded interval. Finally, by using these estimates we obtain $I \to 0$, $n \to \infty$, and the limit is uniform in $T > 0$ on any bounded interval. □

Remark 1. *Recall that*

$$\left[\frac{n}{t} R\left(\frac{n}{t}; A\right)\right]^n x \to T(t)x, \quad \text{as} \quad n \to \infty$$

where $T(t)$ is C_0-semigroup and $R(\lambda; A)$ is the resolvent of its infinitesimal generator. This is related to the Post–Widder real inversion formula

$$f(t) = \lim_{k\to\infty} \frac{(-1)^k}{k!} \left(\frac{k}{t}\right)^{k+1} \hat{f}^{(k)}\left(\frac{k}{t}\right)$$

where $\hat{f}$ is the Laplace transform, see [3]. Using our result, we obtain

$$\int_0^T \left[\left(\frac{n}{t}\right) R\left(\frac{n}{t}; A\right)\right]^n x dt \to \int_0^T T(t) x dt, \quad \text{as} \quad n \to \infty.$$

We see that our theorem gives a similar result for one–time integrated exponentially bounded semigroup $S(s)$, even when this semigroup need not be an integral of some C_0-semigroup.

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Received by the editors November 25, 2003